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THE VORONOVSKAJA TYPE THEOREM FOR THE STANCU BIVARIATE OPERATORS

OVIDIU T. POP

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NATIONAL COLLEGE "MIHAI EMINESCU", 5 MIHAI EMINESCU STREET, SATU MARE 440014, ROMANIA,
VEST UNIVERSITY "VASILE GOLDIŞ" OF ARAD, BRANCH OF SATU MARE, 26 MIHAI VITEAZUL STREET, SATU MARE 440030, ROMANIA
ovidiutiberiu@yahoo.com

ABSTRACT. In this paper, the Voronovskaja type theorem for the Stancu bivariate operators is established. As particular cases, we obtain the Voronovskaja type theorem for the Bernstein and Schurer operators.

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1. INTRODUCTION

In this section, we recall some notions and results which we will use in this article (see [4]).

Define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{iff} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{iff} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.1}$$

For the real number β, we have

$$m + \beta \geq \gamma_\beta \tag{1.2}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0 + \beta = \begin{cases} \max\left\{1+\beta, \{\beta\}\right\}, & \text{iff} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{iff} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.3}$$

For the real numbers α, β, $\alpha \geq 0$, we denote

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{iff} \quad \alpha \leq \beta \\ 1 + \dfrac{\alpha-\beta}{\gamma_\beta}, & \text{iff} \quad \alpha > \beta. \end{cases} \tag{1.4}$$

Remark 1.1. For the real numbers α and β, $\alpha \geq 0$, we have $1 \leq \mu^{(\alpha,\beta)}$.

Lemma 1.1. *For the real numbers α and β, $\alpha \geq 0$, we have*

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.5}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \ldots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.1) - (1.4), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, be defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.6}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$.

These operators are named Bernstein-Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [5]. In [5], the domain of definition for the Bernstein-Stancu operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

Lemma 1.2. *The operators $\left(P_m^{(\alpha,\beta)}\right)_{m \geq m_0}$ verify the following properties*

$$\left(P_m^{(\alpha,\beta)} e_0\right)(x) = 1, \tag{1.7}$$

$$\left(P_m^{(\alpha,\beta)} e_1\right)(x) = x + \frac{\alpha - \beta x}{m+\beta} \tag{1.8}$$

and

$$\left(P_m^{(\alpha,\beta)} e_2\right)(x) = x^2 + \frac{mx(1-x) + (\alpha-\beta x)(2mx + \beta x + \alpha)}{(m+\beta)^2}, \tag{1.9}$$

for any natural number m, $m \geq m_0$ and for any $x \in [0,1]$.

Proof. For the proof see [5] or [6]. ∎

For natural numbers m and s, define $T_{m,s}(x) = \sum_{k=0}^{m}(k-mx)^s p_{m,k}(x)$, for any $x \in [0,1]$. The relations $T_{m,0}(x) = 1$, $T_{m,1}(x) = 0$, $T_{m,2}(x) = mx(1-x)$, $T_{m,3}(x) = mx(1-x)(1-2x)$, $T_{m,4}(x) = 3m^2x^2(1-x)^2 + m\left[x(1-x) - 6x^2(1-x)^2\right]$, for any natural number m and for any $x \in [0,1]$, are known.

Lemma 1.3. a) *There exists* $m(2) \in \mathbb{N}$ *such that*

$$m\left(P_m^{(\alpha,\beta)}\varphi_x^2\right)(x) \leq 1, \tag{1.10}$$

for any $x \in [0,1]$ *and for any natural number* m, $m \geq m(2)$.

b) *There exists* $m(4) \in \mathbb{N}$ *such that*

$$m^2\left(P_m^{(\alpha,\beta)}\varphi_x^4\right) \leq 1, \tag{1.11}$$

for any $x \in [0,1]$*, for any natural number* m*,* $m \geq m(4)$*, where for* $x \in [0,1]$, $\varphi_x : [0,1] \to \mathbb{R}$, $\varphi_x(t) = |t-x|$*, for any* $t \in [0,1]$.

Proof. We have

$$\left(P_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = \left(P_m^{(\alpha,\beta)}e_2\right)(x) - 2x\left(P_m^{(\alpha,\beta)}e_1\right)(x) + x^2\left(P_m^{(\alpha,\beta)}e_0\right)(x)$$

and taking $(1.7)-(1.9)$ into account, we obtain

$$\left(P_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = \frac{mx(1-x) + (\alpha-\beta x)^2}{(m+\beta)^2}.$$

Because $\lim_{m\to\infty} m\left(P_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = x(1-x)$, for any $x \in [0,1]$, there exists $m(2) \in \mathbb{N}$ such that $m\left(P_m^{(\alpha,\beta)}\varphi_x^2\right)(x) - x(1-x) \leq \frac{3}{4}$, for any natural number m, $m \geq m(2)$ and for any $x \in [0,1]$. Taking into account that $x(1-x) \leq \frac{1}{4}$ for any $x \in [0,1]$, the relation (1.10) results. We have

$$\begin{aligned}
\left(P_m^{(\alpha,\beta)}\varphi_x^4\right)(x) &= \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+\alpha}{m+\beta} - x\right)^4 \\
&= \frac{1}{(m+\beta)^4}\sum_{k=0}^{m} p_{m,k}(x)\left[(k-mx) + (\alpha-\beta x)\right]^4 \\
&= \frac{1}{(m+\beta)^4}\big[T_{m,4}(x) + 4(\alpha-\beta x)T_{m,3}(x) + 6(\alpha-\beta x)^2 T_{m,2}(x) \\
&\quad + 4(\alpha-\beta x)^3 T_{m,1}(x) + (\alpha-\beta x)^4 T_{m,0}(x)\big]
\end{aligned}$$

and considering the expressions of $T_{m,0}(x)$, $T_{m,1}(x)$, $T_{m,2}(x)$, $T_{m,3}(x)$, $T_{m,4}(x)$, we obtain

$$\begin{aligned}
\left(P_m^{(\alpha,\beta)}\varphi_x^4\right)(x) &= \frac{1}{(m+\beta)^4}\big\{3m^2x^2(1-x)^2 + m\big[x(1-x) - 6x^2(1-x)^2\big] \\
&\quad + 4(\alpha-\beta x)mx(1-x) + 6(\alpha-\beta x)^2 mx(1-x) + (\alpha-\beta x)^4\big\}.
\end{aligned}$$

Then $\lim_{m\to\infty} m^2\left(P_m^{(\alpha,\beta)}\varphi_x^4\right)(x) = 3x^2(1-x)^2$, for any $x \in [0,1]$, and similarly to (1.10), we get the relation (1.11). ∎

2. PRELIMINARIES

For the real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$, $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, $m_1, m_2, \mu^{(\alpha_1,\beta_1)}$ and $\mu^{(\alpha_2,\beta_2)}$ defined by

$$m_i = \begin{cases} \max\{1, -[\beta_i]\}, & \text{iff} \quad \beta_i \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta_i\}, & \text{iff} \quad \beta_i \in \mathbb{Z} \end{cases}, \tag{2.1}$$

$$\gamma_{\beta_i} = m_i + \beta_i = \begin{cases} \max\{1+\beta_i, \{\beta_i\}\}, & \text{iff} \quad \beta_i \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta_i, 1\}, & \text{iff} \quad \beta_i \in \mathbb{Z} \end{cases}, \tag{2.2}$$

$$\mu^{(\alpha_i,\beta_i)} = \begin{cases} 1, & \text{iff} \quad \alpha_i \leq \beta_i \\ 1 + \dfrac{\alpha_i - \beta_i}{\gamma_{\beta_i}}, & \text{iff} \quad \alpha_i > \beta_i \end{cases} \tag{2.3}$$

where $i \in \{1, 2\}$, let the bivariate operators

$$P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} : C\left(\left[0, \mu^{(\alpha_1,\beta_1)}\right] \times \left(\left[0, \mu^{(\alpha_2,\beta_2)}\right]\right)\right) \to C\left([0,1] \times [0,1]\right)$$

be defined for any function $f \in C\left(\left[0, \mu^{(\alpha_1,\beta_1)}\right] \times \left[0, \mu^{(\alpha_2,\beta_2)}\right]\right)$ by

$$\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right), \tag{2.4}$$

for any natural numbers m, n, $m \geq m_1$, $n \geq m_2$, for any $(x,y) \in [0,1] \times [0,1]$.

In the following, we consider the fixed real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and m_1, m_2, $\mu^{(\alpha_1,\beta_1)}$, $\mu^{(\alpha_2,\beta_2)}$ defined by (2.1) - (2.3).

Lemma 2.1. *If $(x,y) \in [0,1] \times [0,1]$ and m, n are natural numbers, $m \geq m_1$, $n \geq m_2$, then the following equalities hold*

$$\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} e_{00}\right)(x,y) = 1, \tag{2.5}$$

$$\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} e_{10}\right)(x,y) = x + \frac{\alpha_1 - \beta_1 x}{m+\beta_1}, \tag{2.6}$$

$$\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} e_{01}\right)(x,y) = y + \frac{\alpha_2 - \beta_2 y}{n+\beta_2}, \tag{2.7}$$

$$\begin{aligned} &\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} e_{20}\right)(x,y) \\ &= x^2 + \frac{mx(1-x) + (\alpha_1 - \beta_1 x)(2mx + \beta_1 x + \alpha_1)}{(m+\beta_1)^2} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} e_{02}\right)(x,y) \\ &= y^2 + \frac{ny(1-y) + (\alpha_2 - \beta_2 y)(2ny + \beta_2 y + \alpha_2)}{(n+\beta_2)^2}, \end{aligned} \tag{2.9}$$

where $e_{00}(x,y) = 1$, $e_{10}(x,y) = x$, $e_{01}(x,y) = y$, $e_{20}(x,y) = x^2$ and $e_{02}(x,y) = y^2$.

Proof. It follows from Lemma 1.2. ∎

Lemma 2.2. *If m and n are natural numbers, $m \geq m_1$, $n \geq m_2$, then the operator $P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}$ is linear and positive on $[0,1] \times [0,1]$.*

Proof. The proof is immediate. ∎

3. MAIN RESULTS

Lemma 3.1. *Let* $\varphi : \left[0, \mu^{(\alpha_1,\beta_1)}\right] \times \left[0, \mu^{(\alpha_2,\beta_2)}\right] \to \mathbb{R}$ *be a continuous function on* $[0,1]\times[0,1]$. *If* $(x,y)\in[0,1]\times[0,1]$ *and* $\varphi(x,y)=0$, *then*

$$\lim_{m\to\infty}\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\varphi\right)(x,y)=0. \tag{3.1}$$

Proof. Since φ is a continuous function in (x,y), it results that for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$, such that

$$|\varphi(t,\tau)|<\frac{\varepsilon}{4} \tag{3.2}$$

for any $(t,\tau)\in[0,1]\times[0,1]$, $|t-x|<\delta$ and $|\tau-y|<\delta$.
On the other hand, because $\varphi\in C\left([0,1]\times[0,1]\right)$, there exists a positive constant M, such that

$$|\varphi(t,\tau)|\le M \tag{3.3}$$

for any $(t,\tau)\in[0,1]\times[0,1]$.

Next, let m be a natural number, $m\ge\max(m_1,m_2)$ and the fixed numbers ε and δ. We have

$$\left|\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\varphi\right)(x,y)\right| \le\le \sum_{k=0}^{m}\sum_{j=0}^{m}p_{m,k}(x)p_{m,j}(y)\left|\varphi\left(\frac{k+\alpha_1}{m+\beta_1},\frac{j+\alpha_2}{m+\beta_2}\right)\right|. \tag{3.4}$$

Let us divide the set of the sum's indices in the following four classes:

$$I_1(m)=\left\{(k,j):\left|\frac{k+\alpha_1}{m+\beta_1}-x\right|<\delta \quad\text{and}\quad \left|\frac{j+\alpha_2}{m+\beta_2}-y\right|<\delta\right\},$$
$$I_2(m)=\left\{(k,j):\left|\frac{k+\alpha_1}{m+\beta_1}-x\right|\ge\delta \quad\text{and}\quad \left|\frac{j+\alpha_2}{m+\beta_2}-y\right|<\delta\right\},$$
$$I_3(m)=\left\{(k,j):\left|\frac{k+\alpha_1}{m+\beta_1}-x\right|<\delta \quad\text{and}\quad \left|\frac{j+\alpha_2}{m+\beta_2}-y\right|\ge\delta\right\}$$

and

$$I_4(m)=\left\{(k,j):\left|\frac{k+\alpha_1}{m+\beta_1}-x\right|\ge\delta \quad\text{and}\quad \left|\frac{j+\alpha_2}{m+\beta_2}-y\right|\ge\delta\right\}.$$

If we denote

$$\omega_{k,j}(x,y)=p_{m,k}(x)p_{m,j}(y)\left|\varphi\left(\frac{k+\alpha_1}{m+\beta_1},\frac{j+\alpha_2}{m+\beta_2}\right)\right|,\ k,j\in\{1,2,\dots,m\}$$

and

$$S_i=\sum_{(k,j)\in I_i(m)}\omega_{k,j}(x,y),\ i\in\{1,2,3,4\},$$

then relation (3.4) becomes

$$\left|\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\varphi\right)(x,y)\right|\le S_1+S_2+S_3+S_4. \tag{3.5}$$

For the sum S_1, taking (3.2) into account, we have

$$\begin{aligned}S_1 &\le \frac{\varepsilon}{4}\sum_{(k,j)\in I_1(m)}p_{m,k}(x)p_{m,j}(y)\le\frac{\varepsilon}{4}\sum_{k=0}^{m}\sum_{j=0}^{m}p_{m,k}(x)p_{m,j}(y)\\ &=\frac{\varepsilon}{4}\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{00}\right)(x,y),\end{aligned}$$

so $S_1 \leq \frac{\varepsilon}{4}$. From $\left|\frac{k+\alpha_1}{m+\beta_1} - x\right| \geq \delta$ it results that $1 \leq \delta^{-2}\left(\frac{k+\alpha_1}{m+\beta_1} - x\right)^2$. Then, for the sum S_2, taking (3.4) into account, we get

$$\begin{aligned} S_2 &\leq \delta^{-2} \sum_{(k,j)\in I_2(m)} p_{m,k}(x)p_{m,j}(y) \left|\varphi\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{m+\beta_2}\right)\right| \left(\frac{k+\alpha_1}{m+\beta_1} - x\right)^2 \\ &\leq M\delta^{-2} \sum_{(k,j)\in I_2(m)} p_{m,k}(x)p_{m,j}(y) \left(\frac{k+\alpha_1}{m+\beta_1} - x\right)^2 \\ &\leq M\delta^{-2} \sum_{k=0}^{m}\sum_{j=0}^{m} p_{m,k}(x)p_{m,j}(y) \left(\frac{k+\alpha_1}{m+\beta_1} - x\right)^2 \\ &= M\delta^{-2} \sum_{j=0}^{m} p_{m,j}(y) \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k+\alpha_1}{m+\beta_1} - x\right)^2 = M\delta^{-2}\left(P_m^{(\alpha_1,\beta_1)}\varphi_x^2\right)(x), \end{aligned}$$

and so, due to (1.10), we obtain

$$S_2 \leq M\delta^{-2}m^{-1}.$$

In the same way, applying Lemma 1.3, we obtain

$$S_3 \leq M\delta^{-2}m^{-1}$$

and

$$S_4 \leq M\delta^{-4}m^{-2}.$$

Since the numbers ε, δ and M are fixed, there exist the natural numbers m_3, m_4, m_5, $m_i \geq \max\{m_1, m_2\}$, $i \in \{3,4,5\}$, such that $S_2 < \frac{\varepsilon}{4}$ for $m \geq m_3$, $S_3 < \frac{\varepsilon}{4}$ for $m \geq m_4$ and $S_4 < \frac{\varepsilon}{4}$ for $m \geq m_5$.

Let $p(\varepsilon) = \max\{m_3, m_4, m_5\}$. Then, for $\varepsilon > 0$, there exists the natural number $p(\varepsilon)$ such that, for any natural number m, $m \geq p(\varepsilon)$, we have

$$S_1 + S_2 + S_3 + S_4 < \varepsilon. \tag{3.6}$$

From (3.5) and (3.6), we obtain (3.1). ∎

We can now prove the Voronovskaja type theorem for the Stancu bivariate operators.

Theorem 3.2. *Let* $f : \left[0, \mu^{(\alpha_1,\beta_1)}\right] \times \left[0, \mu^{(\alpha_2,\beta_2)}\right] \to \mathbb{R}$ *be a function. If* $(x,y) \in [0,1]\times[0,1]$, f *is two times differentiable on* $[0,1]\times[0,1]$ *and the partial derivatives of the second order of* f *are continuous in* (x,y), *then*

$$\begin{aligned} \lim_{m\to\infty} m\left[\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}f\right)(x,y) - f(x,y)\right] &= (\alpha_1 - \beta_1 x)f'_x(x,y) + (\alpha_2 - \beta_2 y)f'_y(x,y) \\ &+ \frac{1}{2}\left[x(1-x)f''_{x^2}(x,y) + y(1-y)f''_{y^2}(x,y)\right]. \end{aligned} \tag{3.7}$$

Proof. Let m be a natural number, $m \geq \max\{m_1, m_2\}$. By making use of the Taylor formula for $(t,\tau) \in [0,1]\times[0,1]$, we have

$$\begin{aligned} f(t,\tau) = f(x,y) &+ (t-x)f'_x(x,y) + (\tau-y)f'_y(x,y) + \frac{1}{2}\big[(t-x)^2 f''_{x^2}(x,y) \\ &+ 2(t-x)(\tau-y)f''_{xy}(x,y) + (\tau-y)^2 f''_{y^2}(x,y)\big] \\ &+ \omega(t,\tau)\big[(t-x)^2 + (\tau-y)^2\big], \end{aligned}$$

where ω is a continuous function on $[0,1]\times[0,1]$ and $\omega(x,y)=0$. According to Lemma 2.2 and Lemma 2.1, we have

$$\begin{aligned}&\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}f\right)(x,y)\\&=f(x,y)+\frac{\alpha_1-\beta_1x}{m+\beta_1}f'_x(x,y)+\frac{\alpha_2-\beta_2y}{m+\beta_2}f'_y(x,y)\\&+\frac{1}{2}\left[\frac{mx(1-x)+(\alpha_1-\beta_1x)^2}{(m+\beta_1)^2}f''_{x^2}(x,y)+2\frac{\alpha_1-\beta_1x}{m+\beta_1}\cdot\frac{\alpha_2-\beta_2y}{m+\beta_2}f''_{xy}(x,y)\right.\\&\left.+\frac{my(1-y)+(\alpha_2-\beta_2y)^2}{(m+\beta_2)^2}f''_{y^2}(x,y)\right]\\&+\left(P_{n,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\left(\omega(\cdot,*)\left[(\cdot-x)^2+(*-y)^2\right]\right)\right)(x,y),\end{aligned}$$

and hence

$$\begin{aligned}&m\left[\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}f\right)(x,y)-f(x,y)\right]\\&=\frac{m}{m+\beta_1}(\alpha_1-\beta_1x)f'_x(x,y)+\frac{m}{m+\beta_2}(\alpha_2-\beta_2y)f'_y(x,y)\\&+\frac{m^2x(1-x)+m(\alpha_1-\beta_1x)^2}{2(m+\beta_1)^2}f''_{x^2}(x,y)\\&+\frac{2m(\alpha_1-\beta_1y)(\alpha_2-\beta_2y)}{(m+\beta_1)(m+\beta_2)}f''_{xy}(x,y)\\&+\frac{m^2y(1-y)+m(\alpha_2-\beta_2y)^2}{2(m+\beta_2)^2}f''_{y^2}(x,y)\\&+m\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\left(\omega(\cdot,*)\left[(\cdot-x)^2+(*-y)^2\right]\right)\right)(x,y),\end{aligned}\tag{3.8}$$

where "$\cdot$" and "$*$" stand for the first and second variable.

By Cauchy's inequality, it follows that

$$\begin{aligned}&\left|\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\left(\omega(\cdot,*)\left[(\cdot-x)^2+(*-y)^2\right]\right)\right)(x,y)\right|\\&\le\left[\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\omega^2(\cdot,*)\right)(x,y)\right]^{\frac{1}{2}}\\&\cdot\left[\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\left[(\cdot-x)^2+(*-y)^2\right]^2\right)(x,y)\right]^{\frac{1}{2}}.\end{aligned}\tag{3.9}$$

We have

$$\begin{aligned}&m^2\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\left[(\cdot-x)^2+(*-y)^2\right]^2\right)(x,y)\\&=m^2\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\left[(\cdot-x)^4+2(\cdot-x)^2(*-y)^2+(*-y)^4\right]\right)(x,y)\\&=m^2\left(P_m^{(\alpha_1,\beta_1)}\varphi_x^4\right)(x)+2m^2\left(P_m^{(\alpha_1,\beta_1)}\varphi_x^2\right)(x)\left(P_m^{(\alpha_2,\beta_2)}\varphi_y^2\right)(y)\\&+m^2\left(P_m^{(\alpha_2,\beta_2)}\varphi_y^4\right)(y)\end{aligned}$$

and taking Lemma 1.3 into account, we get

$$m^2\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\left[(\cdot-x)^2+(*-y)^2\right]^2\right)(x,y)\le 4\tag{3.10}$$

for any natural number m, $m\ge m_0$, where m_0 is specially chosen, so that the relations of types (1.10) and (1.11) hold for x and y.

From the Lemma 3.1 it follows that

$$\lim_{m\to\infty}\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\omega^2(\cdot,*)\right)(x,y)=0.\tag{3.11}$$

From (3.8) - (3.11), we arrive at the desired result. ∎

Application 3.1. If $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, then we obtain the Bernstein bivariate operators $(B_{m,n})_{m,n\geq 1}$, $B_{m,n} : C([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ defined for any function $f \in C([0,1]\times[0,1])$ by

$$(3.12) \qquad (B_{m,n}f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) f\left(\frac{k}{m},\frac{j}{n}\right),$$

for any non zero natural numbers m, n, for any $(x,y)\in[0,1]\times[0,1]$. From Theorem 3.2, the Voronovskaja type theorem for the Bernstein bivariate operators follows.

Theorem 3.3. *Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be a function. If $(x,y)\in[0,1]\times[0,1]$, f is two times differentiable on $[0,1]\times[0,1]$ and the partial derivatives of the second order of f are continuous in (x,y), then*

$$(3.13) \qquad \lim_{m\to\infty} m\left[(B_{m,m}f)(x,y) - f(x,y)\right] = \frac{1}{2}\left[x(1-x)f''_{x^2}(x,y) + y(1-y)f''_{y^2}(x,y)\right].$$

Application 3.2. If p and q are natural numbers, $\alpha_1 = \alpha_2 = 0$, $\beta_1 = -p$, $\beta_2 = -q$, replace m by $m+p$ and n by $n+q$, then $\gamma_{\beta_1} = \gamma_{-p} = 1$, $\gamma_{\beta_2} = \gamma_{-q} = 1$, $\mu^{(\alpha_1,\beta_1)} = \mu^{(0,-p)} = 1+p$, $\mu^{(\alpha_2,\beta_2)} = \mu^{(0,-q)} = 1+q$, $m_1 = 1$, $m_2 = 1$ and $P^{(0,-p)(0,-q)}_{m+p,n+q} = \widetilde{B}_{m,n,p,q}$. Hence, we obtain the bivariate operators of Bernstein-Schurer $\left(\widetilde{B}_{m,n,p,q}\right)_{m,n\geq 1}$, $\widetilde{B}_{m,n,p,q} : C([0,1+p]\times[0,1+q]) \to C([0,1]\times[0,1])$ defined for any function $f\in C([0,1+p]\times[0,1+q])$ by

$$(3.14) \qquad \left(\widetilde{B}_{m,n,p,q}f\right)(x,y) = \sum_{k=0}^{m+p}\sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y) f\left(\frac{k}{m},\frac{j}{n}\right),$$

for any non zero natural numbers m, n, for any $(x,y)\in[0,1]\times[0,1]$, where $\widetilde{p}_{m,k}(x)$, $\widetilde{p}_{n,j}(x)$ are the fundamental Schurer polynomials, $\widetilde{p}_{m,k}(x) = p_{m+p,k}(x)$ and $\widetilde{p}_{n,j}(y) = p_{n+q,j}(y)$.

From Theorem 3.2, the Voronovskaja type theorem for the Bernstein-Schurer bivariate operators follows (see [2]).

Theorem 3.4. *Let $f : [0,1+p]\times[0,1+q] \to \mathbb{R}$ be a function. If $(x,y)\in[0,1]\times[0,1]$, f is two times differentiable on $[0,1]\times[0,1]$ and the partial derivatives of the second order of f are continuous in (x,y), then*

$$(3.15) \qquad \lim_{m\to\infty} m\left[\left(\widetilde{B}_{m,n,p,q}f\right)(x,y) - f(x,y)\right] = pxf'_x(x,y) + qyf'_y(x,y) + \frac{1}{2}\left[x(1-x)f''_{x^2}(x,y) + y(1-y)f''_{y^2}(x,y)\right].$$

Application 3.3. If p and q are natural numbers, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, we replace m by $m+p$, n by $n+q$, β_1 by $\beta_1 - p$ and β_2 by $\beta_2 - q$, then $P^{(\alpha_1,\beta_1-p)(\alpha_2,\beta_2-q)}_{m+p,n+q} = \widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q}$. So, we obtain the bivariate operators of Schurer-Stancu $\left(\widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q}\right)_{m,n\geq 1}$, $\widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q} : C([0,1+p]\times[0,1+q]) \to C([0,1]\times[0,1])$ defined for any function $f\in C([0,1+p]\times[0,1+q])$ by

$$(3.16) \qquad \left(\widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q}f\right)(x,y) = \sum_{k=0}^{m+p}\sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1},\frac{j+\alpha_2}{n+\beta_2}\right)$$

for any non zero natural numbers m, n, for any $(x,y)\in[0,1]\times[0,1]$ (see [3]).

Theorem 3.5. *Let* $f : [0, 1 + p] \times [0, 1 + q] \to \mathbb{R}$ *be a function. If* $(x, y) \in [0, 1] \times [0, 1]$, f *is two times differentiable on* $[0, 1] \times [0, 1]$ *and the partial derivatives of the second order of* f *are continuous in* (x, y), *then*

$$\begin{aligned} &\lim_{m\to\infty} m\left[\left(\widetilde{S}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p,q} f\right)(x, y) - f(x, y)\right] \\ &= \left[\alpha_1 - (\beta_1 - p)x\right] f'_x(x, y) + \left[\alpha_2 - (\beta_2 - q)y\right] f'_y(x, y) \\ &+ \frac{1}{2}\left[x(1 - x)f''_{x^2}(x, y) + y(1 - y)f''_{y^2}(x, y)\right]. \end{aligned} \tag{3.17}$$

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