The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 3, Issue 1, Article 9, pp. 1-7, 2005

A GENERAL COMMON FIXED POINT THEOREM FOR RECIPROCALLY CONTINUOUS MAPPINGS SATISFYING AN IMPLICIT RELATION

A. DJOUDI AND A. ALIOUCHE

Received 11 April, 2005; accepted 22 December, 2005; published 10 March, 2006.
Communicated by: G. Yuan

FACULTY OF SCIENCE, UNIVERSITY OF ANNABA, P.O. BOX 23000, ANNABA, ALGERIA.
adjoudi@yahoo.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LARBI BEN M'HIDI, OUM-EL-BOUAGHI 04000, ALGERIA.
abdmath@hotmail.com

ABSTRACT. A general common fixed point theorem for compatible mappings satisfying an implicit relation is obtained by replacing the continuity of one mapping by the reciprocal continuity of two mappings.

Key words and phrases: Common fixed point, compatible mappings, reciprocally continuous mappings.

2000 *Mathematics Subject Classification.* 54H25.

ISSN (electronic): 1449-5910

Thanks to Dr. Rhoades, Dr. Telci and Dr. Popa for their papers.

1. INTRODUCTION

In recent years, some interesting common fixed point theorems for contractive type mappings have been reported in the literature, ex, Jachymski [2], Jungck et al [4] and Pant [6] and [7]. These theorems require a compatibility condition, a contractive condition and assume continuity at least one of the mappings and each theorem aims to weaken one or more of these conditions. Recently, Pant [8] introduced the notion of reciprocally continuous which is weaker than the continuity of one of the mappings.

Two self-mappings A and S of a metric space (X, d) are said to be compatible [3] if whenever $\{x_n\} \subset X$ is such that $\lim_{n\to+\infty} Sx_n = \lim Ax_n = t \in X$, it follows that

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0. \tag{1.1}$$

Two self-mappings A and S of a metric space (X, d) are said to be reciprocally continuous [8] if $\lim_{n\to+\infty} ASx_n = At$ and $\lim_{n\to+\infty} SAx_n = St$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

If A and S are both continuous, then they are obviously reciprocally continuous, but the converse is not true. Moreover, in the setting of common fixed point theorems for compatible mappings satisfying contractive conditions, continuity of one of the mappings A or S implies their reciprocal continuity, but not conversely.

The following theorem has been proved in [8].

Theorem 1.1. *Let $\{A_i\}$, $i = 1, 2, ...$, S and T be self-mappings of a complete metric space (X, d) such that*

$$A_1(X) \subset T(X) \text{ and } A_i(X) \subset S(X), \text{ if } i > 1, \tag{1.2}$$

$$d(A_1x, A_2y) \leq \Phi(M_{12}(x, y)), \text{ whenever} M_{12}(x, y) > 0, \tag{1.3}$$

$$d(A_1x, A_iy) < M_{1i}(x, y), \tag{1.4}$$

where

$$M_{1i}(x, y) = \max\left\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), \frac{d(A_1x, Ty) + d(Sx, A_iy)}{2}\right\} \tag{1.5}$$

and $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semi-continuous function such that $\Phi(t) < t$ for each $t > 0$.

Let S be compatible with A_1 and T be compatible with A_k for some $k > 1$. If the mappings in one of the compatible pairs $\{A_1, S\}$ and $\{A_k, T\}$ are reciprocally continuous, then $\{A_i\}$, S and T have a unique common fixed point.

It is our purpose in this paper to prove a common fixed point theorem for reciprocally continuous mappings satisfying an implicit relation. Our theorem generalizes the results of [1], [2], [4], [5], [6], [7], [8], [9], [10] and [11].

2. IMPLICIT RELATIONS

Let $\mathcal{F}$ be the set of all continuous functions $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}^6_+ \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is non- increasing in variables t_5 and t_6.

(F_2) : there exists $0 < \alpha < 1$ such that for all $u, v \geq 0$ with

(F_a) : $F(u, v, u, v, u + v, 0) \leq 0$ or

(F_b) : $F(u, v, v, u, 0, u + v) \leq 0$ we have $u \leq \alpha v$.

(F_3) : $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Example 2.1. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - h\max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\}$, $0 \le h < 1$.
(F_1) : *is clear.*
(F_2) : *Let* $u,v \ge 0$ *and* $F(u,v,u,v,u+v,0) = u - h\max\{u,v,\frac{1}{2}(u+v)\} \le 0$.
If $v \le u$*, then* $u < u$ *a contradiction. Therefore,* $u \le hv$, $\alpha = h < 1$.
Similarly, if $F(u,v,v,u,0,u+v) \le 0$ *then* $u \le \alpha v$, $\alpha = h < 1$.
(F_3) : $F(u,u,0,0,u,u) = u(1-h) > 0$ *for all* $u > 0$.

Example 2.2. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - c_1\max\{t_2^2,t_3^2,t_4^2\} - c_2\max\{t_3t_6,t_4t_5\} - c_3t_5t_6$, $c_1,c_2,c_3 \ge 0$, $c_1+2c_2+c_3 < 1$.
(F_1) : *is clear.*
(F_2) : *Let* $u,v \ge 0$ *and* $F(u,v,u,v,u+v,0) = u^2 - c_1\max\{v^2,u^2\} - c_2v(u+v)\} \le 0$.
If $v \le u$*, then* $u^2 < (c_1+2c_2)u^2 < u^2$ *a contradiction. Therefore,* $u \le \sqrt{c_1+2c_2}v$, $\alpha = \sqrt{c_1+2c_2} < 1$.
Similarly, if $F(u,v,v,u,0,u+v) \le 0$ *then* $u \le \alpha v$*,* $\alpha = \sqrt{c_1+2c_2} < 1$.
(F_3) : $F(u,u,0,0,u,u) = u^2(1-(c_1+c_3)) > 0$ *for all* $u > 0$.

Example 2.3. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$, $a,b,c,d \ge 0$, $a+b+c+d < 1$.
(F_1) : *is clear.*
(F_2) : *Let* $u,v \ge 0$ *and* $F(u,v,u,v,u+v,0) = u^3 - au^2v - bu^2v = u^2(u-(a+b)v) \le 0$. *Then* $u \le (a+b)v$*,* $\alpha = a+b < 1$.
Similarly, if $F(u,v,v,u,0,u+v) \le 0$ *then* $u \le \alpha v$*,* $\alpha = a+b < 1$.
(F_3) : $F(u,u,0,0,u,u) = u^3(1-(a+c+d)) > 0$ *for all* $u > 0$.

Example 2.4. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - c\frac{t_3^2t_4^2+t_5^2t_6^2}{t_2+t_3+t_4+1}$, $0 < c < 1$.
(F_1) : *is clear.*
(F_2) : *Let* $u,v \ge 0$ *and* $F(u,v,u,v,u+v,0) = u^3 - c\frac{u^2v^2}{2v+u+1} \le 0$. *Then,* $u < c\frac{v^2}{2v+u+1} < cv$, $\alpha = c$.
Similarly, if $F(u,v,v,u,0,u+v) \le 0$ *then* $u < \alpha v$*,* $\alpha = c$.
(F_3) : $F(u,u,0,0,u,u) = u^3\frac{(1-c)u+1}{u+1} > 0$ *for all* $u > 0$.

Example 2.5. $F(t_1,t_2,t_3,t_4,t_5,t_6) = (1+pt_2)t_1 - p\max\{t_3t_4,t_5t_6\} - h\max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\}$, $0 \le h < 1$, $p \ge 0$.
(F_1) : *is clear.*
(F_2) : *Let* $u,v \ge 0$ *and* $F(u,v,u,v,u+v,0) = (1+pv)u - puv - h\max\{u,v,\frac{1}{2}(u+v)\} \le 0$. *If* $v \le u$*, then* $u < u$ *a contradiction. Therefore,* $u \le hv$, $\alpha = h < 1$.
Similarly, if $F(u,v,v,u,0,u+v) \le 0$ *then* $u \le \alpha v$, $\alpha = h < 1$.
(F_3) : $F(u,u,0,0,u,u) = u(1-h) > 0$ *for all* $u > 0$.

Example 2.6. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - h\max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6),b\sqrt{t_5t_6}\}$, $0 < h < 1$, $0 < b < 1$.
(F_1) : *is clear.*
(F_2) : *As in Example 2.1.*
(F_3) : $F(u,u,0,0,u,u) = u - h\max\{u,bu\} = u(1-h) > 0$ *for all* $u > 0$.

3. MAIN RESULTS

Theorem 3.1. *Let* $\{A_i\}$, $i = 1,2,...,$ S *and* T *be self-mappings of a metric space* (X,d) *satisfying the inequality*

(3.1) $$F(d(A_1x,A_iy),d(Sx,Ty),d(A_1x,Sx),d(A_iy,Ty),d(A_1x,Ty),d(Sx,A_iy)) \le 0$$

$i = 1, 2, ...$, for all x, y in X where F satisfies (F_3). Then, $\{A_i\}$, S and T have at most one common fixed point.

Proof. Suppose that $\{A_i\}$, S and T have two common fixed points z and z' such that $z \neq z'$. Then, using (3.1) we have for $i > 1$

$$F(d(A_1z, A_iz'), d(Sz, Tz'), d(A_1z, Sz), d(A_iz', Tz'), d(A_1z, Tz'), d(Sz, A_iz') = F(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z, z' \leq 0 \tag{3.2}$$

a contradiction of (F_3). ∎

Theorem 3.2. *Let $\{A_i\}$, $i = 1, 2, ...$, S and T be self-mappings of a complete metric space (X, d) such that (1.2) holds and there exists $F \in \mathcal{F}$ such that inequality (3.1) holds for all x, y in X. Let S be compatible with A_1 and T be compatible with A_k for some $k > 1$. If the mappings in one of the compatible pairs $\{A_1, S\}$ and $\{A_k, T\}$ are reciprocally continuous, then $\{A_i\}$, S and T have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X, then by (1.2), there exists a point $x_1 \in X$ such that $A_1x_0 = Tx_1$. For this point x_1 we can choose a point x_2 such that $A_ix_1 = Sx_2$ and soon. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = A_1x_{2n} = Tx_{2n+1},\ y_{2n+1} = Sx_{2n+2} = A_ix_{2n+1} \tag{3.3}$$

for every $n = 0, 1, 2, ...$

Using (3.1) and (3.3) it follows that

$$\begin{aligned} 0 \geq\ & F(d(A_1x_{2n}, A_ix_{2n+1}), d(Sx_{2n}, A_ix_{2n+1}), d(A_1x_{2n}, Sx_{2n}), \\ & d(A_ix_{2n+1}, Tx_{2n+1}), d(A_1x_{2n}, Tx_{2n+1}), d(Sx_{2n}, A_ix_{2n+1})) \end{aligned} \tag{3.4}$$

$$\begin{aligned} 0 \geq\ & F(d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), \\ & d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n+1})). \end{aligned} \tag{3.5}$$

Using (F_1) we get

$$\begin{aligned} 0 \geq\ & F(d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), \\ & d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})). \end{aligned} \tag{3.6}$$

By (F_a), there exists $0 < \alpha < 1$ such that

$$d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n-1}, y_{2n}). \tag{3.7}$$

Similarly, we get

$$d(y_{2n+1}, y_{2n+2}) \leq \alpha d(y_{2n}, y_{2n+1}). \tag{3.8}$$

Using (3.7) and (3.8) we obtain

$$d(y_n, y_{n+1}) \leq \alpha^n d(y_{n-1}, y_n). \tag{3.9}$$

Therefore, $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, it converges to a point $z \in X$. Hence, the subsequences $\{A_1x_{2n}\}$, $\{A_ix_{2n+1}\}$, $\{Sx_{2n+2}\}$ and $\{Tx_{2n+1}\}$, $i > 1$ converge also to z.

Now, suppose that the pair $\{A_1, S\}$ is compatible and A_1 and S are reciprocally continuous. Then

$$\lim_{n\to\infty} d(SA_1x_{2n}, A_1Sx_{2n}) = 0 \tag{3.10}$$

and

$$\lim_{n\to\infty} SA_1x_{2n} = Sz,\ \lim_{n\to\infty} A_1Sx_{2n} = A_1z. \tag{3.11}$$

By (3.10) and (3.11) we have $A_1 z = Sz$.

If $A_1 z \neq z$, using (3.1) we have

$$F(d(A_1 z, A_i x_{2n+1}), d(Sz, Tx_{2n+1}), d(A_1 z, Sz), d(A_i x_{2n+1}, Tx_{2n+1}), d(A_1 z, Tx_{2n+1}), d(Sz, A_i x_{2n+1})) \leq 0. \tag{3.12}$$

Letting n tend to infinity in (3.12) we obtain

$$F(d(A_1 z, z), d(A_1 z, z), 0, 0, d(A_1 z, z), d(A_1 z, z)) \leq 0 \tag{3.13}$$

a contradiction of (F_3). Then, $A_1 z = Sz = z$.

Since $A_1(X) \subset T(X)$, there exists $v \in X$ such that $A_1 z = Tv = z$.

If $z \neq A_k v$, using (3.1) we get

$$F(d(A_1 x_{2n}, A_k v), d(Sx_{2n}, Tv), d(A_1 x_{2n}, Sx_{2n}), d(A_k v, Tv), d(A_1 x_{2n}, Tv), d(Sx_{2n}, A_k v)) \leq 0. \tag{3.14}$$

Letting n tend to infinity in (3.14) we obtain

$$F(d(z, A_k v), 0, 0, d(z, A_k v), 0, d(z, A_k v)) \leq 0. \tag{3.15}$$

By (F_b) we have $z = A_k v = Tv$. Since the pair $\{A_k, T\}$ is compatible we have $TA_k v = A_k Tv$, i.e, $A_k z = Tz$. If $A_k z \neq z$, using (3.1) we have

$$F(d(A_1 z, A_k z), d(Sz, Tz), d(A_1 z, Sz), d(A_k z, Tz), d(A_1 z, Tz), d(Sz, A_k z)) \leq 0. \tag{3.16}$$

So,

$$F(d(z, A_k z), d(z, A_k z), 0, 0, d(z, A_k z), d(z, A_k z)) \leq 0, \tag{3.17}$$

a contradiction of (F_3). Then $A_k z = Tz = z$.

If $A_i z \neq z$, using (3.1) we get

$$F(d(A_1 z, A_i z), d(Sz, Tz), d(A_1 z, Sz), d(A_i z, Tz), d(A_1 z, Tz), d(Sz, A_i z)) \leq 0. \tag{3.18}$$

Therefore,

$$F(d(z, A_i z), 0, 0, d(z, A_i z), 0, d(z, A_i z)) \leq 0, \tag{3.19}$$

a contradiction of (F_b). Then $A_i z = Tz = z$, $i > 1$

Then, $A_1 z = Sz = A_i z = Tz = z$, $i > 1$. Hence, $\{A_i\}$, S and T have a common fixed point z in X.

Now, suppose that the pair $\{A_k, T\}$ is compatible and A_k and T are reciprocally continuous, then

$$\lim_{n\to\infty} d(TA_k x_{2n}, A_k T x_{2n}) = 0 \tag{3.20}$$

and

$$\lim_{n\to\infty} TA_k x_{2n+1} = Tz, \lim_{n\to\infty} A_k T x_{2n+1} = A_k z. \tag{3.21}$$

By (3.20) and (3.21) we have $A_k z = Tz$.

If $A_k z \neq z$, using (3.1) we have

$$F(d(A_1 x_{2n}, A_k z), d(Sx_{2n}, Tz), d(A_1 x_{2n}, Sx_{2n}), d(A_k z, Tz), d(A_1 x_{2n}, Tz), d(Sx_{2n}, A_k z)) \leq 0. \tag{3.22}$$

Letting n tend to infinity in (3.22) we obtain

$$F(d(z, A_k z), d(z, A_k z), 0, 0, d(z, A_k z), d(z, A_k z)) \leq 0, \tag{3.23}$$

a contradiction of (F_3). Then $A_k z = Tz = z$.

Since $A_k(X) \subset S(X)$, there exists $u \in X$ such that $A_k z = Su = z$.

If $z \neq A_1 u$, using (3.1) we get

$$\begin{aligned} F(d(A_1 u, A_k x_{2n+1}), d(Su, T x_{2n+1}), d(A_1 u, Su), \\ d(A_k x_{2n+1} T x_{2n+1}), d(A_1 u, T x_{2n+1}), d(Su, A_k x_{2n+1})) \quad \leq \quad 0. \end{aligned} \tag{3.24}$$

Letting n tend to infinity in (3.24) we obtain

$$F(d(A_1 u, z), 0, d(A_1 u, z), 0, d(A_1 u, z), 0) \leq 0. \tag{3.25}$$

By (F_a) we have $z = A_1 u = Su$. Since the pair $\{A, S\}$ is compatible we have $SA_1 u = A_1 Su$, i.e, $A_1 z = Sz$. If $A_1 z \neq z$, using (3.1) we get

$$\begin{aligned} F(d(A_1 z, A_k z), d(Sz, Tz), d(A_1 z, Sz), \\ d(A_k z, Tz), d(A_1 z, Tz), d(Sz, A_k z)) \quad \leq \quad 0. \end{aligned} \tag{3.26}$$

So,

$$F(d(A_1 z, z), d(A_1 z, z), 0, 0, d(A_1 z, z), d(A_1 z, z)) \leq 0, \tag{3.27}$$

a contradiction of (F_3). Then $A_1 z = Sz = z$. Therefore, $A_1 z = Sz = A_k z = Tz = z$.

Similarly, we can prove that $A_i z = Tz = z$, $i > 1$.

Therefore, $A_1 z = Sz = A_i z = Tz = z$, $i > 1$. Hence, $\{A_i\}$, S and T have a common fixed point z in X. The uniqueness of z follows from Theorem **??**. ∎

Corollary 3.3. *Let A, B, S and T be self-mappings of a complete metric space (X, d) satisfying the following conditions*

$$A(X) \subset T(X) \ \textit{ and } \ B(X) \subset S(X) \tag{3.28}$$

$$\begin{aligned} F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), \\ d(By, Ty), d(Ax, Ty), d(Sx, By)) \quad \leq \quad 0 \end{aligned} \tag{3.29}$$

for all x, y in X and $F \in \digamma$. Suppose that the pairs $\{A, S\}$ and $\{B, T\}$ are compatible. If the mappings A and S or B and T are reciprocally continuous, then A, B, S and T have a unique common fixed point.

Remark 3.1. Corollary 3.3 generalizes Theorem 3.1 of [11] and a theorem of [10].

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