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DIFFERENTIAL SANDWICH THEOREMS FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let q_1 and q_2 be univalent in $\Delta := \{z : |z| < 1\}$ with $q_1(0) = q_2(0) = 1$. We give some applications of first order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions f with $f(0) = f'(0) = 1$ to satisfy $q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z)$.

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1. INTRODUCTION

Let $\mathcal{H}$ be the class of functions analytic in $\Delta := \{z : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \ldots$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is called to be superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [6] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [6], Bulbo acă considered certain classes of first order differential superordinations [3] as well as superordination-preserving integral operators [2]. Ali *et al.* [1] have used the results of Bulboacă [3] to obtain sufficient conditions for normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in Δ. Also, Tuneski [7] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{f'(z)^2}$.

In the present paper, we obtain sufficient conditions for the normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z).$$

where q_1 and q_2 are given univalent functions in Δ. Also we obtain results for functions defined by using Carlson-Shaffer operator, Ruscheweyh and Sălăgean derivatives.

Let the function $\varphi(a, c; z)$ be given by

$$\varphi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0, -1, -2, \ldots; z \in \Delta),$$

where $(x)_n$ is the *Pochhammer symbol* defined by

$$(x)_n := \begin{cases} 1, & n = 0; \\ x(x+1)(x+2)\ldots(x+n-1), & n \in \mathbb{N} := \{1, 2, 3, \ldots\}. \end{cases}$$

Corresponding to the function $\varphi(a, c; z)$, Carlson and Shaffer [4] introduced a linear operator $L(a, c)$, which is defined by the following Hadamard product (or convolution):

$$L(a, c)f(z) := \varphi(a, c; z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}.$$

We note that

$$L(a, a)f = f, \quad L(2, 1)f = zf', \quad L(\delta + 1, 1)f = D^{\delta} f,$$

where $D^\delta f$ is the *Ruscheweyh derivative* of f. The *Sălăgean derivative* of a function f, denoted by $\mathcal{D}^m f$, is defined by

$$\mathcal{D}^m f(z) = f(z) * \left(z + \sum_{n=2}^{\infty} n^m z^n \right).$$

2. PRELIMINARIES

In our present investigation, we shall need the following definition and results.

Definition 2.1. [6, Definition 2, p. 817] Denote by Q, the set of all functions q that are analytic and injective on $\overline{\Delta} - E(q)$, where

$$E(q) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(q)$.

Theorem 2.1. [5, Theorem 3.4h, p. 132] *Let q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in Δ, and*
(2) $\Re \frac{zh'(z)}{Q(z)} > 0$ *for $z \in \Delta$.*

If p is analytic in Δ with $p(\Delta) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p \prec q$ and q is the best dominant.

By taking $\theta(w) := \alpha w$ and $\phi(w) := \gamma$ in Theorem 2.1, we get

Lemma 2.2. *Let q be univalent in Δ with $q(0) = 1$. Let $\alpha, \gamma \in \mathbb{C}$. Further assume that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\{0, -\Re(\alpha/\gamma)\}.$$

If p is analytic in Δ, and

$$\alpha p(z) + \gamma zp'(z) \prec \alpha q(z) + \gamma zq'(z),$$

then $p \prec q$ and q is the best dominant.

Theorem 2.3. [3] *Let q be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\Re\left[\vartheta'(q(z))/\varphi(q(z))\right] > 0$ *for $z \in \Delta$, and*
(2) *$zq'(z)\varphi(q(z))$ is starlike univalent in Δ.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.1}$$

then $q \prec p$ and q is the best subordinant.

By taking $\theta(w) := \alpha w$ and $\phi(w) := \gamma$ in Theorem 2.3, we get the following extension of [6, Theorem 8, p. 822]:

Lemma 2.4. *Let q be convex univalent in Δ, $q(0) = 1$. Let $\alpha, \gamma \in \mathbb{C}$ and $\Re(\alpha/\gamma) > 0$. If $p \in \mathcal{H}[q(0), 1] \cap Q$, $\alpha p + \gamma zp'$ is univalent in Δ, and*

$$\alpha q(z) + \gamma zq'(z) \prec \alpha p(z) + \gamma zp'(z),$$

then $q \prec p$ and q is the best subordinant.

3. SUBORDINATION AND SUPERORDINATION FOR ANALYTIC FUNCTIONS

By using Lemma 2.2, we first prove the following.

Theorem 3.1. *Let q be univalent in Δ with $q(0) = 1$. Let $\alpha, \gamma \in \mathbb{C}$. Further assume that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\{0, -\Re(1/\gamma)\}. \tag{3.1}$$

If $f \in \mathcal{A}$ satisfies

$$\gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)} \prec q(z) + \gamma zq'(z),$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{f(z)}{zf'(z)}.$$

Then a computation shows that

$$\gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)} = p(z) + \gamma zp'(z).$$

The subordination (3.1) becomes

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z)$$

and Theorem 3.1 follows by an application of Lemma 2.2. ∎

Example 3.1. *When $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$ and $\gamma = 1$, Theorem 3.1 gives the following: If $f \in \mathcal{A}$, then*

$$1 - \frac{f''(z)f(z)}{\{f'(z)\}^2} \prec \frac{(A-B)z}{(1+Bz)^2} + \frac{1+Az}{1+Bz} \Rightarrow \frac{f(z)}{zf'(z)} \prec \frac{1+Az}{1+Bz}.$$

Also if $f \in \mathcal{A}$, then

$$1 - \frac{f''(z)f(z)}{\{f'(z)\}^2} \prec \frac{2z}{(1-z)^2} + \frac{1+z}{1-z} \Rightarrow \Re\frac{zf'(z)}{f(z)} > 0$$

and

$$\left|\frac{f''(z)f(z)}{\{f'(z)^2\}}\right| < 2\lambda \Rightarrow \left|\frac{f(z)}{zf'(z)} - 1\right| < \lambda \quad (0 < \lambda \le 1).$$

Theorem 3.2. *Let q be convex univalent in Δ. If $f \in \mathcal{A}$, $f(z)/zf'(z) \in \mathcal{H}[1,1] \cap Q$, $\gamma \in \mathbb{C}$ with $\Re\gamma > 0$, $\gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)}$ is univalent in Δ, and*

$$q(z) + \gamma zq'(z) \prec \gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)},$$

then

$$q(z) \prec \frac{f(z)}{zf'(z)}$$

and q is the best subordinant.

Proof. Theorem 3.2 follows by an application of Lemma 2.4. ∎

Example 3.2. *By taking* $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ *in Theorem 3.2, we get the following result. Let* q *be convex univalent in* Δ*. Let* $f \in \mathcal{A}$*,* $f(z)/zf'(z) \in \mathcal{H}[1,1] \cap Q$*,* $\gamma \in \mathbb{C}$ *with* $\Re\gamma > 0$*, and* $\gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)}$ *is univalent in* Δ*. Then*

$$\frac{(A-B)\gamma z}{(1+Bz)^2} + \frac{1+Az}{1+Bz} \prec \gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)} \Rightarrow \frac{1+Az}{1+Bz} \prec \frac{f(z)}{zf'(z)}.$$

Corollary 3.3. *Let* $\alpha, \gamma \in \mathbb{C}$*. Let* q_1 *be convex univalent in* Δ *and satisfies (3.1). Let* q_2 *be univalent in* Δ *,* $q_2(0) = 1$*. If* $0 \neq f(z)/zf'(z) \in \mathcal{H}[1,1] \cap Q$*,* $\gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)}$ *is univalent in* Δ*, and*

$$q_1(z) + \gamma z q_1'(z) \prec \gamma\left[1 - \frac{f''(z)f(z)}{\{f'(z)\}^2}\right] + (1-\gamma)\frac{f(z)}{zf'(z)} \prec q_2(z) + \gamma z q_2'(z),$$

then $q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$ *and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

Theorem 3.4. *Let* q *be univalent in* Δ *with* $q(0) = 1$*. Let* $\gamma \in \mathbb{C}$*. Further assume that (3.1) holds. If* $f \in \mathcal{A}$ *satisfies*

$$\frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2\left(\frac{z}{f(z)}\right)'' \prec q(z) + \gamma z q'(z),$$

then

$$\frac{z^2 f'(z)}{\{f(z)\}^2} \prec q(z)$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) := \frac{z^2 f'(z)}{\{f(z)\}^2}.$$

Then a computation shows that

$$\frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2\left(\frac{z}{f(z)}\right)'' = p(z) + \gamma z p'(z).$$

Theorem 3.4 now follows as an application of Lemma 2.2. ∎

Example 3.3. *Taking* $q(z) = (1 + Az)/(1 + Bz)$ *in Theorem 3.4, we have the following result. Let* $-1 \leq B < A \leq 1$ *. If* $f \in \mathcal{A}$*,* $\gamma \in \mathbb{C}$ *with* $\Re\gamma > 0$ *and*

$$\frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2\left(\frac{z}{f(z)}\right)'' \prec \gamma\frac{(A-B)z}{(1+Bz)^2} + \frac{1+Az}{1+Bz},$$

then

$$\frac{z^2 f'(z)}{\{f(z)\}^2} \prec \frac{1+Az}{1+Bz}$$

and $(1 + Az)/(1 + Bz)$ *is the best dominant.*

Theorem 3.5. *Let* q *be univalent in* Δ *with* $q(0) = 1$*. Let* $\gamma \in \mathbb{C}$ *with* $\Re(\gamma) > 0$*. If* $f \in \mathcal{A}$*,* $z^2 f'(z)/f(z)^2 \in \mathcal{H}[1,1] \cap Q$*,* $\frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2\left(\frac{z}{f(z)}\right)''$ *is univalent in* Δ*, then*

$$q(z) + \gamma z q'(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2\left(\frac{z}{f(z)}\right)'',$$

implies

$$q(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2}$$

and q is the best subordinant.

Proof. Theorem 3.5 follows by an application of Lemma 2.4. ∎

Corollary 3.6. *Let $\alpha, \gamma \in \mathbb{C}$. Let q_1 be convex univalent in Δ and satisfies $\Re\gamma > 0$. Let q_2 be univalent in Δ , $q_2(0) = 1$ and satisfies (3.1). If $f \in \mathcal{A}$, $\frac{z^2 f'(z)}{\{f(z)\}^2} \in \mathcal{H}[1,1] \cap Q$, $\frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2 \left(\frac{z}{f(z)}\right)''$ is univalent in Δ, and*

$$q_1(z) + \gamma z q_1'(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} - \gamma z^2 \left(\frac{z}{f(z)}\right)'' \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

4. APPLICATIONS TO CARLSON-SHAFFER OPERATOR

Theorem 4.1. *Let q be convex univalent in Δ with $q(0) = 1$, $\gamma \in \mathbb{C}$. Further, assume that (3.1) holds. If $f \in \mathcal{A}$, and*

$$a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - (a+1)\frac{L(a+2,c)f(z)\, L(a,c)f(z)}{(L(a+1,c)f(z))^2} \prec q(z) + \gamma z q'(z), \tag{4.1}$$

then

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{L(a,c)f(z)}{L(a+1,c)f(z)}. \tag{4.2}$$

From (4.2), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - \frac{z(L(a+1,c)f(z))'}{L(a+1,c)f(z)}. \tag{4.3}$$

By using the identity:

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z)$$

and (4.2) in (4.3), we obtain

$$\frac{zp'(z)}{p(z)} = -(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \frac{a}{p(z)} + 1.$$

The subordination (4.1) becomes

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z)$$

and therefore our result follows as an application of Lemma 2.2. ∎

Example 4.1. *When* $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ *and* $\gamma = 1$*, Theorem 4.1 gives the following: If* $f \in \mathcal{A}$*, and*

$$a + 2\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - (a+1)\frac{L(a+2,c)f(z)\,L(a,c)f(z)}{(L(a+1,c)f(z))^2} \prec \gamma\frac{(A-B)z}{(1+Bz)^2} + \frac{1+Az}{1+Bz},$$

then

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec \frac{1+Az}{1+Bz},$$

and in particular, if

$$a + 2\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - (a+1)\frac{L(a+2,c)f(z)\,L(a,c)f(z)}{(L(a+1,c)f(z))^2} \prec \gamma\frac{2z}{(1-z)^2} + \frac{1+z}{1-z},$$

then

$$\Re\frac{L(a+1,c)f(z)}{L(a,c)f(z)} > 0.$$

Theorem 4.2. *Let* q *be convex univalent in* Δ*. Let* $\gamma \in \mathbb{C}$ *with* $\Re\gamma > 0$*. If* $f \in \mathcal{A}$*,* $\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \in \mathcal{H}[1,1] \cap Q$*,*

$$a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - (a+1)\frac{L(a+2,c)f(z)\,L(a,c)f(z)}{(L(a+1,c)f(z))^2}$$

is univalent in Δ*, and*

$$q(z) + \gamma zq'(z) \prec a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - (a+1)\frac{L(a+2,c)f(z)\,L(a,c)f(z)}{(L(a+1,c)f(z))^2},$$

then

$$q(z) \prec \frac{L(a,c)f(z)}{L(a+1,c)f(z)}$$

and q *is the best subordinant.*

Corollary 4.3. *Let* $\alpha, \gamma \in \mathbb{C}$*. Let* q_1 *be convex univalent in* Δ *and satisfies* $\Re\gamma > 0$*. Let* q_2 *be univalent in* Δ *,* $q_2(0) = 1$ *and satisfies (3.1). If* $f \in \mathcal{A}$*,* $\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \in \mathcal{H}[1,1] \cap Q$*,*

$$a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - (a+1)\frac{L(a+2,c)f(z)\,L(a,c)f(z)}{(L(a+1,c)f(z))^2}$$

is univalent in Δ*, and*

$$q_1(z)+\gamma zq'(z) \prec a+(1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)}-(a+1)\frac{L(a+2,c)f(z)\,L(a,c)f(z)}{(L(a+1,c)f(z))^2} \prec q_2(z)+\gamma zq_2'(z),$$

then

$$q_1(z) \prec \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

For $a = \delta + 1$ and $\gamma = 1$, we get,

Example 4.2. *Let* $\alpha, \gamma \in \mathbb{C}$*. Let* q_1 *be convex univalent in* Δ*,* $q_1(0) = 1$ *and satisfies* $\Re\gamma > 0$*. Let* q_2 *be univalent in* Δ *,* $q_2(0) = 1$ *and satisfies (3.1). If* $f \in \mathcal{A}$*,* $\frac{D^\delta f(z)}{D^{\delta+1}f(z)} \in \mathcal{H}[1,1] \cap Q$*,*

$$(1+\delta) + 2\frac{D^\delta f(z)}{D^{\delta+1}f(z)} - (2+\delta)\frac{D^{\delta+2}f(z)\,D^\delta f(z)}{(D^{\delta+1}f(z))^2},$$

is univalent in Δ*, and*

$$q_1(z) + zq'(z) \prec (1+\delta) + 2\frac{D^\delta f(z)}{D^{\delta+1} f(z)} - (2+\delta)\frac{D^{\delta+2} f(z)\, D^\delta f(z)}{(D^{\delta+1} f(z))^2} \prec q_2(z) + zq_2'(z),$$

then

$$q_1(z) \prec \frac{D^\delta f(z)}{D^{\delta+1} f(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

Theorem 4.4. *Let* q *be convex univalent in* Δ*,* $\gamma \in \mathbb{C}$*. Further, assume that (3.1) holds. If* $f \in \mathcal{A}$ *satisfies*

$$-2a\gamma + \{1+(a-1)\gamma\}\frac{L(a,c)f(z)}{L(a+1,c)f(z)} + (a+1)\gamma\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2} \prec q(z) + \gamma z q'(z),$$

then

$$z\frac{L(a+1,c)f(z)}{\{L(a,c)f(z)\}^2} \prec q(z)$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) := z\frac{L(a+1,c)f(z)}{\{L(a,c)f(z)\}^2}.$$

Then a computation shows that

$$p(z) + \gamma z p'(z) = -2a\gamma + \{1+(a-1)\gamma\}\frac{L(a,c)f(z)}{L(a+1,c)f(z)} + (a+1)\gamma\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2}.$$

The proof of the theorem follows by an application of Lemma 2.2. ∎

Theorem 4.5. *Let* q *be convex univalent in* Δ*,* $\gamma \in \mathbb{C}$ *with* $\Re\gamma > 0$*. If* $f \in \mathcal{A}$*,* $\frac{zL(a+1,c)f(z)}{\{L(a,c)f(z)\}^2} \in \mathcal{H}[1,1] \cap Q$*,*

$$-2a\gamma + \{1+(a-1)\gamma\}\frac{L(a,c)f(z)}{L(a+1,c)f(z)} + (a+1)\gamma\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2},$$

is univalent in Δ*, and*

$$q(z) + \gamma z q'(z) \prec -2a\gamma + \{1+(a-1)\gamma\}\frac{L(a,c)f(z)}{L(a+1,c)f(z)} + (a+1)\gamma\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2},$$

then

$$q(z) \prec z\frac{L(a+1,c)f(z)}{\{L(a,c)f(z)\}^2}.$$

and q *is the best subordinant.*

Corollary 4.6. *Let* $\alpha, \gamma \in \mathbb{C}$*. Let* q_1 *be convex univalent in* Δ *and satisfies* $\Re\gamma > 0$*. Let* q_2 *be univalent in* Δ *,* $q_2(0) = 1$ *and satisfies (3.1). If* $f \in \mathcal{A}$*,* $\frac{zL(a+1,c)f(z)}{\{L(a,c)f(z)\}^2} \in \mathcal{H}[1,1] \cap Q$*,*

$$-2a\gamma + \{1+(a-1)\gamma\}\frac{L(a,c)f(z)}{L(a+1,c)f(z)} + (a+1)\gamma\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2}$$

is univalent in Δ*, and*

$$q_1(z) + \gamma z q_1'(z) \prec -2a\gamma + \{1 + (a-1)\gamma\} \frac{L(a,c)f(z)}{L(a+1,c)f(z)} + (a+1)\gamma \frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{zL(a+1,c)f(z)}{\{L(a,c)f(z)\}^2} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

5. APPLICATIONS TO SĂLĂGEAN DERIVATIVE

Theorem 5.1. *Let* q *be convex univalent in* Δ *with* $q(0) = 1$ *and let* $\gamma \in \mathbb{C}$*. Further, assume that (3.1) holds. If* $f \in \mathcal{A}$*, and*

$$(1-\gamma)\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} + \gamma \left\{1 - \frac{\mathcal{D}^{m+2} f(z)\, \mathcal{D}^m f(z)}{(\mathcal{D}^{m+1} f(z))^2}\right\} \prec q(z) + \gamma z q'(z), \tag{5.1}$$

then

$$\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} \prec q(z)$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) := \frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} \tag{5.2}$$

By taking logarithmic derivative of $p(z)$ given by (5.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{z(\mathcal{D}^m f(z))'}{\mathcal{D}^m f(z)} - \frac{z(\mathcal{D}^{m+1} f(z))'}{\mathcal{D}^{m+1} f(z)}. \tag{5.3}$$

By using the identity:

$$z(\mathcal{D}^m f(z))' = \mathcal{D}^{m+1} f(z)$$

and (5.2) in (5.3), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\mathcal{D}^{m+1} f(z)}{\mathcal{D}^m f(z)} - \frac{\mathcal{D}^{m+2} f(z)}{\mathcal{D}^{m+1} f(z)}.$$

The subordination (5.1) becomes

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z)$$

and therefore our result follows as an application of Lemma 2.2. ∎

Theorem 5.2. *Let* q *be convex univalent in* Δ*. Let* $\gamma \in \mathbb{C}$*. Assume that* $\Re\gamma > 0$ *. If* $f \in \mathcal{A}$*,* $\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} \in \mathcal{H}[1,1] \cap Q$*,*

$$(1-\gamma)\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} + \gamma \left\{1 - \frac{\mathcal{D}^{m+2} f(z)\, \mathcal{D}^m f(z)}{(\mathcal{D}^{m+1} f(z))^2}\right\}$$

is univalent in Δ*, and*

$$q(z) + \gamma z q'(z) \prec (1-\gamma)\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} + \gamma \left\{1 - \frac{\mathcal{D}^{m+2} f(z)\, \mathcal{D}^m f(z)}{(\mathcal{D}^{m+1} f(z))^2}\right\},$$

then

$$q(z) \prec \frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)}$$

and q is the best subordinant.

Theorem 5.3. *Let $\gamma \in \mathbb{C}$. Let q_1 be convex univalent in Δ and satisfies $\Re\gamma > 0$. Let q_2 be univalent in Δ, $q_2(0) = 1$ and satisfies (3.1). If $f \in \mathcal{A}$, $\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} \in \mathcal{H}[1,1] \cap Q$,*

$$(1-\gamma)\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} + \gamma\left\{1 - \frac{\mathcal{D}^{m+2} f(z)\,\mathcal{D}^m f(z)}{(\mathcal{D}^{m+1} f(z))^2}\right\}$$

is univalent in Δ, and

$$q_1(z) + \gamma z q'(z) \prec (1-\gamma)\frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} + \gamma\left\{1 - \frac{\mathcal{D}^{m+2} f(z)\,\mathcal{D}^m f(z)}{(\mathcal{D}^{m+1} f(z))^2}\right\} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{\mathcal{D}^m f(z)}{\mathcal{D}^{m+1} f(z)} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

Theorem 5.4. *Let q be convex univalent in Δ and $\gamma \in \mathbb{C}$ with $\Re\gamma > 0$. If $f \in \mathcal{A}$ satisfies*

$$\{1+\gamma\}z\frac{\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2} + \gamma z\frac{\mathcal{D}^{m+2} f(z)}{\{\mathcal{D}^m f(z)\}^2} - 2\gamma z\frac{\{\mathcal{D}^{m+1} f(z)\}^2}{\{\mathcal{D}^m f(z)\}^3} \prec q(z) + \gamma z q'(z),$$

then

$$z\frac{\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2} \prec q(z).$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := z\frac{\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2}.$$

Then a computation shows that

$$\{1+\gamma\}z\frac{\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2} + \gamma z\frac{\mathcal{D}^{m+2} f(z)}{\{\mathcal{D}^m f(z)\}^2} - 2\gamma z\frac{\{\mathcal{D}^{m+1} f(z)\}^2}{\{\mathcal{D}^m f(z)\}^3} = p(z) + \gamma z p'(z).$$

Our result follows now by an application of Lemma 2.2. ∎

Theorem 5.5. *Let q be convex univalent in Δ. Let $\gamma \in \mathbb{C}$. Assume that $\Re\gamma > 0$. If $f \in \mathcal{A}$, $\frac{z\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2} \in \mathcal{H}[1,1] \cap Q$,*

$$\{1+\gamma\}z\frac{\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2} + \gamma z\frac{\mathcal{D}^{m+2} f(z)}{\{\mathcal{D}^m f(z)\}^2} - 2\gamma z\frac{\{\mathcal{D}^{m+1} f(z)\}^2}{\{\mathcal{D}^m f(z)\}^3}$$

is univalent in Δ, and

$$q(z) + \gamma z q'(z) \prec \{1+\gamma\}z\frac{\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2} + \gamma z\frac{\mathcal{D}^{m+2} f(z)}{\{\mathcal{D}^m f(z)\}^2} - 2\gamma z\frac{\{\mathcal{D}^{m+1} f(z)\}^2}{\{\mathcal{D}^m f(z)\}^3},$$

then

$$q(z) \prec \frac{z\mathcal{D}^{m+1} f(z)}{\{\mathcal{D}^m f(z)\}^2}$$

and q is the best subordinant .

Theorem 5.6. *Let* $\gamma \in \mathbb{C}$. *Let* q_1 *be convex univalent in* Δ *and satisfies* $\Re\gamma > 0$. *Let* q_2 *be univalent in* Δ, $q_1(0) = 1$. *Assume that (3.1) holds. If* $f \in \mathcal{A}$, $\frac{z\mathcal{D}^{m+1}f(z)}{\{\mathcal{D}^m f(z)\}^2} \in \mathcal{H}[1,1] \cap Q$,

$$\{1+\gamma\}z\frac{\mathcal{D}^{m+1}f(z)}{\{\mathcal{D}^m f(z)\}^2} + \gamma z\frac{\mathcal{D}^{m+2}f(z)}{\{\mathcal{D}^m f(z)\}^2} - 2\gamma z\frac{\{\mathcal{D}^{m+1}f(z)\}^2}{\{\mathcal{D}^m f(z)\}^3}$$

is univalent in Δ, *and*

$$q_1(z)+\gamma z q_1'(z) \prec \{1+\gamma\}z\frac{\mathcal{D}^{m+1}f(z)}{\{\mathcal{D}^m f(z)\}^2}+\gamma z\frac{\mathcal{D}^{m+2}f(z)}{\{\mathcal{D}^m f(z)\}^2}-2\gamma z\frac{\{\mathcal{D}^{m+1}f(z)\}^2}{\{\mathcal{D}^m f(z)\}^3} \prec q_2(z)+\gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{z\mathcal{D}^{m+1}f(z)}{\{\mathcal{D}^m f(z)\}^2} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

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