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ON THE OPTIMAL BUCKLING LOADS OF CLAMPED COLUMNS

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ABSTRACT. We consider the problem of determining the optimal shape of a clamped column of given length and volume, without minimum cross section constraints. We prove that the necessary condition of optimality derived by Olhoff and Rasmussen [9] is sufficient when $0 < \alpha < 1$. The number α appears in Equation (2.1). For the case $\alpha = 1$ it is shown that the value 48 is optimal. We also determine the exact values of the optimal shape at the extremities, and take advantage of a robust nonlinear ordinary differential equation solver COLSYS to compute the optimal buckling load with a high accuracy.

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1. INTRODUCTION

An interesting problem with a long history is to determine the shape of the strongest clamped column with given length and volume. The problem known as Lagrange's problem was considered by Tadjbakhsh and Keller [11] in 1962, analytically without cross section constraints. The derived optimal shape presented singularities at two interior points, and the largest buckling load was found to be $\lambda = 16\pi^2/3(\approx 52.6379)$. In 1977, Olhoff and Rasmussen [9] reconsidered the problem and pointed out that Tadjbakhsh and Keller's solution based on a single-modal formulation is not optimal, since it corresponds to a true buckling load much lower than $4\pi^2$. They suggested that it is necessary to formulate the problem with a double critical load, such that there exist two different forms of stability loss of the optimal column. Basing on their bimodal formulation, Olhoff and Rasmussen obtained a column having a non-vanishing cross sectional area and with a critical buckling load $\lambda = 52.3563 < 16\pi^2$. However they did not succeed to prove the validity of their column. Olhoff and Rasmussen's solution was defended by many authors, notably by Masur [8], Seiranian [10], and Cox and Overton [4], nevertheless a proof of existence is still lacking.

2. TECHNICAL RESULTS

Consider a column with circular and identically oriented cross section $\Omega(x)$, having its centroid on the x-axis with a radius $R(x)$. The moment $I(x)$ is precisely the second moment of area of the cross section about the x-axis, perpendicular to the plane of buckling (y, z). Suppose that the Young's modulus E is constant and for every x belonging to the axis of the column, the surface mass density $\rho_x(y, z)$ is radial, i.e., depends only on the distance of x to the point (y, z). That is for example

$$\rho_x(y, z) = k(y^2 + z^2)^\gamma,$$

where k and γ are constants independent of x and (y, z). In this case we have

$$I(x) = \int\int_{\Omega(x)} \rho_x(y, z)(y^2 + z^2)\, dy\, dz = C_\gamma\, A(x)^{\gamma+2},$$

where $A(x)$ denotes the cross-sectional area and C_γ is a constant depending only on γ. We may also change the geometry of $\Omega(x)$ to collect a number of examples where I varies as some power of A. We proceed then to consider the general case where $EI = \sigma^\alpha$, and α is an arbitrary nonzero real number. The equation of equilibrium of a column clamped at both extremities $x = 0$ and $x = 1$ is

$$(\sigma^\alpha u'')'' + \lambda u'' = 0, \qquad 0 < x < 1, \tag{2.1}$$

subject to the boundary conditions

$$u(0) = u'(0) = 0, \qquad u(1) = u'(1) = 0. \tag{2.2}$$

Lagrange's problem is equivalent to finding a nonnegative function $\sigma(x)$ that maximizes the first eigenvalue $\lambda_1(\sigma)$ of problem (2.1)-(2.2), and satisfies the constant-volume condition

$$\int_0^1 \sigma(x)\, dx = 1. \tag{2.3}$$

Let μ_n denote the nth eigenvalue of the second-order problem

$$(\sigma^\alpha v')' + \mu v' = 0, \qquad v(0) = v(1) = 0, \tag{2.4}$$

and U the set of all σ in $L^\infty(0, 1)$ having a positive lower bound. A function u will be said to be even [resp. odd] if $u(x) = u(1 - x)$ [resp. $u(x) = -u(1 - x)$] for all x in $(0, 1)$.

In [7], we have proved the following results.

Proposition 2.1. *Let σ be an arbitrary member of U (σ is not necessarily even). Then we have*

$$\lambda_1(\sigma) \leq \mu_2(\sigma).$$

Proposition 2.2. *Let σ be an arbitrary member of U. If $\lambda_1(\sigma)$ is double, then*

$$\lambda_1(\sigma) = \mu_2(\sigma).$$

If furthermore σ is even, then $\lambda_1(\sigma)$ admits, up to a scalar multiple, a unique odd and a unique even eigenfunction.

One can then deduce that the two buckling modes y_1 and y_2 associated with the optimal column plotted in Figure 2 [12] are incorrect, because both are even. The main result given in [7] is the following

Theorem 2.3. *There exists $\sigma \in U$ even such that $\lambda_1(\sigma)$ does not possess a positive eigenfunction.*

This result contradicts Theorem 2.2 by Cox and Overton [4], in which they claimed that the symmetric clamped-clamped column possesses a positive first eigenfunction. This claim was the source of many confusions. It has been implicitly used by many authors, in particular in [3], [12] and later in [6], and led to erroneous conclusions. It can be easily proved that it also leads to the optimality of Tadjbakhsh and Keller's shape [6].

Theorem 2.3 is based on the fact that when σ is even $\lambda_1(\sigma)$ can be represented as

$$\lambda_1(\sigma) = \min\{\lambda_{(2)}(\sigma), \nu_{(2)}(\sigma)\}. \tag{2.5}$$

where $\lambda_{(2)}(\sigma)$ and $\nu_{(2)}(\sigma)$ are respectively the second eigenvalues of the problems:

$$w'' + \lambda\sigma^{-\alpha}w = 0, \qquad 0 < x < 1/2, \tag{2.6}$$

$$2w(0) + w'(0) = 0, \qquad w(1/2) = 0, \tag{2.7}$$

and

$$w'' + \nu\sigma^{-\alpha}w = 0, \qquad 0 < x < 1/2, \tag{2.8}$$

$$w'(0) = 0, \qquad w'(1/2) = 0. \tag{2.9}$$

Here and throughout we identify σ with its restriction to the interval $(0, 1/2)$. We note that problem (2.6)-(2.7) has a zero first eigenvalue with a corresponding eigenfunction $w = x - 1/2$. From (2.5) we may distinguish three cases:

(i) if $\lambda_{(2)}(\sigma) < \nu_{(2)}(\sigma)$ then $\lambda_1(\sigma)$ is simple and $u_1(\sigma)$ is odd.
(ii) if $\lambda_{(2)}(\sigma) > \nu_{(2)}(\sigma)$ then $\lambda_1(\sigma)$ is simple and $u_1(\sigma)$ is even; we have $\lambda_1(\sigma) = \mu_2(\sigma)$.
(iii) if $\lambda_{(2)}(\sigma) = \nu_{(2)}(\sigma)$ then $\lambda_1(\sigma)$ is double.

3. Sufficient Conditions

We recall that the necessary condition of optimality derived by Olhoff and Rasmussen [9] has the following form

$$\sigma^{\alpha-1}(\delta_1|u_1''|^2 + \delta_2|u_2''|^2) = 1, \tag{3.1}$$

where δ_1 and δ_2 are nonnegative numbers satisfying $\delta_1 + \delta_2 = 1$, and u_1 and u_2 are two linearly independent eigenfunctions corresponding to $\lambda_1(\sigma)$. Unlike the single-modal formulation, (3.1) rules out the possibility of optimal columns with vanishing cross sectional area since u_1'' and u_2'' do not vanish simultaneously. The aim of this section is to prove that (3.1) is also a sufficient condition of optimality when $0 < \alpha < 1$. Let $\bar{U}$ denote the set of all $\sigma \in U$ satisfying (2.3), and assume that (3.1) holds for some even function $\tilde{\sigma} \in \bar{U}$. We may assume that the corresponding eigenfunctions $\tilde{u}_1$ and $\tilde{u}_2$ are respectively even and odd. Let σ be an arbitrary member of U. According to ([4], Theorem 2.4) there exists an even function $\bar{\sigma}$ such that $\lambda_1(\sigma) \leq \lambda_1(\bar{\sigma})$. We notice that this result can be proved independent of Theorem 2.2. Let $p = 1/(1-\alpha)$ and $\tilde{u} = \sqrt{\delta_1}\tilde{u}_1 + \sqrt{\delta_2}\tilde{u}_2$. Put $\xi = \int_0^1 (\tilde{u}')^2\,dx$. Then we have

$$\begin{aligned}\lambda_1(\bar{\sigma}) &\leq \xi^{-1}\int_0^1 \bar{\sigma}^\alpha |\tilde{u}''|^2 \\ &= \xi^{-1}\int_0^1 \bar{\sigma}^\alpha (\delta_1|\tilde{u}_1''|^2 + \delta_2|\tilde{u}_2''|^2)\,dx.\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}\lambda_1(\bar{\sigma}) &\leq \xi^{-1}\left(\int_0^1 (\delta_1|\tilde{u}_1''|^2 + \delta_2|\tilde{u}_2''|^2)^p\,dx\right)^{1/p} \\ &= \xi^{-1}\int_0^1 \tilde{\sigma}^\alpha (\delta_1|\tilde{u}_1''|^2 + \delta_2|\tilde{u}_2''|^2)\,dx \\ &= \lambda_1(\tilde{\sigma}).\end{aligned}$$

It is easily verified that $\tilde{\sigma}$ is also a maximizer of $\lambda_1(\sigma)$ over the set V of all nonnegative functions $\sigma \in L^\infty(0,1)$ satisfying (2.3) and such that problem (2.1)-(2.2) has a discrete spectrum. In fact, we have for any $\sigma \in V$

$$\lambda_1(\sigma) \leq \lambda_1(\sigma+\varepsilon) \leq \lambda_1(\tilde{\sigma})\left(\int_0^1 (\sigma+\varepsilon)\,dx\right)^\alpha,$$

where ε is a positive number. Letting $\varepsilon \to 0$ yields $\lambda_1(\sigma) \leq \lambda_1(\tilde{\sigma})$.

4. Optimal Solution

In this section, we derive an algorithm for determining the optimal buckling load. We first recall the proof of Theorem 4.4 in [4] showing that the optimal solution $\tilde{\sigma}$ is infinitely differentiable over $(0,1)$. Multiplying by $\tilde{\sigma}^{\alpha+1}$ the necessary condition (3.1) written for $\tilde{\sigma}$, $\tilde{u}_1$ and $\tilde{u}_2$, we get

$$\delta_1(\tilde{\sigma}^\alpha|\tilde{u}_1''|)^2 + \delta_2(\tilde{\sigma}^\alpha|\tilde{u}_2''|)^2 = \tilde{\sigma}^{\alpha+1}. \tag{4.1}$$

On the other hand, we have

$$\tilde{\sigma}^\alpha \tilde{u}_1'' = l_1 - \lambda_1(\tilde{\sigma})\,\tilde{u}_1,$$
$$\tilde{\sigma}^\alpha \tilde{u}_2'' = l_2 - \lambda_1(\tilde{\sigma})\,\tilde{u}_2,$$

where l_1 and l_2 are affine function of x. >From these latter relations we find, on recalling the inclusion $H_0^2 \subset C^1$, that $(\tilde{\sigma}^\alpha\tilde{u}_1'')^2$ and $(\tilde{\sigma}^\alpha\tilde{u}_2'')^2$ are C^1 and hence from (4.1) that $\tilde{\sigma}$ is C^1 since it is positive. We then conclude that $\tilde{u}''$ is C^1, that is $\tilde{u}$ is C^3. Repeating this exact argument we find that $\tilde{\sigma} \in C^5$, and continued repetitions leads to $\tilde{\sigma} \in C^\infty(0,1)$.

In view of the symmetry, the eigenfunctions $\tilde{u}_1$ and $\tilde{u}_2$ satisfy the differential equations

$$(\tilde{\sigma}^\alpha\tilde{u}_1'')' + \lambda_1\tilde{u}_1' = 0,$$

and

$$(\tilde{\sigma}^\alpha\tilde{u}_2'')' + \lambda_1\tilde{u}_2' = C_0,$$

where C_0 is a nonzero constant. Now a differentiation of both sides of (4.1) yields

$$-(\alpha+1)\tilde{\sigma}' - 2\lambda_1[\delta_1\tilde{u}_1''\tilde{u}_1' + \delta_2\tilde{u}_2''\tilde{u}_2'] + 2C_0\delta_2\tilde{u}_2'' = 0.$$

Hence,

$$\tilde{\sigma}(x) = -\frac{\lambda_1}{\alpha+1}(\delta_1|\tilde{u}_1'|^2 + \delta_2|\tilde{u}_2'|^2) + \frac{2C_0\delta_2}{\alpha+1}\tilde{u}_2' + C_1,$$

where C_1 is a constant. Integrating this equation over $(0,1)$ and taking into account (3.1) brings

$$C_1 = (\alpha+2)/(\alpha+1),$$

and therefore

$$\tilde{\sigma}(0) = (\alpha+2)/(\alpha+1) = \tilde{\sigma}(1). \tag{4.2}$$

Condition (4.2) provides important informations to numerical computations. We shall use the ODE solver COLSYS [1]-[2] to solve our problem. Before this we must find an equivalent formulation which will be accepted by COLSYS. That is, we must convert our optimization problem into an equivalent standard nonlinear ODE problem, see [12] for a similar approach. To reduce the computation efforts, the symmetry can be exploited in a subtle way. Putting $w_i = \sqrt{\delta_i}\tilde{\sigma}^\alpha \tilde{u}_i''$, $i = 1, 2$, we get

$$w_i'' + \lambda\sigma^{-\alpha}w_i = 0, \qquad i = 1, 2,$$

$$\sigma^{\alpha+1} = w_1^2 + w_2^2, \qquad 0 < x < 1/2, \tag{4.3}$$

and w_1 and w_2 satisfy respectively (2.9) and (2.7). Let

$$W_1(x) = \int_0^x \tilde{\sigma}^{-\alpha}(t)w_1(t)\,dt,$$

$$W_2(x) = \int_0^x \tilde{\sigma}^{-\alpha}(t)(t-1/2)w_2(t)\,dt.$$

The fact that λ_1 is not zero is assured by one of the following conditions:

$$W_1(1/2) = 0 \quad \text{or} \quad W_2(1/2) = 0.$$

Our optimization problem is now replaced by the system of nonlinear equations summarized below:

ODE's: $0 < x < 1/2$,

$$\lambda' = 0,$$
$$W_1' = (w_1^2 + w_2^2)^\gamma w_1,$$
$$w_i'' = -\lambda(w_1^2 + w_2^2)^\gamma w_i, \quad i = 1, 2,$$

where $\gamma = -\alpha/(\alpha+1)$, with the six boundary conditions:

$$\begin{array}{ll} w_1'(0) = 0, & W_1(1/2) - 0, \\ 2w_2(0) + w_2'(0) = 0, & w_1'(1/2) = 0, \\ w_1^2(0) + w_2^2(0) = \tilde{\sigma}(0)^{\alpha+1}, & w_2(1/2) = 0, \end{array}$$

where $\sigma(0)$ is given by (4.2). Since the bimodal formulation does not suffer from singularity, the system is now ready to be input to COLSYS for direct solution, without any special treatment. COLSYS requires to be provided with the following parameters:
NCOLP = number of collocation points per sub-intervals;
NSUBI = number of sub-intervals in the initial mesh;
TOLER = error tolerance on each solution component.

The cases below were computed with the parameters

$$NCOLP = 4, \qquad NSUBI = 4, \qquad TOLER = 10^{-7}.$$

The following initial guess is provided for nonlinear iterations

$$\lambda = 50,$$

$$W_1(x) = x(1-x),$$

$$w_1 = x - 1/2,$$

$$w_2 = -6x^2 + 5x - 1.$$

Table 4.1: Numerical results with $\alpha = 2$ and $\alpha = 3$, and comparison with the results of Olhoff and Rasmussen [9] *and Masur* [8].

α	2	[9]	[8]	3
λ_{opt}	52.35625	52.3563	52.3565	54.82542
σ_{max}	1.33394	-	1.33392	1.25167
σ_{min}	0.22582	0.226	0.22583	0.37107
x_*	0.2466	-	0.2467	0.2430

Table 4.1 shows the results of the computations for $\alpha = 2$ and $\alpha = 3$. The third and the fourth columns of the table list the numerical results obtained in [9] and [8], respectively, for $\alpha = 2$. The number $x = x_*$ is the location where the minimum value of σ is attained. It can be seen from the comparison in the table that the results produced by the present approach are in good agreement with those of [9] and [8]. In particular we obtained the same optimal buckling load as that in [9]. For completeness, we plotted in Figure 4.1 the optimal shape σ obtained in both cases.

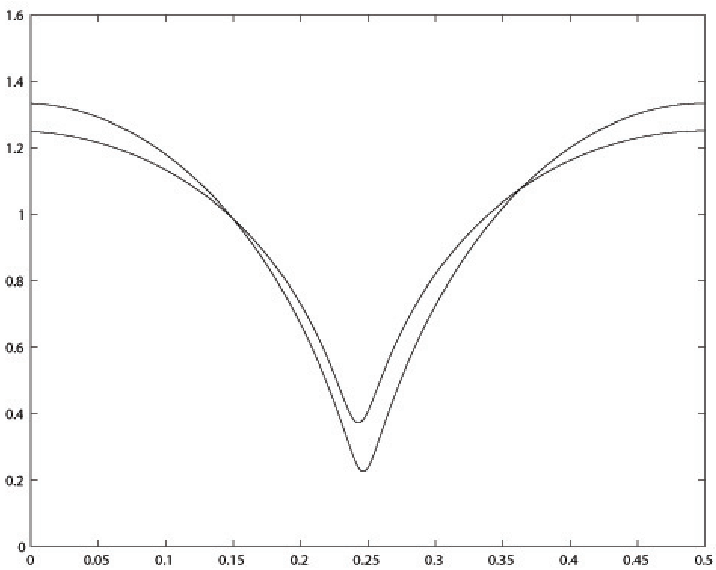

Figure 4.1: The optimal shapes obtained using COLSYS for $\alpha = 2$ and $\alpha = 3$ in the interval $[0, \frac{1}{2}]$.

5. THE CASE $\alpha = 1$

This case has received considerable attention. A number of workers have considered this case as a critical case, where the optimal buckling load changes multiplicity, and claimed that

$$|\tilde{u}''(x)| = Const.$$

is a necessary condition of optimality. The following theorem shows that this case turns out to be singular.

Theorem 5.1. *If $\alpha = 1$ then $\lambda_1(\sigma) \leq 48$ for all $\sigma \in V$. Moreover, equality is attained by the function*

$$\tilde{\sigma} = \begin{cases} 3/2(1-16x^2) & \text{if } 0 \leq x \leq 1/4, \\ 3/2[1-16(x-1/2)^2] & \text{if } 1/4 < x < 3/4, \\ 3/2[1-16(1-x)^2] & \text{if } 3/4 < x < 1. \end{cases}$$

Its corresponding first eigenmode is

$$\tilde{u} = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1/4, \\ 1/8-(x-1/2)^2 & \text{if } 1/4 < x < 3/4, \\ (x-1/2)^2 & \text{if } 3/4 < x < 1. \end{cases}$$

Proof. We have for every $\sigma \in V$,

$$\inf_{u \in H_0^2} \frac{\int_0^1 \sigma u''^2\,dx}{\int_0^1 u'^2\,dx} \leq \frac{\int_0^1 \sigma \tilde{u}''^2\,dx}{\int_0^1 \tilde{u}'^2\,dx} = \frac{2\int_0^1 \sigma(x)\,dx}{\int_0^1 \tilde{u}'^2\,dx} = 48.$$

Hence $\lambda_1(\sigma) \leq 48$. Moreover, it is easily checked that the differential equation

$$(\tilde{\sigma}\tilde{u}'')'' + \lambda \tilde{u}'' = 0$$

holds everywhere on $(0,1)$ for $\lambda = 48$. To prove the theorem, it suffices to show that the spectrum of the problem

$$(\tilde{\sigma} u'')'' + \lambda u'' = 0, \qquad 0 < x < 1, \tag{5.1}$$

$$u(0) = u'(0) = 0, \qquad u(1) = u'(1) = 0, \tag{5.2}$$

is discrete and that $\lambda_1(\tilde{\sigma}) = 48$. Note that an eigenfunction of (5.1)-(5.2) satisfies in particular

$$\int_0^1 \tilde{\sigma} u''^2\,dx < \infty.$$

Put $\bar{v} = u'$. Then $\bar{v}$ satisfies

$$(\tilde{\sigma}\bar{v}')' + \lambda \bar{v} = C, \qquad C = Const.$$

If $\lambda = 0$ then there exists a constant D such that

$$\tilde{\sigma}\bar{v}' = Cx + D$$

everywhere on $(0,1)$. The integral $\int_0^{1/4} \tilde{\sigma}\bar{v}'^2\,dx$ is finite only if

$$4D + C = 0.$$

In this case $\int_{3/4}^{1} \tilde{\sigma}\bar{v}'^2\,dx = +\infty$ so long as C and D are not both zero. If $C = D = 0$ then the conditions $\bar{v}(0) = \bar{v}(1) = 0$ and $\int_0^1 \bar{v}\,dx = 0$ imply that $\bar{v}$ is identically zero on $(0,1)$. As a result, $\lambda = 0$ cannot be an eigenvalue.

Let now $v_0 = \bar{v} - C/\lambda$ so that

$$(\tilde{\sigma}v_0')' + \lambda v_0 = 0, \tag{5.3}$$

everywhere on $(0,1)$. We shall look at this equation only on the half interval $(1/4, 3/4)$. The only solutions v of (5.3) defined on $(1/4,3/4)$ and such that $\int_{1/4}^{3/4}(1-16x^2)v'^2\,dx < \infty$ are polynomial functions, called *Legendre* polynomials, see [5]. Indeed, the spectrum of (5.3) restricted to $(1/4,3/4)$ consists of the sequence $\lambda_n = 24n(n+1),\ n \in \mathbb{N}$. Each eigenvalue is simple and its corresponding eigenfunction is a polynomial of degree n. We have for example, $\lambda_0 = 0,\ v = Const.$, $\lambda_0 = 48,\ v = (x - 1/2)$, etc. Hence, the spectrum of (5.1)-(5.2) is a subset of

$$S = \{24n(n+1),\ n \in \mathbb{N}^*\}.$$

It follows that $\lambda_1(\tilde{\sigma}) = 48$ and that $\tilde{\sigma}$ is optimal. It is finally clear from the analysis that the optimal buckling 48 is simple. ∎

The solution $\tilde{\sigma}$ was classified by some authors as non-optimal, since, in their opinion, substitution of the test function $u(x) = x^2$ for $0 \le x \le 1/4$, $u(x) = 1/8(1-2x)$ for $1/4 < x < 3/4$, and $u(x) = -(1-x)^2$ for $3/4 \le x \le 1$, into a Rayleigh quotient involving $\tilde{\sigma}$ leads to the value 27.43. However, as it was pointed out by Seiranian [10], this assertion is not valid since it neglects the order of the singularity of $\tilde{\sigma}$ at points $1/4$ and $3/4$.

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