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POSITIVE PERIODIC TIME-SCALE SOLUTIONS FOR FUNCTIONAL DYNAMIC EQUATIONS

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ABSTRACT. Using Krasnoselskii's fixed point theorem, we establish the existence of positive periodic solutions to two pairs of related nonautonomous functional delta dynamic equations on periodic time scales, and then extend the discussion to higher-dimensional equations. Two pairs of corresponding nabla equations are also provided in an analogous manner.

Key words and phrases: Periodic boundary value problem, nabla dynamic equation, delta dynamic equation.

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1. PRELIMINARIES

In this article, we investigate the existence of positive periodic solutions for the two pairs of first-order nonautonomous functional delta dynamic equations

$$y^{\Delta}(t) = -p(t)y^{\sigma}(t) + \lambda h(t)f(y(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}} \tag{1.1}$$

and

$$y^{\Delta}(t) = -p(t)y(t) + \lambda h(t)f(y(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.2}$$

or

$$x^{\Delta}(t) = -\left(\ominus p(t)\right)x^{\sigma}(t) - \lambda h(t)f(x(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}} \tag{1.3}$$

and

$$x^{\Delta}(t) = \ominus(-p)(t)x(t) - \lambda h(t)f(x(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.4}$$

where for all cases p, h and τ are right-dense continuous T-periodic functions. We assume that $\lambda, T > 0$, and that h and p are nonnegative with h and p not identically zero in $[t_0, T+t_0]_{\mathbb{T}}$; additionally, we must have $1-\mu(t)p(t) > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ in (1.2) and (1.4). Throughout we also assume that $f \in C([0,\infty),[0,\infty))$ with $f(u) > 0$ for $u > 0$, such that the following limits exist for f:

$$\ell_0 := \lim_{u\to 0} f(u)/u \in [0,\infty], \qquad \ell_\infty := \lim_{u\to\infty} f(u)/u \in [0,\infty].$$

Moreover, the time scale $\mathbb{T}$ is itself periodic, that is $t+T \in \mathbb{T}$ and $\mu(t) = \mu(t+T)$ for all $t \in \mathbb{T}$ (for more on time scales and time-scale notation, please consult the Appendix, Section 7). Since y is defined on the time scale $\mathbb{T}$, τ is a function such that $t-\tau(t) \in \mathbb{T}$. Functional dynamic equations with periodic delays appear in a number of models in mathematical ecology. One example would be to interpret (1.1) and (1.2) as standard Malthus population decay models in the spirit of $y' = -p(t)y$, or (1.3) and (1.4) as growth models, all subject to a perturbation with periodic delay. In this context an important question is whether this model supports positive periodic solutions. Such a question has been studied extensively by a number of authors in the continuous and discrete cases; see for example [5, 6, 11, 14] and the references therein. In the most recent papers, for the restricted case of $\mathbb{T} = \mathbb{Z}$, [10] considers equations (1.2) and (1.4), while [9] extends (1.2) and (1.4) to systems such as (5.2) and (5.4). To our knowledge, no one has considered concurrently, even for $\mathbb{T} = \mathbb{Z}$, all four equations (1.1) through (1.4) and their related systems (5.1) through (5.4), much less the corresponding nabla equations (6.1) through (6.4) and nabla systems (6.5) through (6.8).

There are other approaches to the existence of solutions for dynamic equations on time scales than those featured in this note; for alternative approaches to the existence of solutions and multiple solutions to dynamic equations on time scales, including periodic solutions to problems on periodic time scales, consult [1, 12, 13]. In this paper, we will obtain existence criteria for T-periodic solutions of delta equations (1.1) through (1.4), nabla equations (6.1) through (6.4), and their corresponding systems (5.1) through (5.4) and (6.5) through (6.8), respectively, by means of a well-known fixed point theorem [8], stated here for reference.

Theorem 1.1. *Let E be a Banach space and let $P \subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open balls of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $L : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that*

(1) $\|Lu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Lu\| \geq \|u\|$ *for* $P \cap \partial\Omega_2$*, or that*
(2) $\|Lu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Lu\| \leq \|u\|$ *for* $P \cap \partial\Omega_2$.

Then L has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2. EXISTENCE RESULTS FOR (1.1)

First we consider (1.1). By the simple [3, Theorem 1.16], useful formula $w^\sigma = w + \mu w^\Delta$, equation (1.1) is equivalent to

$$y^\Delta(t) = \frac{-p(t)y(t) + \lambda h(t) f(y(t-\tau(t)))}{1+\mu(t)p(t)}.$$

Alternatively, using Theorem 7.2 in Section 7, we will rewrite (1.1) as

$$(e_p(t,t_0)y(t))^\Delta = \lambda e_p(t,t_0)h(t)f(y(t-\tau(t))). \tag{2.1}$$

To better understand the form of solutions to (1.1), we integrate the equivalent equation (2.1) from t to $t+T$ and use the fundamental theorem to obtain

$$e_p(t+T,t_0)y(t+T) - e_p(t,t_0)y(t) = \lambda \int_t^{t+T} e_p(s,t_0)h(s)f(y(s-\tau(s)))\Delta s.$$

Since y is to be a T-periodic solution we have $y(t+T) = y(t)$, allowing us to solve for y as

$$y(t) = \lambda \int_t^{t+T} K(t,s)h(s)f(y(s-\tau(s)))\Delta s, \quad K(t,s) := \frac{e_p(s,t)}{e_p(t_0+T,t_0)-1}; \tag{2.2}$$

we have used the semigroup property in Theorem 7.2 (5), and the fact that p and the time scale $\mathbb{T}$ are periodic imply that e_p satisfies $e_p(t+T,t) = e_p(a+T,a)$ for $a,t \in \mathbb{T}$. Note that the denominator in $K(t,s)$ is not zero since we have assumed that $p(t_1) > 0$ for some $t_1 \in [t_0, t_0+T]_\mathbb{T}$. It is straightforward to realize that any function y that satisfies (2.2) is also a T-periodic solution of (1.1). By the properties of the time-scale exponential and the definition of $K(t,s)$, we have

$$m := \frac{1}{e_p(t_0+T,t_0)-1} \le K(t,s) \le \frac{e_p(t_0+T,t_0)}{e_p(t_0+T,t_0)-1} =: M, \quad s \in [t,t+T]_\mathbb{T}, \tag{2.3}$$

and

$$0 < \frac{m}{M} = e_{\ominus p}(t_0+T,t_0) \le \frac{K(t,s)}{K(t,t+T)} \le 1, \quad t \in [t_0,\infty)_\mathbb{T}, \quad s \in [t,t+T]_\mathbb{T}. \tag{2.4}$$

Now let $\mathcal{B}$ be the set of all real T-periodic continuous functions, augmented with the usual linear structure and the supremum norm

$$\|y\| = \sup_{t \in [t_0,t_0+T]_\mathbb{T}} |y(t)|.$$

Then $\mathcal{B}$ is a Banach space with cone

$$\mathcal{S} = \left\{ y \in \mathcal{B} : y(t) \ge \frac{m}{M}\|y\|, \quad t \in [t_0,\infty)_\mathbb{T} \right\}.$$

Define a mapping $L : \mathcal{B} \to \mathcal{B}$ by

$$(Ly)(t) = \lambda \int_t^{t+T} K(t,s)h(s)f(y(s-\tau(s)))\Delta s.$$

Lemma 2.1. *The operator L defined above satisfies $L\mathcal{S} \subset \mathcal{S}$.*

Proof. It follows using standard reasoning that L is completely continuous on bounded subsets of $\mathcal{S}$, and for $y \in \mathcal{S}$,

$$(Ly)(t) \le \lambda M \int_{t_0}^{t_0+T} h(s)f(y(s-\tau(s)))\Delta s, \quad t \in [t_0,\infty)_\mathbb{T},$$

so that

$$(Ly)(t) \geq \lambda m \int_{t_0}^{t_0+T} h(s) f(y(s-\tau(s)))\Delta s \geq \frac{m}{M}\|Ly\|.$$

∎

Lemma 2.2. *Assume that there exist two distinct positive numbers a and b such that*

$$\max_{0\leq u\leq a} f(u) \leq \frac{a}{\lambda A}, \quad \min_{\frac{mb}{M}\leq u\leq b} f(u) \geq \frac{b}{\lambda B}, \tag{2.5}$$

where

$$A := \max_{t\in[t_0,t_0+T]_{\mathbb{T}}} \int_{t_0}^{t_0+T} K(t,s)h(s)\Delta s, \quad B := \min_{t\in[t_0,t_0+T]_{\mathbb{T}}} \int_{t_0}^{t_0+T} K(t,s)h(s)\Delta s. \tag{2.6}$$

Then there exists $\overline{y} \in \mathcal{S}$ which is a fixed point of L and satisfies $\min\{a,b\} \leq \|\overline{y}\| \leq \max\{a,b\}$.

Proof. Note that $A, B > 0$. Let $\mathcal{S}_\zeta = \{w \in \mathcal{S} : \|w\| < \zeta\}$. Assume that $a < b$. Then, for any $y \in \mathcal{S}$ which satisfies $\|y\| = a$, in view of (2.5), we have

$$(Ly)(t) \leq \left(\lambda \int_t^{t+T} K(t,s)h(s)\Delta s\right) \cdot \frac{a}{\lambda A} \leq \lambda A \cdot \frac{a}{\lambda A} = a. \tag{2.7}$$

That is, $\|Ly\| \leq \|y\|$ for $y \in \partial\mathcal{S}_a$. For any $y \in \mathcal{S}$ which satisfies $\|y\| = b$, we have

$$(Ly)(t) \geq \left(\lambda \int_t^{t+T} K(t,s)h(s)\Delta s\right) \cdot \frac{b}{\lambda B} \geq \lambda B \cdot \frac{b}{\lambda B} = b. \tag{2.8}$$

That is, we have $\|Ly\| \geq \|y\|$ for $y \in \partial\mathcal{S}_b$. In view of Theorem 1.1, there exists $\overline{y} \in \mathcal{S}$ which satisfies $a \leq \|\overline{y}\| \leq b$ such that $L\overline{y} = \overline{y}$. If $a > b$, (2.7) is replaced by $(Ly)(t) \geq b$ in view of (2.5), and (2.8) is replaced by $(Ly)(t) \leq a$ in view of (2.5). The same conclusion then follows. ∎

Theorem 2.3. *Assume $\ell_0 = \infty$ and $\ell_\infty = \infty$. Then for any $\lambda \in (0, \lambda^\dagger)$, equation* (1.1) *has at least two positive periodic solutions, where*

$$\lambda^\dagger := \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0\leq u\leq r} f(u)},$$

for A defined in (2.6).

Proof. Let $q(r) := r/(A \max_{0\leq u\leq r} f(u))$. Clearly $q \in C((0,\infty),(0,\infty))$. By the choices $\ell_0 = \infty$ and $\ell_\infty = \infty$, we see further that $\lim_{r\to 0} q(r) = \lim_{r\to\infty} q(r) = 0$. Thus, there exists $r_0 > 0$ such that $q(r_0) = \max_{r>0} q(r) = \lambda^\dagger$. For any $\lambda \in (0, \lambda^\dagger)$, by the intermediate value property, there exist $a_1 \in (0, r_0)$ and $a_2 \in (r_0, \infty)$ such that $q(a_1) = q(a_2) = \lambda$. Thus, we have $f(u) \leq a_1/(\lambda A)$ for $u \in [0, a_1]$ and $f(u) \leq a_2/(\lambda A)$ for $u \in [0, a_2]$. On the other hand, there must exist $b_1 \in (0, a_1)$ and $b_2 \in (a_2, \infty)$ such that $f(u)/u \geq M/(\lambda m B)$ for $u \in (0, b_1] \cup [\frac{mb_2}{M}, \infty)$. That is, $f(u) \geq b_1/(\lambda B)$ for $u \in [\frac{mb_1}{M}, b_1]$ and $f(u) \geq b_2/(\lambda B)$ for $u \in [\frac{mb_2}{M}, b_2]$. An application of Lemma 2.2 leads to two distinct solutions of (1.1). ∎

Corollary 2.4. *If either $\ell_0 = \infty$ or $\ell_\infty = \infty$, then for any $0 < \lambda < \lambda^\dagger$, equation* (1.1) *has at least one positive periodic solution.*

Theorem 2.5. *Assume $\ell_0 = 0$ and $\ell_\infty = 0$. Then for any $\lambda > \lambda^{\dagger\dagger}$, equation* (1.1) *has at least two positive periodic solutions, where*

$$\lambda^{\dagger\dagger} := \frac{1}{B} \inf_{r>0} \frac{r}{\min_{\frac{mr}{M}\leq u\leq r} f(u)},$$

for B defined in (2.6).

Proof. Let $v(r) := r/(B \min_{\frac{mr}{M} \le u \le r} f(u))$. Clearly, $v \in C((0,\infty),(0,\infty))$. By the choices $\ell_0 = 0$ and $\ell_\infty = 0$, we see that $\lim_{r\to 0} v(r) = \lim_{r\to\infty} v(r) = \infty$. Thus, there exists $r_0 > 0$ such that $v(r_0) = \min_{r>0} v(r) = \lambda^{\dagger\dagger}$. For any $\lambda > \lambda^{\dagger\dagger}$, there exist $b_1 \in (0, r_0)$ and $b_2 \in (r_0, \infty)$ such that $v(b_1) = v(b_2) = \lambda$. Thus, we have $f(u) \ge b_1/(\lambda B)$ for $u \in [\frac{mb_1}{M}, b_1]$ and $f(u) \ge b_2/(\lambda B)$ for $u \in [\frac{mb_2}{M}, b_2]$. On the other hand, since $\ell_0 = 0$ we see that $f(0) = 0$ and that there exists $a_1 \in (0, b_1)$ such that $f(u)/u \le 1/(\lambda A)$ for $u \in (0, a_1]$. Thus, we have $f(u) \le a_1/(\lambda A)$. From $\ell_\infty = 0$, we see that there exists $a \in (b_2, \infty)$ such that $f(u)/u \le 1/(\lambda A)$ for $u \in [a, \infty)$. Let $\varpi := \max_{0\le u\le a} f(u)$. Then we have $f(u) \le a_2/(\lambda A)$ for $u \in [0, a_2]$, where $a_2 > a$ and $a_2 \ge \lambda \varpi A$. An application of Lemma 2.2 leads to two distinct solutions of (1.1). ∎

Corollary 2.6. *If either $\ell_0 = 0$ or $\ell_\infty = 0$, then for any $\lambda > \lambda^{\dagger\dagger}$, equation* (1.1) *has at least one positive periodic solution.*

Corollary 2.7. *Assume that either $\ell_0 = \infty$ and $\ell_\infty = 0$, or $\ell_\infty = \infty$ and $\ell_0 = 0$. Then for any $\lambda > 0$, equation* (1.1) *has a positive periodic solution.*

Proof. Suppose first that $\ell_0 = \infty$ and $\ell_\infty = 0$ hold. If $\sup_{0\le u<\infty} f(u) = D < \infty$, then $\lambda^\dagger \ge (1/A)\sup_{r>0}(r/D) = \infty$. If f is unbounded, then there exist a sequence $\{r_n\}$ such that $f(r_n) = \max_{0\le u\le r_n} f(u)$ and $\lim_{n\to\infty} r_n = \infty$. Since $\ell_\infty = 0$, we have $\lambda^\dagger \ge (1/A)\sup(r_n/f(r_n)) = \infty$. Thus, we have proved $\lambda^\dagger = \infty$. In this case, our assertion follows from the remark following Theorem 2.3. If $\ell_\infty = \infty$ and $\ell_0 = 0$ hold, then we have $\lim_{u\to\infty} f(u) = \infty$. Let $\{r_n\}$ satisfy $\lim_{n\to\infty} r_n = \infty$ and $f(mr_n/M) = \min_{\frac{mr_n}{M}\le u\le r_n} f(u)$. Since $\ell_\infty = \infty$, we have $\lambda^{\dagger\dagger} \le (1/B)\inf(r_n/f(mr_n/M)) = 0$. Thus, $\lambda^{\dagger\dagger} = 0$. In this case, our assertion follows from Corollary 2.6. ∎

Theorem 2.8. *Assume $\ell_0, \ell_\infty \in (0,\infty)$. Then, for each λ satisfying either*

$$\frac{M}{mB\ell_\infty} < \lambda < \frac{1}{A\ell_0} \quad \textit{or} \quad \frac{M}{mB\ell_0} < \lambda < \frac{1}{A\ell_\infty}, \tag{2.9}$$

equation (1.1) *has a positive periodic solution.*

Proof. Assuming (2.9) holds, let $\varepsilon > 0$ be such that

$$\frac{M}{mB(\ell_\infty - \varepsilon)} \le \lambda \le \frac{1}{A(\ell_0 + \varepsilon)}.$$

Since $\ell_0 > 0$, there exists $\beta_1 > 0$ such that $f(u) \le (\ell_0 + \varepsilon)u$ for $0 < u \le \beta_1$. So, for $y \in \mathcal{S}$ with $\|y\| = \beta_1$, we have

$$\begin{aligned}(Ly)(t) &\le \lambda(\ell_0+\varepsilon)\int_t^{t+T} K(t,s)h(s)y(s-\tau(s))\Delta s \\ &\le \lambda(\ell_0+\varepsilon)\|y\|\int_{t_0}^{t_0+T} K(t,s)h(s)\Delta s \\ &\le \lambda(\ell_0+\varepsilon)A\|y\| \le \|y\|.\end{aligned}$$

Next, since $\ell_\infty > 0$, there exists $\bar\beta_2 > 0$ such that $f(u) \ge (\ell_\infty - \varepsilon)u$ for $u \ge \bar\beta_2$. If $\beta_2 = \max\{2\beta_1, \frac{m}{M}\bar\beta_2\}$, then for $y \in \mathcal{S}$ with $\|y\| = \beta_2$,

$$\begin{aligned}(Ly)(t) &\ge \lambda(\ell_\infty-\varepsilon)\int_t^{t+T} K(t,s)h(s)y(s-\tau(s))\Delta s \\ &\ge \lambda(\ell_\infty-\varepsilon)\frac{m}{M}\|y\|\int_{t_0}^{t_0+T} K(t,s)h(s)\Delta s \\ &\ge \lambda(\ell_\infty-\varepsilon)\frac{m}{M}B\|y\| \ge \|y\|.\end{aligned}$$

In view of Lemma 2.2, we see that equation (1.1) has a positive periodic solution. The other case can be handled in a similar manner. ∎

Corollary 2.9. *Assume that either* $\ell_0 = \infty$ *and* $\ell_\infty \in (0, \infty)$*, or* $\ell_\infty = \infty$ *and* $\ell_0 \in (0, \infty)$*. Then for any* $0 < \lambda < 1/(A\ell_0)$ *or* $0 < \lambda < 1/(A\ell_\infty)$*, equation* (1.1) *has a positive periodic solution.*

Corollary 2.10. *Assume that either* $\ell_0 = 0$ *and* $\ell_\infty \in (0, \infty)$*, or* $\ell_\infty = 0$ *and* $\ell_0 \in ((0, \infty)$*. Then for any* $M/(mB\ell_\infty) < \lambda < \infty$ *or* $M/(mB\ell_0) < \lambda < \infty$*, equation* (1.1) *has a positive periodic solution.*

3. EXISTENCE RESULTS FOR (1.3)

Similarly, we can also discuss equation (1.3). By (1.3) and Theorem 7.2, we have

$$x(t) = \lambda \int_t^{t+T} G(t,s)h(s)f(x(s-\tau(s)))\Delta s,$$

where

$$G(t,s) := \frac{e_{\ominus p}(s,t_0)}{e_{\ominus p}(t,t_0) - e_{\ominus p}(t+T,t_0)} = \frac{e_p(t+T,s)}{e_p(t_0+T,t_0)-1}$$

satisfies

$$m = K(t,t) = G(t_0, t_0+T) = G(t, t+T) \le G(t,s) \le G(t,t) = K(t_0, t_0+T) = M,$$

and

$$\frac{m}{M} = \frac{G(t,t+T)}{G(t,t)} \le \frac{G(t,s)}{G(t,t)} \le 1.$$

Let

$$A^* := \max_{t\in[t_0,t_0+T]_{\mathbb{T}}} \int_{t_0}^{t_0+T} G(t,s)h(s)\Delta s, \quad B^* := \min_{t\in[t_0,t_0+T]_{\mathbb{T}}} \int_{t_0}^{t_0+T} G(t,s)h(s)\Delta s.$$

Then we have the following results.

Theorem 3.1. *Suppose either* $\ell_0 = \infty$ *or* $\ell_\infty = \infty$*. Then for any* $\lambda \in (0, \overline{\lambda})$*, equation* (1.3) *has a positive periodic solution, where*

$$\overline{\lambda} = \frac{1}{A^*} \sup_{r>0} \frac{r}{\max_{0\le u\le r} f(u)}.$$

Theorem 3.2. *Suppose* $\ell_0 = \infty$ *and* $\ell_\infty = \infty$*. Then for any* $\lambda \in (0, \overline{\lambda})$*, equation* (1.3) *has at least two positive periodic solutions.*

Theorem 3.3. *Suppose either* $\ell_0 = 0$ *or* $\ell_\infty = 0$*. Then for any* $\lambda > \underline{\lambda}$*, equation* (1.3) *has a positive periodic solution, where*

$$\underline{\lambda} = \frac{1}{B^*} \inf_{r>0} \frac{r}{\min_{\frac{mr}{M}\le u\le r} f(u)}.$$

Example 3.1. *Fix* $\eta > 0$ *and let* $\mathbb{T} = \eta\mathbb{Z}$*. For fixed constants* $p, \lambda > 0$*, choose* $\tau(t) \equiv n\eta$ *for some* $n \in \mathbb{N}$*, and let* h *be a positive right-dense continuous* T*-periodic function on this time scale. Take* $f(u) := u^2$*, so that* $\ell_0 = 0$ *and* $\ell_\infty = \infty$*. Then we have that* (1.3) *is equivalent to*

$$x(t+\eta) = (1+\eta p)x(t) - (1+\eta p)\eta\lambda h(t)(x(t-n\eta))^2. \tag{3.1}$$

As developed earlier, T-periodic solutions x of this equation satisfy

$$x(t) = \eta\lambda \sum_{j=t/\eta}^{(t+T)/\eta-1} G(t, j\eta)h(j\eta)f(x(j\eta - n\eta)),$$

where

$$G(t, s) = \frac{(1+\eta p)^{(t+T-s)/\eta}}{(1+\eta p)^{T/\eta} - 1}, \qquad \frac{m}{M} = \frac{G(t, t+T)}{G(t,t)} = (1+\eta p)^{-T/\eta}.$$

Since $\overline{\lambda} = \infty$ and $\underline{\lambda} = 0$, both Theorem 3.1 and Theorem 3.3 predict that (3.1) has a positive periodic solution for any $\lambda > 0$.

Theorem 3.4. *Suppose $\ell_0 = 0$ and $\ell_\infty = 0$. Then for any $\lambda > \underline{\lambda}$, equation* (1.3) *has at least two positive periodic solutions.*

Corollary 3.5. *Suppose either $\ell_0 = \infty$ and $\ell_\infty = 0$, or $\ell_\infty = \infty$ and $\ell_0 = 0$. Then for any $\lambda > 0$, equation* (1.3) *has a positive periodic solution.*

Theorem 3.6. *Suppose $\ell_0 \in (0, \infty)$ and $\ell_\infty \in (0, \infty)$. Then, for each λ satisfying*

$$\frac{M}{mB^*\ell_\infty} < \lambda < \frac{1}{A^*\ell_0} \quad \textit{or} \quad \frac{M}{mB^*\ell_0} < \lambda < \frac{1}{A^*\ell_\infty},$$

equation (1.3) *has a positive periodic solution.*

Corollary 3.7. *Suppose either $\ell_0 = \infty$ and $\ell_\infty \in (0, \infty)$, or $\ell_\infty = \infty$ and $\ell_0 \in (0, \infty)$. Then for any $0 < \lambda < 1/(A^*\ell_\infty)$ or $0 < \lambda < 1/(A^*\ell_0)$ equation* (1.3) *has a positive periodic solution.*

Corollary 3.8. *Suppose either $\ell_0 = 0$ and $\ell_\infty \in (0, \infty)$, or $\ell_\infty = 0$ and $\ell_0 \in (0, \infty)$. Then for any $M/(mB^*\ell_\infty) < \lambda < \infty$ or $M/(mB^*\ell_0) < \lambda < \infty$ equation* (1.3) *has a positive periodic solution.*

4. EXISTENCE RESULTS FOR (1.2) AND (1.4)

In a manner analogous to that of the previous sections, we get the existence of positive solutions to (1.2) and (1.4); remember the additional assumption here of $1 - \mu(t)p(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. For (1.2), that is

$$y^\Delta(t) = -p(t)y(t) + \lambda h(t)f(y(t - \tau(t))), \quad t \in [t_0, \infty)_{\mathbb{T}},$$

a periodic solution y exists if and only if y satisfies

$$y(t) = \lambda \int_t^{t+T} \gamma(t, s)h(s)f(y(s - \tau(s)))\Delta s, \quad \gamma(t, s) := \frac{e_{-p}(t+T, \sigma(s))}{1 - e_{-p}(t_0 + T, t_0)}.$$

Then

$$\frac{e_{-p}(t+T, \sigma(t))}{1 - e_{-p}(t_0 + T, t_0)} = \gamma(t, t) \le \gamma(t, s) \le \gamma(t, t+T) = \frac{e_{-p}(t+T, \sigma(t+T))}{1 - e_{-p}(t_0 + T, t_0)},$$

so that

$$\frac{m}{M} := \frac{\gamma(t,t)}{\gamma(t, t+T)} = \frac{1}{e_{-p}(\sigma(t_0), \sigma(t_0) + T)} = e_{-p}(t_0 + T, t_0) < 1.$$

Likewise, for (1.4),

$$x^\Delta(t) = \ominus(-p)(t)x(t) - \lambda h(t)f(x(t - \tau(t))), \quad t \in [t_0, \infty)_{\mathbb{T}},$$

a solution x would have to satisfy

$$x(t) = \lambda \int_t^{t+T} \Gamma(t, s)h(s)f(x(s - \tau(s)))\Delta s, \quad \Gamma(t, s) := \frac{e_{-p}(\sigma(s), t)}{1 - e_{-p}(t_0 + T, t_0)}.$$

In this case the kernel bounds are

$$\frac{e_{-p}(\sigma(t+T),t)}{1-e_{-p}(t_0+T,t_0)} = \Gamma(t,t+T) \le \Gamma(t,s) \le \Gamma(t,t) = \frac{e_{-p}(\sigma(t),t)}{1-e_{-p}(t_0+T,t_0)},$$

so that, interestingly, we again have

$$\frac{m}{M} := \frac{\Gamma(t,t+T)}{\Gamma(t,t)} = \frac{e_{-p}(\sigma(t+T),t)}{e_{-p}(\sigma(t),t)} = e_{-p}(t_0+T,t_0). \tag{4.1}$$

Compare (2.4) with (4.1); we point out that

$$e_{-p}(t_0+T,t_0) = e_{\ominus p}(t_0+T,t_0)$$

if $\mathbb{T} = \mathbb{R}$.

5. Extending to a System

In this section, we investigate the existence of positive periodic solutions for the two pairs of higher-dimensional nonautonomous functional delta dynamic equations

$$y^\Delta(t) = -P(t)y^\sigma(t) + \lambda h(t) f(y(t-\tau(t))), \quad t \in [t_0,\infty)_\mathbb{T} \tag{5.1}$$

and

$$y^\Delta(t) = -P(t)y(t) + \lambda h(t) f(y(t-\tau(t))), \quad t \in [t_0,\infty)_\mathbb{T}, \tag{5.2}$$

or

$$x^\Delta(t) = -\left(\ominus P(t)\right) x^\sigma(t) - \lambda h(t) f(x(t-\tau(t))), \quad t \in [t_0,\infty)_\mathbb{T} \tag{5.3}$$

and

$$x^\Delta(t) = \ominus(-P)(t)x(t) - \lambda h(t) f(x(t-\tau(t))), \quad t \in [t_0,\infty)_\mathbb{T}, \tag{5.4}$$

where

$$P(t) = \text{diag}[p_1(t), p_2(t), \dots, p_n(t)], \quad h(t) = \text{diag}[h_1(t), h_2(t), \dots, h_n(t)];$$

for all cases p_i, h_i ($1 \le i \le n$) and τ are right-dense continuous T-periodic functions. We assume that $\lambda, T > 0$, and that h_i and p_i are nonnegative with h_i and p_i not identically zero in $[t_0, T+t_0]_\mathbb{T}$; additionally, we must have $1-\mu(t)p_i(t) > 0$ for all $t \in [t_0,\infty)_\mathbb{T}$ in (5.2) and (5.4) for $1 \le i \le n$. Throughout we also assume that $f : \mathbb{R}^n_+ \to \mathbb{R}^+$ is continuous where $\mathbb{R}^n_+ = \{(y_1,\dots,y_n)^T \in \mathbb{R}^n : y_i \ge 0, 0 \le i \le n\}$ and $\mathbb{R}^+ = \{y \in \mathbb{R} : y > 0\}$. For a related discrete version of this discussion, see [9].

Define $K(t,s) = \text{diag}[K_1(t,s), K_2(t,s), \dots, K_n(t,s)]$ where

$$K_i(t,s) := \frac{e_{p_i}(s,t)}{e_{p_i}(t_0+T,t_0)-1}.$$

Then using equation (2.2) we have that

$$y(t) = \lambda \int_t^{t+T} K(t,s)h(s)f(y(s-\tau(s)))\Delta s$$

is a T-periodic solution of equation (5.1). Also using equations (2.3), (2.4) we have for $s \in [t,t+T]_\mathbb{T}$ that

$$m_i := \frac{1}{e_{p_i}(t_0+T,t_0)-1} \le K_i(t,s) \le \frac{e_{p_i}(t_0+T,t_0)}{e_{p_i}(t_0+T,t_0)-1} =: M_i, \tag{5.5}$$

and

$$0 < \frac{m_i}{M_i} = e_{\ominus p_i}(t_0+T,t_0) \le \frac{K(t,s)}{K(t,t+T)} \le 1, \quad t \in [t_0,\infty)_\mathbb{T}, \quad s \in [t,t+T]_\mathbb{T}. \tag{5.6}$$

Let

$$\gamma = \min\left\{\frac{m_i}{M_i} : 1 \le i \le n\right\}.$$

Then $\gamma \in (0,1)$. Again let $\mathcal{B}$ be the Banach space of continuous and T-periodic functions augmented with the supremum norm

$$||y|| = \max_{1\le i\le n} |y_i|_0, \text{ where } |y_i|_0 = \sup_{t\in[t_0,t_0+T]_{\mathbb{T}}} |y_i(t)|.$$

Define a cone $\mathcal{S}$ by

$$\mathcal{S} = \{y \in \mathcal{B} : y_i(t) \ge \gamma|y_i|_0, 1 \le i \le n\},$$

and for a positive number r, define Ω_r by

$$\Omega_r = \{y \in \mathcal{S} : |y_i|_0 < r : 1 \le i \le n\}.$$

Note that $\partial\Omega_r = \{y \in \mathcal{S} : |y_i|_0 = r : 1 \le i \le n\}$. Define a mapping $L : \mathcal{B} \to \mathcal{B}$ by

$$(Ly)(t) = \lambda \int_t^{t+T} K(t,s)h(s)f(y(s-\tau(s)))\Delta s,$$

and denote

$$(Ly) = (L_1y, L_2y, \dots, L_ny)^T.$$

For the remainder of this section we will use the following notations:

$$q_i = \min_{t_0\le u\le t_0+T} h_i(u), \quad r_i = \max_{t_0\le u\le t_0+T} h_i(u)$$

$$l_i^0 = \lim_{u\to 0} \tfrac{f_i(u)}{u} \in [0,\infty], \quad l_i^\infty = \lim_{u\to\infty} \frac{f_i(u)}{u} \in [0,\infty],$$

for $1 \le i \le n$ and

$$q = \min_{1\le i\le n} q_i \quad r = \max_{1\le i\le n} r_i$$

$$m = \min_{1\le i\le n} m_i \quad M = \max_{1\le i\le n} M_i.$$

Lemma 5.1. *The operator L satisfies $L\mathcal{S} \subset \mathcal{S}$.*

Proof. It follows using standard reasoning that L is completely continuous on bounded subsets of $\mathcal{S}$. Then for $y \in \mathcal{S}$,

$$(L_iy)(t) \le \lambda M_i \int_{t_0}^{t_0+T} h_i(s)f_i(y(s-\tau(s)))\Delta s, \quad t \in [t_0,\infty)_{\mathbb{T}},$$

so that

$$(L_iy)(t) \ge \lambda m_i \int_{t_0}^{t_0+T} h(s)f(y(s-\tau(s)))\Delta s \ge \frac{m_i}{M_i}|L_iy|_0 \ge \gamma|L_iy|_0,$$

and so $L\mathcal{S} \subset \mathcal{S}$. ∎

Lemma 5.2. *For $1 \le i \le n$, assume that there exist distinct positive numbers a_i and b_i such that*

$$\max_{0\le u\le a_i} f_i(u) \le \frac{a_i}{\lambda A_i}, \qquad \min_{\frac{m_ib_i}{M_i}\le u\le b_i} f_i(u) \ge \frac{b_i}{\lambda B_i} \tag{5.7}$$

where

$$A_i := \max_{t\in[t_0,t_0+T]_{\mathbb{T}}} \int_{t_0}^t K_i(t,s)h_i(s)\Delta s, \qquad B_i := \min_{t\in[t_0,t_0+T]_{\mathbb{T}}} \int_{t_0}^t K_i(t,s)h_i(s)\Delta s. \tag{5.8}$$

Let $a = \max\{a_i : 1 \leq i \leq n\}$ *and* $b = \min\{b_i : 1 \leq i \leq n\}$*. Then there exists* $\overline{y} \in \mathcal{S}$ *which is a fixed point of* L *and satisfies*

$$\min\{a, b\} \leq \|\overline{y}\| \leq \max\{a, b\}.$$

Proof. As in Lemma 2.2, if $a < b$, then for any $y \in \partial\Omega_{a_i}$ we have $(L_i y) \leq a_i \leq a$ for $1 \leq i \leq n$, that is $\|Ly\| \leq \|y\|$. Also for $y \in \Omega_{b_i}$, $(L_i y) \geq b_i \geq b$ for $1 \leq i \leq n$, that is $\|Ly\| \geq \|y\|$. In view of Theorem 1.1, there exists $\overline{y} \in \mathcal{S}$ which satisfies $a \leq \|\overline{y}\| \leq b$ such that $L\overline{y} = \overline{y}$. If $a > b$, then the same conclusion follows. ∎

Similar to the theorems in Section 2, the following results may be proven.

Theorem 5.3. *Assume* $l_i^0 = \infty$ *and* $l_i^\infty = \infty$ *for* $1 \leq i \leq n$*. Then for any* $\lambda \in (0, \lambda^*)$*, equation* (5.1) *has at least two positive solutions, where*

$$\lambda^* := \min_{1\leq i\leq n} \left(\frac{1}{A_i} \sup_{r>0} \frac{r}{\max\limits_{0\leq u\leq r} f_i(u)} \right),$$

for A_i *as defined in* (5.8).

Corollary 5.4. *If either* $l_i^0 = \infty$ *for* $1 \leq i \leq n$*, or* $l_i^\infty = \infty$ *for* $1 \leq i \leq n$*, then for any* $0 < \lambda < \lambda^*$*, equation* (5.1) *has at least one positive periodic solution.*

Theorem 5.5. *Assume* $l_i^0 = 0$ *and* $l_i^\infty = 0$ *for* $1 \leq i \leq n$*. Then for any* $\lambda > \lambda^{**}$*, equation* (5.1) *has at least two positive periodic solutions, where*

$$\lambda^{**} = \max_{1\leq i\leq n} \left(\frac{1}{B_i} \inf_{r>0} \frac{r}{\min\limits_{\frac{m_i r}{M_i}\leq u\leq r} f(u)} \right),$$

for B_i *as defined in* (5.8).

Corollary 5.6. *If either* $l_i^0 = 0$ *for* $1 \leq i \leq n$*, or* $l_i^\infty = 0$ *for* $1 \leq i \leq n$*, then for any* $\lambda > \lambda^{**}$*, equation* (5.1) *has at least one positive periodic solution.*

Corollary 5.7. *Assume that either* $l_i^0 = \infty$ *and* $l_i^\infty = 0$ *for* $1 \leq i \leq n$*, or* $l_i^0 = 0$ *and* $l_i^\infty = \infty$ *for* $1 \leq i \leq n$*. Then for any* $\lambda > 0$ *equation* (5.1) *has a positive periodic solution.*

Similarly one may discuss existence results for equation (5.3).

6. CORRESPONDING NABLA EQUATIONS

Consider briefly nabla dynamic equations, introduced in [2] and explored in [4, Chapter 3]. Just as in the previous sections, we may also investigate the existence of positive periodic solutions for the two pairs of first-order nonautonomous functional nabla dynamic equations

$$y^\nabla(t) = -q(t)y^\rho(t) + \lambda h(t)f(y(t-\tau(t))), \quad t \in [t_0, \infty)_\mathbb{T} \tag{6.1}$$

and

$$y^\nabla(t) = -q(t)y(t) + \lambda h(t)f(y(t-\tau(t))), \quad t \in [t_0, \infty)_\mathbb{T}, \tag{6.2}$$

or

$$x^\nabla(t) = -\left(\ominus_\nu q(t)\right) x^\rho(t) - \lambda h(t)f(x(t-\tau(t))), \quad t \in [t_0, \infty)_\mathbb{T} \tag{6.3}$$

and

$$x^\nabla(t) = \ominus_\nu(-q)(t)x(t) - \lambda h(t)f(x(t-\tau(t))), \quad t \in [t_0, \infty)_\mathbb{T}, \tag{6.4}$$

where for these cases q, h and τ are left-dense continuous T-periodic functions. Note that $\nu(t) = t - \rho(t)$ and $\ominus_\nu q := \frac{-q}{1-\nu q}$. Again we assume that $\lambda, T > 0$, and that h and q are nonnegative with h and q not identically zero in $[t_0, T + t_0]_{\mathbb{T}}$; additionally, we must have $1 - \nu(t)q(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ in (6.1) and (6.3). Moreover, the time scale $\mathbb{T}$ is periodic in the sense that $t + T \in \mathbb{T}$ and $\nu(t) = \nu(t + T)$ for all $t \in \mathbb{T}$. Using the nabla exponential function $\hat{e}$ and its properties, a T-periodic solution y of (6.1) satisfies

$$y(t) = \lambda \int_t^{t+T} \hat{K}(t,s)h(s)f(y(s-\tau(s)))\nabla s, \quad \hat{K}(t,s) := \frac{\hat{e}_q(s,t)}{\hat{e}_q(t_0+T,t_0)-1},$$

while a T-periodic solution x of (6.3) satisfies

$$x(t) = \lambda \int_t^{t+T} \hat{G}(t,s)h(s)f(x(s-\tau(s)))\nabla s$$

for

$$\hat{G}(t,s) := \frac{\hat{e}_{\ominus_\nu q}(s,t_0)}{\hat{e}_{\ominus_\nu q}(t,t_0) - \hat{e}_{\ominus_\nu q}(t+T,t_0)} = \frac{\hat{e}_q(t+T,s)}{\hat{e}_q(t_0+T,t_0)-1}.$$

Similarly for the other pair, y is a T-periodic solution of (6.2) if and only if

$$y(t) = \lambda \int_t^{t+T} \hat{\gamma}(t,s)h(s)f(y(s-\tau(s)))\nabla s, \quad \hat{\gamma}(t,s) := \frac{\hat{e}_{-q}(t+T,\rho(s))}{1-\hat{e}_{-q}(t_0+T,t_0)},$$

and x is a T-periodic solution of (6.4) if and only if

$$x(t) = \lambda \int_t^{t+T} \hat{\Gamma}(t,s)h(s)f(x(s-\tau(s)))\nabla s, \quad \hat{\Gamma}(t,s) := \frac{\hat{e}_{-q}(\rho(s),t)}{1-\hat{e}_{-q}(t_0+T,t_0)}.$$

One may also consider the higher-dimensional nabla dynamic equations

(6.5) $$y^\nabla(t) = -Q(t)y^\rho(t) + \lambda h(t)f(y(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}}$$

and

(6.6) $$y^\nabla(t) = -Q(t)y(t) + \lambda h(t)f(y(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}},$$

or

(6.7) $$x^\nabla(t) = -\left(\ominus_\nu Q(t)\right) x^\rho(t) - \lambda h(t)f(x(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}}$$

and

(6.8) $$x^\nabla(t) = \ominus_\nu(-Q)(t)x(t) - \lambda h(t)f(x(t-\tau(t))), \quad t \in [t_0,\infty)_{\mathbb{T}},$$

where

$$Q(t) = \text{diag}[q_1(t), q_2(t), \ldots, q_n(t)], \quad h(t) = \text{diag}[h_1(t), h_2(t), \ldots, h_n(t)];$$

for all cases q_i, h_i ($1 \le i \le n$) and τ are left-dense continuous T-periodic functions. We assume that $\lambda, T > 0$, and that h_i and q_i are nonnegative with h_i and p_i not identically zero in $[t_0, T+t_0]_{\mathbb{T}}$; additionally, we must have $1 - \nu(t)q_i(t) > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ in (6.6) and (6.8) for $1 \le i \le n$. Throughout we also assume that $f : \mathbb{R}^n_+ \to \mathbb{R}^+$ is continuous where $\mathbb{R}^n_+ = \{(y_1,\ldots,y_n)^T \in \mathbb{R}^n : y_i \ge 0, 0 \le i \le n\}$ and $\mathbb{R}^+ = \{y \in \mathbb{R} : y > 0\}$.

7. APPENDIX ON TIME SCALES

A time scale is simply any nonempty closed set of real numbers, and the time-scale calculus is the unification and extension of discrete calculus, quantum calculus, continuous calculus, and indeed arbitrary real-number calculus to a new, more general and overarching theory [7]. The definitions that follow here will serve as a short summary of the time-scale calculus; they can be found in [3] and [4] and the references therein.

Definition 7.1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\},) \text{ for all } t \in \mathbb{T}.$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} > -\infty$. Define the graininess function $\mu : \mathbb{T} \to \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

Throughout this work the assumption is made that $\mathbb{T}$ is unbounded above and has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$ and define the time scale interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$, then the point t is right-scattered, while if $\rho(t) < t$, then t is left-scattered. If $\sigma(t) = t$, then t is right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is left-dense. The following defines the so-called delta derivative.

Definition 7.2. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t such that, for all $s \in U$,

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| \le \epsilon |\sigma(t) - s|.$$

Call $y^\Delta(t)$ the (delta) derivative of $y(t)$ at t.

Definition 7.3. If $F^\Delta(t) = f(t)$, then define the (Cauchy) delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Similar definitions hold for the nabla derivative and integral. The following theorem is due to Hilger [7].

Theorem 7.1. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*

(1) *If f is differentiable at t, then f is continuous at t.*
(2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(3) *If f is differentiable and t is right-dense, then*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(4) *If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.*

Next we define the important concept of right-dense continuity. An important fact concerning right-dense continuity is that every right-dense continuous function has a delta antiderivative [3, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists.

Definition 7.4. The function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous, denoted $f \in C_{rd}(\mathbb{T};\mathbb{R})$, provided f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s\to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

A function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0, \forall t \in \mathbb{T}$. Let

$$\mathcal{R} := \{p \in C_{rd}(\mathbb{T};\mathbb{R}) : 1 + \mu(t)p(t) \neq 0, t \in \mathbb{T}\}.$$

Also, $p \in \mathcal{R}^+$ iff $1 + \mu(t)p(t) > 0,\ \forall t \in \mathbb{T}$. Then if $p \in \mathcal{R}, t_0 \in \mathbb{T}$, one can define the generalized exponential function $e_p(t, t_0)$ to be the unique solution of the initial value problem

$$x^\Delta = p(t)x, \quad x(t_0) = 1.$$

Many of the properties of this generalized exponential function $e_p(t, t_0)$ listed in Theorem 7.2 below are employed throughout this work.

Theorem 7.2. [3, Theorem 2.36] *If* $p, q \in \mathcal{R}$ *and* $s, t \in \mathbb{T}$, *then*

(1) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(2) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(3) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p := \frac{-p}{1+\mu p}$;
(4) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(5) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(6) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$, *where* $p \oplus q := p + q + \mu pq$;
(7) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$.

Again a similar list of properties for the nabla exponential function $\hat{e}$ exists; see [4, Chapter 3].

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