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BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFUSION-WAVE EQUATION

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ABSTRACT. Non homogeneous fractional diffusion-wave equation has been solved under linear/nonlinear boundary conditions. As the order of time derivative changes from 0 to 2, the process changes from slow diffusion to classical diffusion to mixed diffusion-wave behaviour.

Numerical examples presented here confirm this inference. Orthogonality of eigenfunctions in case of fractional Stürm-Liouville problem has been established.

Key words and phrases: Caputo derivative, Fractional diffusion-wave equation, Mittag-Leffler function, anomalous diffusion.

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1. INTRODUCTION

The time fractional diffusion-wave equation [1] is obtained from the classical diffusion or wave equation by replacing the first-or second order time derivative by a fractional derivative of order α with $0 < \alpha < 1$ or $1 < \alpha < 2$, respectively [8]. It represents anomalous subdiffusion if $0 < \alpha < 1$, and anomalous super diffusion in case of $1 < \alpha < 2$. It is a well established fact that this equation models various phenomena. Nigmatullin [10] has employed the fractional diffusion equation to describe diffusion in media with fractal geometry. Mainardi [8] has pointed out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media. Metzler and Klafter [9] have demonstrated that fractional diffusion equation describes a non-Markovian diffusion process with a memory. Ginoa *et al* [4] have presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials. Recently Agrawal [1] has solved fractional-diffusion equation defined in a bounded space domain using finite sine transform technique. This equation has also been solved using Adomian decomposition method [2, 5].

In the present paper we solve nonhomogeneous fractional diffusion-wave equation under homogeneous/nonhomogeneous boundary conditions using the method of separation of variables to get analytical solutions. Some numerical solutions have been obtained for derivatives of fractional order. It is observed that as α increases from 0 to 2, the process changes from slow diffusion to classical diffusion to diffusion-wave to classical wave process.

The paper has been organized as follows. In Section 2 nonhomogeneous fractional diffusion-wave equation with boundary conditions has been solved by variation of parameters method to get analytical solution. Section 3 deals with diffusion-wave equation in higher dimensions. In Section 4, nonhomogeneous boundary conditions have been explored. Some Numerical examples have been presented in Section 5 and fractional Stürm-Liouville problem has been studied in Section 6.

2. NONHOMOGENEOUS FRACTIONAL DIFFUSION-WAVE EQUATION

We consider the following nonhomogeneous fractional diffusion-wave equation:

$$D_t^\alpha u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + q(t), \quad 0 < x < \pi,\ t > 0,\ 0 < \alpha \le 2, \tag{2.1}$$

where D_t^α denotes Caputo fractional derivative with respect to t variable and k denotes a constant coefficient, x and t are the space and time variables, $q(t)$ is assumed to be a continuous function of t. The Caputo fractional derivative of order α, is defined as:

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,t)}{\partial t^m}d\tau, \quad m-1 < \alpha \le m, m \in I\!N,\ t > o.$$

Note that for $\alpha = 1$ and for $\alpha = 2$, (2.1) represents the standard diffusion and the wave equation respectively (homogeneous if $q(t) \equiv 0$ and non-homogeneous otherwise). In the present paper we consider the cases $0 < \alpha < 1$ and $1 < \alpha < 2$, which represent slow diffusion and diffusion-wave respectively [7]. We consider (2.1) along with the boundary conditions given below:

$$u(0,t) = u(\pi,t) = 0, \quad t \ge 0, \tag{2.2}$$

$$u(x,0) = f(x), \quad 0 < x < \pi, \tag{2.3}$$

$$u_t(x,0) = 0, \quad 0 < x < \pi. \tag{2.4}$$

Equation (2.1) ($0 < \alpha < 1$), together with boundary conditions (2.2) and (2.3), yields boundary value problem for fractional diffusion. Since (2.1) is nonhomogeneous, we use the method of variation of parameters [3]. In this method first we solve the corresponding homogeneous equation (putting $q(t) \equiv 0$ in (2.1)), together with the boundary conditions, by separation of variables method. Assume $u(x,t) = X(x)T(t)$, then (2.1) along with conditions (2.2) and (2.3) yields

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(\pi) = 0, \tag{2.5}$$

and

$$D_t^\alpha T(t) + \lambda k T(t) = 0, \quad t \geq 0. \tag{2.6}$$

The Stürm-Liouville problem given by (2.5) has eigenvalues $\lambda_n = n^2$ and the corresponding eigenfunctions $X_n(x) = \sin nx$, $(n = 1, 2, \cdots)$. The solution of (2.6) for the case $\lambda = n^2$ is (upto a constant multiple) $T_n(t) = E_\alpha(-n^2 k t^\alpha)$, where E_α denotes the Mittag-Leffler function [6, 11]. Now we seek a solution of the nonhomogeneous problem which is of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx. \tag{2.7}$$

We assume that the series (2.7) can be differentiated term by term. Note [3]

$$1 = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} \sin nx, \quad 0 < x < \pi. \tag{2.8}$$

Hence, in view of (2.1), we get

$$\sum_{n=1}^{\infty} \left[D_t^\alpha B_n(t) + kn^2 B_n(t)\right] \sin nx = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} q(t) \sin nx. \tag{2.9}$$

By identifying the coefficients in the sine series on each side of this equation, we get

$$D_t^\alpha B_n(t) + kn^2 B_n(t) = \frac{2[1-(-1)^n]}{n\pi} q(t), \quad n = 1, 2, \cdots . \tag{2.10}$$

Using (2.3),

$$\sum_{n=1}^{\infty} B_n(0) \sin nx = f(x), \quad 0 < x < \pi, \tag{2.11}$$

which yields

$$B_n(0) = b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx, \quad (n = 1, 2, \cdots). \tag{2.12}$$

For each value of n, (2.10) and (2.12) make up a fractional initial value problem, having the solution [6]

$$B_n(t) = b_n E_{\alpha,1}(-n^2 k t^\alpha) + \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-n^2 k t^\alpha) \frac{2[1-(-1)^n]}{n\pi} q(t-\tau)\, d\tau, \quad (n = 1, 2, \cdots). \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.7), we get

$$\begin{aligned} u(x,t) = \quad & \frac{2}{\pi} \sum_{n=1}^{\infty} E_\alpha(-n^2 k t^\alpha) \sin nx \int_0^\pi f(r) \sin nr \, dr \\ & + \sum_{n=1}^{\infty} \sin nx \frac{2[1-(-1)^n]}{n\pi} \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-n^2 k t^\alpha) q(t-\tau)\, d\tau. \end{aligned} \tag{2.14}$$

Note: Equation (2.1) together with (2.2),(2.3) and (2.4) form boundary value problem for fractional wave equation. Solving similarly and observing that $B'_n(0) = 0$, we get the solution as given in (2.14). $q(t) \equiv 0$ in (2.14) corresponds to the case discussed by Agrawal [1].

3. FRACTIONAL DIFFUSION-WAVE IN HIGHER DIMENSIONS

In this section we consider

$$D_t^\alpha u = a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \quad 0<x,\, y<\pi,\, t>0,\, 0<\alpha\le 2, \tag{3.1}$$

where a denotes a constant coefficient. We consider (3.1) along with the following boundary conditions.

$$u(x,0,t)=u(x,\pi,t)=u(0,y,t)=u(\pi,y,t)=0, \quad t\ge 0, \tag{3.2}$$

$$u(x,y,0)=f(x,y), \quad 0\le x,y\le\pi, \tag{3.3}$$

$$u_t(x,y,0)=0, \quad 0\le x,y\le\pi. \tag{3.4}$$

We assume that the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ are also continuous. Functions of the type $U=X(x)Y(y)T(t)$ satisfy (3.1) if

$$\frac{D_t^\alpha T(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=-\lambda, \tag{3.5}$$

where λ is a separation constant. (3.5), implies:

$$\frac{Y''(y)}{Y(y)}=-\lambda-\frac{X''(x)}{X(x)}=-\mu, \tag{3.6}$$

where μ is another separation constant. In view of (3.1) we get,

$$X''(x)+(\lambda-\mu)X(x)=0, \quad X(0)=0,\, X(\pi)=0, \tag{3.7}$$

and

$$Y''(y)+\mu Y(y)=0, \quad Y(0)=0,\, Y(\pi)=0. \tag{3.8}$$

(3.5), together with (3.4) gives:

$$D_t^\alpha T(t)+\lambda a^2T(t)=0, \quad T'(0)=0. \tag{3.9}$$

The Stürm-Liouville problem given in (3.8) has eigenvalues $\mu=m^2 \quad (m=1,2,\cdots)$ and the corresponding eigenfunctions are $Y_m(y)=\sin my$. Similarly the Stürm-Liouville problem given in (3.7) has eigenvalues $\lambda-\mu=n^2 \quad (n=1,2,\cdots)$ and the corresponding eigenfunctions are $X_n(x)=\sin nx$. Thus (3.9) takes the form:

$$D_t^\alpha T(t)+a^2(m^2+n^2)T(t)=0, \quad T'(0)=0, \quad m=1,2,\cdots,\, n=1,2,\cdots \tag{3.10}$$

For any fixed positive integers m and n, the solution of (3.10) is (except for a constant factor) $T_{mn}(t)=E_\alpha\left(-a^2(m^2+n^2)\,t^\alpha\right)$[6]. The formal solution of the boundary value problem is, therefore

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}\sin nx\,\sin my\,E_\alpha\left(-a^2(m^2+n^2)\,t^\alpha\right), \tag{3.11}$$

where the coefficients b_{mn} need to be determined so that

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}\sin nx\sin my, \quad 0\le x,y\le\pi. \tag{3.12}$$

By grouping terms in this double sine series so as to display the total coefficient of $\sin nx$ for each n, one can write formally

$$f(x,y)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}B_{mn}\sin my\right)\sin nx, \tag{3.13}$$

for each fixed $y\,(0 \le y \le \pi)$, (3.13) is a Fourier series representation of the function $f(x, y)$, with variable $x\,(0 \le x \le \pi)$, provided that

$$\sum_{m=1}^{\infty} B_{mn} \sin my = \frac{2}{\pi} \int_0^{\pi} f(x, y) \sin nx \, dx \quad (n = 1, 2, \cdots). \tag{3.14}$$

The right-hand side here is a sequence of functions $F_n(y)\,(n = 1, 2, \cdots)$, each represented by its Fourier sine series (3.14) on the interval $y\,(0 \le y \le \pi)$ where

$$B_{mn} = \frac{2}{\pi} \int_0^{\pi} F_n(y) \sin my \, dy \quad (m = 1, 2, \cdots). \tag{3.15}$$

Hence the coefficients B_{mn} have the values

$$B_{mn} = \frac{4}{\pi^2} \int_0^{\pi} \sin my \int_0^{\pi} f(x, y) \sin nx \;\, dx \, dy. \tag{3.16}$$

In view of (3.16), (3.11) gives
(3.17)

$$u(x, y, t) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_\alpha \left(-a^2(m^2 + n^2)\, t^\alpha\right) \sin nx \sin my \int_0^{\pi} \sin mr \int_0^{\pi} f(s, r) \sin ns \, ds \, dr$$

4. NONHOMOGENEOUS BOUNDARY CONDITIONS

We consider the following homogeneous fractional diffusion-wave equation

$$D_t^\alpha u = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\, t > 0,\, 0 < \alpha \le 2, \tag{4.1}$$

along with the nonhomogeneous boundary value conditions:

$$u(0, t) = 0, , \quad t > 0, \tag{4.2}$$

$$u(x, 0) = 0, \quad 0 < x < 1, \tag{4.3}$$

$$K u_x(1, t) = A, \quad t > 0. \tag{4.4}$$

For $1 < \alpha \le 2$, the initial condition:

$$u_t(x, 0) = 0, \quad 0 < x < 1, \tag{4.5}$$

should be added. Let

$$u(x, t) = U(x, t) + \Phi(x). \tag{4.6}$$

Equations(4.1)–(4.6) yield

$$\begin{aligned}
D_t^\alpha U &= k \left[\frac{\partial^2 U}{\partial x^2} + \Phi''(x)\right], \qquad 0 < x < 1,\, t > 0,\, 0 < \alpha \le 2, \\
U(0, t) + \Phi(0) &= 0, \\
K\left[U_x(1, t) + \Phi'(1)\right] &= A, \\
U(x, 0) + \Phi(x) &= 0, \\
U_t(x, 0) &= 0, \qquad \text{(for } 1 < \alpha \le 2).
\end{aligned} \tag{4.7}$$

Assume

$$\Phi''(x) = 0 \text{ and } \Phi(0) = 0,\; K\Phi'(1) = A. \tag{4.8}$$

(4.8) yields a boundary value problem for $U(x,t)$ that does have two-point boundary conditions leading to a Stürm-Liouville problem:

$$
\begin{aligned}
&D_t^\alpha U = k\frac{\partial^2 U}{\partial x^2}, \quad (0 < x < 1,\, t > 0),\, 0 < \alpha \leq 2\\
&U(0,t) = 0,\\
&U_x(1,t) = 0,\\
&U(x,0) = -\Phi(x),\\
&U_t(x,0) = 0, \quad (\text{for } \ 1 < \alpha \leq 2).
\end{aligned} \tag{4.9}
$$

(4.8) implies that

$$\Phi(x) = \frac{A}{K}x. \tag{4.10}$$

Let $U = X(x)T(t)$. Then

$$U(x,t) = \sum_{n=1}^{\infty} E_\alpha(-[\frac{(2n-1)\pi}{2}]^2 k\, t^\alpha)\, \phi_n(x), \tag{4.11}$$

where $\phi_n(x) = \frac{(2n-1)\pi}{\sqrt{2}} \sin x$. The BVP given in (4.9) has been solved in Section 2, and has the following solution.

$$u(x,t) = \frac{A}{K}\left[x + 8\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2\pi^2} E_\alpha(-[\frac{(2n-1)\pi}{2}]^2 k\, t^\alpha) \sin\frac{(2n-1)\pi}{2}x\right]. \tag{4.12}$$

5. ILLUSTRATIVE EXAMPLES

Example. Consider the following nonhomogeneous fractional diffusion-wave equation along with the boundary conditions given below:

$$
\begin{aligned}
&D_t^\alpha u = \frac{\partial^2 u}{\partial x^2} + t, \quad 0 < \alpha \leq 2,\, t > 0,\\
&u(0,t) = u(\pi,t) = 0, \quad t \geq 0,\\
&u(x,0) = f(x), \quad 0 < x < \pi,\\
&u_t(x,0) = 0, \quad 0 < x < \pi,
\end{aligned}
$$

where

$$f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2},\\ \pi - x & \frac{\pi}{2} < x < \pi. \end{cases} \tag{5.1}$$

In Figs. 1, 2, 3 and 4 we plot $u(x,t)$ for $0 \le t \le 1$ and various values of α.

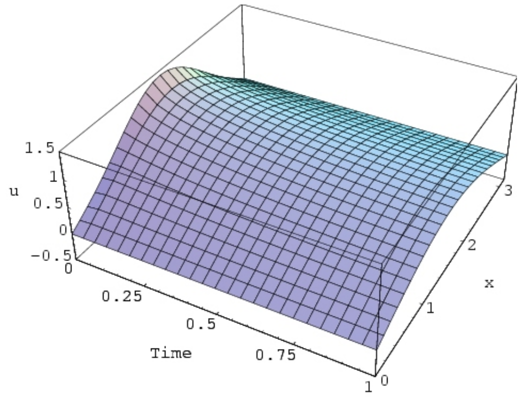

Fig. 1 ($\alpha = .75$)

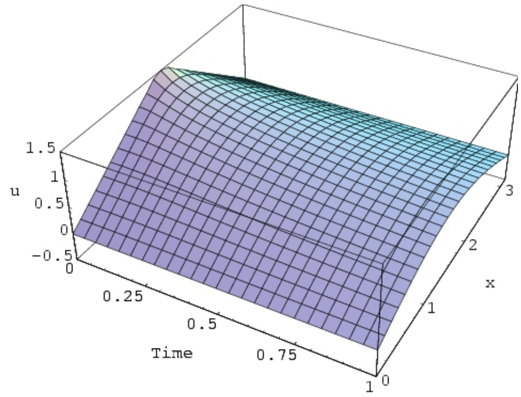

Fig. 2 ($\alpha = 1$)

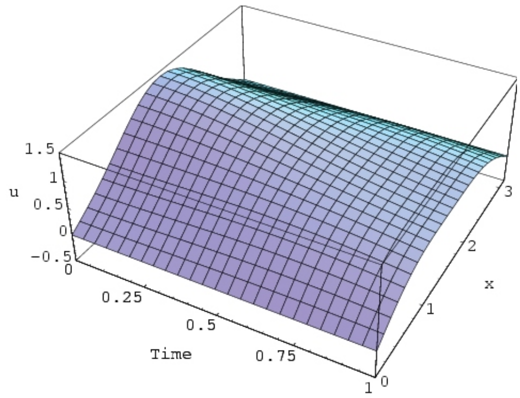

Fig. 3 ($\alpha = 1.5$)

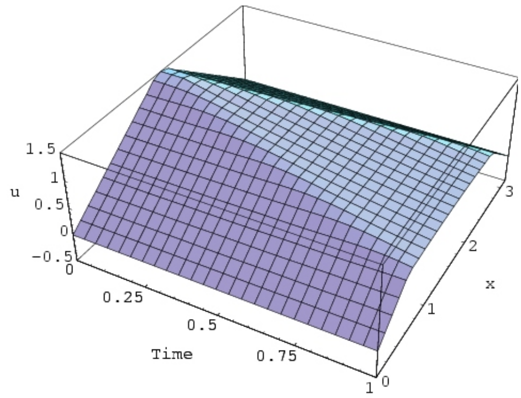

Fig. 4 ($\alpha = 2$)

Comment: As the order of the time derivative changes from 0 to 2, the process changes from slow diffusion to classical diffusion to mixed diffusion wave behaviour.

6. FRACTIONAL STÜRM-LIOUVILLE PROBLEM

Consider the following BVP

$$\left[p(x)\, y^{(\beta)}\right]' + \lambda\, q(x)\, y = 0, \quad 0 < \beta < 1,\ y(a) = y(b) = 0, \tag{6.1}$$

where $y^{(\beta)} = \frac{1}{\Gamma(1-\beta)} \int_a^x (x-t)^{-\beta} y'(t)dt$. Let y_n and y_m satisfy (6.1) for the values $\lambda = \lambda_n$ and $\lambda = \lambda_m$ respectively, *i.e.*

$$\left[p(x)\, y_n^{(\beta)}\right]' + \lambda_n\, q(x)\, y_n = 0, \quad y_n(a) = y_n(b) = 0, \tag{6.2}$$

$$\left[p(x)\, y_m^{(\beta)}\right]' + \lambda_m\, q(x)\, y_m = 0, \quad y_m(a) = y_m(b) = 0. \tag{6.3}$$

Multiplying (6.2) by y_m and (6.3) by y_n respectively, integrating and subtracting, we get

$$\begin{aligned} &\int_a^b \left\{ y_n(x) \left[p(x)\, y_m^{(\beta)}(x)\right]' - y_m(x) \left[p(x)\, y_n^{(\beta)}(x)\right]' \right\} dx = \\ &-\int_a^b \left[p(x)\, y_m^{(\beta)}(x)\, y_n'(x) - p(x)\, y_n^{(\beta)}(x)\, y_m'(x)\right] dx = \\ &(\lambda_m - \lambda_n) \int_a^b q(x)\, y_n(x)\, y_m(x)\, dx. \end{aligned} \tag{6.4}$$

Note

$$
\begin{aligned}
\left|\int_a^b p(x)y_n^{(\beta)}(x)y_m'(x)dx\right| &= \left|\int_a^b \left[\int_a^x \frac{(x-t)^{-\beta}}{\Gamma(1-\beta)}y_n'(t)dt\right] p(x)\, y_m'(x)dx\right| \\
&\le \frac{M}{\Gamma(1-\beta)}\left|\int_a^b \left[\int_a^x (x-t)^{-\beta}\, dt\right] y_m'(x)\, dx\right| \\
&\le \frac{M}{\Gamma(1-\beta)}\left|\int_a^b \frac{(x-a)^{1-\beta}}{1-\beta}\, y_m'(x)\, dx\right| \\
&\le \frac{M(b-a)^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\left|\int_a^b y_m'(x)\, dx\right| \\
&= 0, \text{ as } y_m(a) = y_m(b) = 0.
\end{aligned} \tag{6.5}
$$

Similarly $\left|\int_a^b p(x)y_m^{(\beta)}(x)y_n'(x)\,dx\right| = 0$. Hence $(\lambda_m - \lambda_n)\int_a^b q(x)y_n(x)y_m(x)\,dx = 0$. Thus the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

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