BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFUSION-WAVE EQUATION

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ABSTRACT. Non homogeneous fractional diffusion-wave equation has been solved under linear/nonlinear boundary conditions. As the order of time derivative changes from 0 to 2, the process changes from slow diffusion to classical diffusion to mixed diffusion-wave behaviour. Numerical examples presented here confirm this inference. Orthogonality of eigenfunctions in case of fractional Sturm-Liouville problem has been established.

Key words and phrases: Caputo derivative, Fractional diffusion-wave equation, Mittag-Leffler function, anomalous diffusion.

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1. Introduction

The time fractional diffusion-wave equation \([1]\) is obtained from the classical diffusion or wave equation by replacing the first-or second order time derivative by a fractional derivative of order \(\alpha\) with \(0 < \alpha < 1\) or \(1 < \alpha < 2\), respectively \([8]\). It represents anomalous subdiffusion if \(0 < \alpha < 1\), and anomalous super diffusion in case of \(1 < \alpha < 2\). It is a well established fact that this equation models various phenomena. Nigmatullin \([10]\) has employed the fractional diffusion equation to describe diffusion in media with fractal geometry. Mainardi \([8]\) has pointed out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media. Metzler and Klafter \([9]\) have demonstrated that fractional diffusion equation describes a non-Markovian diffusion process with a memory. Giona et al \([4]\) have presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials. Recently Agrawal \([1]\) has solved fractional-diffusion equation defined in a bounded space domain using finite sine transform technique. This equation has also been solved using Adomian decomposition method \([2, 5]\).

In the present paper we solve nonhomogeneous fractional diffusion-wave equation under homogeneous/nonhomogeneous boundary conditions using the method of separation of variables to get analytical solutions. Some numerical solutions have been obtained for derivatives of fractional order. It is observed that as \(\alpha\) increases from 0 to 2, the process changes from slow diffusion to classical diffusion to diffusion-wave to classical wave process.

The paper has been organized as follows. In Section 2 nonhomogeneous fractional diffusion-wave equation with boundary conditions has been solved by variation of parameters method to get analytical solution. Section 3 deals with diffusion-wave equation in higher dimensions. In Section 4 nonhomogeneous boundary conditions have been explored. Some Numerical examples have been presented in Section 5 and fractional Stürm-Liouville problem has been studied in Section 6.

2. Nonhomogeneous Fractional Diffusion-Wave Equation

We consider the following nonhomogeneous fractional diffusion-wave equation:

\[
D^\alpha_t u(x,t) = k \frac{\partial^2 u(x,t)}{\partial x^2} + q(t), \quad 0 < x < \pi, \ t > 0, \ 0 < \alpha \leq 2,
\]

where \(D^\alpha_t\) denotes Caputo fractional derivative with respect to \(t\) variable and \(k\) denotes a constant coefficient, \(x\) and \(t\) are the space and time variables, \(q(t)\) is assumed to be a continuous function of \(t\). The Caputo fractional derivative of order \(\alpha\), is defined as:

\[
D^\alpha_t u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \quad m - 1 < \alpha \leq m, \ m \in \mathbb{N}, \ t > 0.
\]

Note that for \(\alpha = 1\) and for \(\alpha = 2\), \((2.1)\) represents the standard diffusion and the wave equation respectively (homogeneous if \(q(t) \equiv 0\) and non-homogeneous otherwise). In the present paper we consider the cases \(0 < \alpha < 1\) and \(1 < \alpha < 2\), which represent slow diffusion and diffusion-wave respectively \([7]\). We consider \((2.1)\) along with the boundary conditions given below:

\[
(2.2) \quad u(0,t) = u(\pi,t) = 0, \quad t \geq 0,
\]

\[
(2.3) \quad u(x,0) = f(x), \quad 0 < x < \pi,
\]

\[
(2.4) \quad u_t(x,0) = 0, \quad 0 < x < \pi.
\]

Equation \((2.1)\) \((0 < \alpha < 1)\), together with boundary conditions \((2.2)\) and \((2.3)\), yields boundary value problem for fractional diffusion. Since \((2.1)\) is nonhomogeneous, we use the method of variation of parameters \([3]\). In this method first we solve the corresponding homogeneous
equation (putting \( q(t) \equiv 0 \) in (2.1)), together with the boundary conditions, by separation of variables method. Assume \( u(x, t) = X(x)T(t) \), then (2.1) along with conditions (2.2) and (2.3) yields
\[
X''(x) + \lambda X(x) = 0, \quad X(0) = X(\pi) = 0, \quad \lambda \neq 0,
\]
and
\[
D_\alpha^t T(t) + \lambda kT(t) = 0, \quad t \geq 0.
\]
The Sturm-Liouville problem given by (2.5) has eigenvalues \( \lambda_n = n^2 \) and the corresponding eigenfunctions \( X_n(x) = \sin nx, \quad (n = 1, 2, \ldots) \). The solution of (2.6) for the case \( \lambda = n^2 \) is (upto a constant multiple) \( T_n(t) = E_\alpha(-n^2kt^\alpha) \), where \( E_\alpha \) denotes the Mittag-Leffler function \([6,11]\). Now we seek a solution of the nonhomogeneous problem which is of the form
\[
(2.7) \quad u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin nx.
\]
We assume that the series (2.7) can be differentiated term by term. Note \([3]\)
\[
(2.8) \quad 1 = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} \sin nx, \quad 0 < x < \pi.
\]
Hence, in view of (2.1), we get
\[
(2.9) \quad \sum_{n=1}^{\infty} \left[ D_\alpha^t B_n(t) + kn^2 B_n(t) \right] \sin nx = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} q(t) \sin nx.
\]
By identifying the coefficients in the sine series on each side of this equation, we get
\[
(2.10) \quad D_\alpha^t B_n(t) + kn^2 B_n(t) = \frac{2[1 - (-1)^n]}{n\pi} q(t), \quad n = 1, 2, \ldots.
\]
Using (2.3),
\[
(2.11) \quad \sum_{n=1}^{\infty} B_n(0) \sin nx = f(x), \quad 0 < x < \pi,
\]
which yields
\[
(2.12) \quad B_n(0) = b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx, \quad (n = 1, 2, \ldots).
\]
For each value of \( n \), (2.10) and (2.12) make up a fractional initial value problem, having the solution \([6]\)
\[
(2.13) \quad B_n(t) = b_n E_{\alpha,1}(-n^2kt^\alpha) + \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-n^2kt^\alpha) \frac{2[1 - (-1)^n]}{n\pi} q(t - \tau) \, d\tau, \quad (n = 1, 2, \ldots).
\]
Substituting (2.12) and (2.13) in (2.7), we get
\[
(2.14) \quad u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} E_{\alpha}(-n^2kt^\alpha) \sin nx \int_0^\pi f(r) \sin nr \, dr + \sum_{n=1}^{\infty} \sin nx \frac{2[1 - (-1)^n]}{n\pi} \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-n^2kt^\alpha) q(t - \tau) \, d\tau.
\]
Note: Equation (2.1) together with (2.2), (2.3) and (2.4) form boundary value problem for fractional wave equation. Solving similarly and observing that \( B_n'(0) = 0 \), we get the solution as given in (2.14). \( q(t) \equiv 0 \) in (2.14) corresponds to the case discussed by Agrawal \([1]\).
3. Fractional Diffusion-Wave in Higher Dimensions

In this section we consider

\[ D_t^\alpha u = a^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad 0 < x, y < \pi, \ t > 0, \ 0 < \alpha \leq 2, \]

where \( a \) denotes a constant coefficient. We consider (3.1) along with the following boundary conditions.

\[ u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0, \quad t \geq 0, \]
\[ u(x, y, 0) = f(x, y), \quad 0 \leq x, y \leq \pi, \]
\[ u_t(x, y, 0) = 0, \quad 0 \leq x, y \leq \pi. \]

We assume that the partial derivatives \( f_x(x, y) \) and \( f_y(x, y) \) are also continuous. Functions of the type \( U = X(x)Y(y)T(t) \) satisfy (3.1) if

\[ \frac{D_t^\alpha T(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda, \]

where \( \lambda \) is a separation constant. (3.5), implies:

\[ \frac{Y''(y)}{Y(y)} = -\lambda - \frac{X''(x)}{X(x)} = -\mu, \]

where \( \mu \) is another separation constant. In view of (3.1) we get,

\[ X''(x) + (\lambda - \mu)X(x) = 0, \quad X(0) = 0, X(\pi) = 0, \]

and

\[ Y''(y) + \mu Y(y) = 0, \quad Y(0) = 0, Y(\pi) = 0. \]

(3.5), together with (3.4) gives:

\[ D_t^\alpha T(t) + \lambda a^2 T(t) = 0, \quad T'(0) = 0. \]

The Sturm-Liouville problem given in (3.8) has eigenvalues \( \mu = m^2 \) \((m = 1, 2, \ldots)\) and the corresponding eigenfunctions are \( Y_m(y) = \sin my \). Similarly the Sturm-Liouville problem given in (3.7) has eigenvalues \( \lambda - \mu = n^2 \) \((n = 1, 2, \ldots)\) and the corresponding eigenfunctions are \( X_n(x) = \sin nx \). Thus (3.9) takes the form:

\[ D_t^\alpha T(t) + a^2 (m^2 + n^2) T(t) = 0, \quad T'(0) = 0, \quad m = 1, 2, \ldots, \ n = 1, 2, \ldots \]

For any fixed positive integers \( m \) and \( n \), the solution of (3.10) is (except for a constant factor)

\[ T_{mn}(t) = E_\alpha \left( -a^2 (m^2 + n^2) t^\alpha \right) \]

The formal solution of the boundary value problem is, therefore

\[ u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin nx \sin my E_\alpha \left( -a^2 (m^2 + n^2) t^\alpha \right), \]

where the coefficients \( b_{mn} \) need to be determined so that

\[ f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin nx \sin my, \quad 0 \leq x, y \leq \pi. \]

By grouping terms in this double sine series so as to display the total coefficient of \( \sin nx \) for each \( n \), one can write formally

\[ f(x, y) = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} B_{mn} \sin my \right) \sin nx, \]
for each fixed \( y (0 \leq y \leq \pi) \), (3.13) is a Fourier series representation of the function \( f(x, y) \), with variable \( x (0 \leq x \leq \pi) \), provided that

\[
\sum_{m=1}^{\infty} B_{mn} \sin my = \frac{2}{\pi} \int_{0}^{\pi} f(x, y) \sin nx \, dx \quad (n = 1, 2, \ldots).
\]

The right-hand side here is a sequence of functions \( F_n(y) \) \((n = 1, 2, \ldots)\), each represented by its Fourier sine series (3.14) on the interval \( y (0 \leq y \leq \pi) \) where

\[
B_{mn} = \frac{2}{\pi} \int_{0}^{\pi} F_n(y) \sin my \, dy \quad (m = 1, 2, \ldots).
\]

Hence the coefficients \( B_{mn} \) have the values

\[
B_{mn} = \frac{4}{\pi^2} \int_{0}^{\pi} \sin my \int_{0}^{\pi} f(x, y) \sin nx \, dx \, dy.
\]

In view of (3.16), (3.11) gives

\[
u(x, y, t) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_\alpha \left(-a^2(m^2 + n^2) t^\alpha\right) \sin nx \sin my \int_{0}^{\pi} \sin nr \int_{0}^{\pi} f(s, r) \sin ns \, ds \, dr
\]

4. **Nonhomogeneous boundary conditions**

We consider the following homogeneous fractional diffusion-wave equation

\[
D_\alpha^t u = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0, \ 0 < \alpha \leq 2,
\]

along with the nonhomogeneous boundary value conditions:

(4.2) \( u(0, t) = 0, \quad t > 0 \)

(4.3) \( u(x, 0) = 0, \quad 0 < x < 1, \)

(4.4) \( K u_x(1, t) = A, \quad t > 0. \)

For \( 1 < \alpha \leq 2 \), the initial condition:

(4.5) \( u_t(x, 0) = 0, \quad 0 < x < 1, \)

should be added. Let

(4.6) \( u(x, t) = U(x, t) + \Phi(x). \)

Equations (4.1)–(4.6) yield

\[
D_\alpha^t U = k \left[ \frac{\partial^2 U}{\partial x^2} + \Phi''(x) \right], \quad 0 < x < 1, \ t > 0, \ 0 < \alpha \leq 2,
\]

(4.7) \( U(0, t) + \Phi(0) = 0, \)

\( K [U_x(1, t) + \Phi'(1)] = A, \)

\( U(x, 0) + \Phi(x) = 0, \)

\( U_t(x, 0) = 0, \quad (for \ 1 < \alpha \leq 2). \)

Assume

(4.8) \( \Phi''(x) = 0 \) and \( \Phi(0) = 0, \ K\Phi'(1) = A. \)
(4.8) yields a boundary value problem for $U(x, t)$ that does have two-point boundary conditions leading to a Stürm-Liouville problem:

$$D_t^\alpha U = k \frac{\partial^2 U}{\partial x^2}, \quad (0 < x < 1, \ t > 0), \ 0 < \alpha \leq 2$$

(4.9)

$$U(0, t) = 0,$$

$$U_x(1, t) = 0,$$

$$U(x, 0) = -\Phi(x),$$

$$U_t(x, 0) = 0, \quad (\text{for } 1 < \alpha \leq 2).$$

(4.8) implies that

$$\Phi(x) = \frac{A}{K} x.$$

Let $U = X(x)T(t)$. Then

$$U(x, t) = \sum_{n=1}^{\infty} E_\alpha \left(-\frac{(2n-1)^2\pi^2}{2}k t^\alpha\right) \phi_n(x),$$

where $\phi_n(x) = \frac{(2n-1)\pi}{\sqrt{2}} \sin x$. The BVP given in (4.9) has been solved in Section 2, and has the following solution.

$$u(x, t) = \frac{A}{K} \left[ x + 8 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2\pi^2} E_\alpha \left(-\frac{(2n-1)^2\pi^2}{2}k t^\alpha\right) \sin \frac{(2n-1)^2\pi^2}{2} x \right].$$

5. Illustrative Examples

Example. Consider the following nonhomogeneous fractional diffusion-wave equation along with the boundary conditions given below:

$$D_t^\alpha u = \frac{\partial^2 u}{\partial x^2} + t, \quad 0 < \alpha \leq 2, \ t > 0,$$

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0,$$

$$u(x, 0) = f(x), \quad 0 < x < \pi,$$

$$u_t(x, 0) = 0, \quad 0 < x < \pi,$$

where

$$f(x) = \begin{cases} 
  x & 0 < x < \frac{\pi}{2}, \\
  \pi - x & \frac{\pi}{2} < x < \pi.
\end{cases}$$

(5.1)
In Figs. 1, 2, 3 and 4 we plot $u(x, t)$ for $0 \leq t \leq 1$ and various values of $\alpha$.

**Comment:** As the order of the time derivative changes from 0 to 2, the process changes from slow diffusion to classical diffusion to mixed diffusion wave behaviour.

### 6. Fractional Sturm-Liouville Problem

Consider the following BVP

\[(6.1) \quad [p(x) y^{(\beta)}(x)]' + \lambda q(x) y(x) = 0, \quad 0 < \beta < 1, \quad y(a) = y(b) = 0,\]

where $y^{(\beta)} = \frac{1}{\Gamma(1-\beta)} \int_a^x (x-t)^{-\beta} y'(t) dt$. Let $y_n$ and $y_m$ satisfy (6.1) for the values $\lambda = \lambda_n$ and $\lambda = \lambda_m$ respectively, i.e.

\[(6.2) \quad [p(x) y_n^{(\beta)}(x)]' + \lambda_n q(x) y_n(x) = 0, \quad y_n(a) = y_n(b) = 0,\]

\[(6.3) \quad [p(x) y_m^{(\beta)}(x)]' + \lambda_m q(x) y_m(x) = 0, \quad y_m(a) = y_m(b) = 0.\]

Multiplying (6.2) by $y_n$ and (6.3) by $y_m$ respectively, integrating and subtracting, we get

\[
(6.4) \quad \int_a^b \left\{ y_n(x) [p(x) y_m^{(\beta)}(x)]' - y_m(x) [p(x) y_n^{(\beta)}(x)]' \right\} dx =
\]

\[
- \int_a^b \left[ p(x) y_m^{(\beta)}(x) y'_n(x) - p(x) y_n^{(\beta)}(x) y'_m(x) \right] dx =
\]

\[
(\lambda_m - \lambda_n) \int_a^b q(x) y_n(x) y_m(x) dx.
\]
Note

\[ \left| \int_a^b p(x)y_n^{(\beta)}(x)y_m'(x)dx \right| = \left| \int_a^b \left[ \int_a^x \frac{(x-t)^{-\beta}}{\Gamma(1-\beta)} y_n'(t)dt \right] p(x)y_m'(x)dx \right| \]

\[ \leq \frac{M}{\Gamma(1-\beta)} \left| \int_a^b \left[ \int_a^x (x-t)^{-\beta} dt \right] y_m'(x)dx \right| \]

\[ \leq \frac{M}{\Gamma(1-\beta)} \left| \int_a^b \frac{(x-a)^{1-\beta}}{1-\beta} y_m'(x)dx \right| \]

\[ \leq \frac{M(b-a)^{1-\beta}}{(1-\beta)\Gamma(1-\beta)} \left| \int_a^b y_n'(x)dx \right| \]

\[ = 0, \quad \text{as } y_m(a) = y_m(b) = 0. \]

Similarly \[ \int_a^b p(x)y_m^{(\beta)}(x)y_n'(x)dx = 0. \] Hence \( (\lambda_m - \lambda_n) \int_a^b q(x)y_n(x)y_m(x)dx = 0. \) Thus the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

**References**


