The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 3, Issue 1, Article 15, pp. 1-8, 2006

GENERAL EXTENSION OF HARDY-HILBERT'S INEQUALITY (I)

W. T. SULAIMAN

Received 29 October, 2004; accepted 9 July, 2005; published 19 May, 2006.

waadsulaiman@hotmail.com

COLLEGE OF COMPUTER SCIENCE AND MATHEMATICS, UNIVERSITY OF MOSUL, IRAQ.

ABSTRACT. A generalization for Hardy-Hilbert's inequality that extends the recent results of Yang and Debnath [6], is given.

Key words and phrases: Hardy-Hilbert inequalities, Integral inequalities.

2000 *Mathematics Subject Classification.* 26D15.

ISSN (electronic): 1449-5910

The author would like to thank the referee for valuable suggestions that have been implemented in the final version of the manuscript.

1. INTRODUCTION

Let f and g be real functions, such that

$$0 < \int_0^\infty f^2(t)\,dt < \infty \text{ and } 0 < \int_0^\infty g^2(t)\,dt < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^\infty f^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(t)\,dt\right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant π is the best possible. A double series inequality associated with (1.1) is as follows :

If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}, \tag{1.2}$$

where the constant factor π is the best possible.

Hardy-Hilbert's inequalities (1.1) and (1.2) are important in mathematical analysis and its applications (cf. [1, Chap. 9]).

Recently Hu [2] and [3] gave two distinct improvements of (1.1) , and Gao [4] gave a strengthened version of (1.2). By introducing parameter $\lambda \in (0, 1]$ and estimating the weight function, Yang [5] gave a generalization of (1.1), as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^\infty t^{1-\lambda}f^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^\infty t^{1-\lambda}g^2(t)\,dt\right)^{\frac{1}{2}}, \tag{1.3}$$

where $B(p, q)$ is the beta function.

By introducing some other parameters, Yang and Debnath [6], established the following results:

Theorem 1.1. *If $f, g \geq 0$, $A, B > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - min\{p, q\}$, such that*

$$0 < \int_0^\infty t^{1-\lambda}f^p(t)\,dt < \infty, 0 < \int_0^\infty t^{1-\lambda}g^q(t)\,dt < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda}dxdy$$

$$< \frac{k_\lambda(p)}{A^{\phi_\lambda(p)}B^{\phi_\lambda(q)}}\left(\int_0^\infty x^{1-\lambda}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{1-\lambda}g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{1.4}$$

where

$$k_\lambda(p) = B(\phi_\lambda(p), \phi_\lambda(q)), \phi_\lambda(r) = \frac{r+\lambda-2}{r} \ (r = p, q),$$

and the constant factor $\frac{k_\lambda(p)}{A^{\phi_\lambda(p)}B^{\phi_\lambda(q)}}$ *is the best possible.*

Theorem 1.2. *If* $a_n, b_n > 0 (n \in N)$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - min\{p, q\} < \lambda \leq 2$, $A, B > 0$ *are such that* $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^\lambda} < \frac{k_\lambda(p)}{A^{\phi_\lambda(p)}B^{\phi_\lambda(q)}}\left(\sum_{n=1}^{\infty} n^{1-\lambda}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{1-\lambda}b_n^q\right)^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor $\frac{k_\lambda(p)}{A^{\phi_\lambda(p)}B^{\phi_\lambda(q)}}$ *is the best possible.*

The aim of this paper is to give further generalization for the inequality (1.4).

2. GENERALIZATION

We start with the following lemma

Lemma 2.1. *Let* F, G, $L(f, g)$, $M(f)$, $N(g)$*, be positive functions,* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *such that*

$$0 < \int_a^b M^p(f(t))F^p(t)\,dt < \infty, 0 < \int_c^d N^q(g(t))G^q(t)\,dt < \infty,$$

then the following inequalities

$$\begin{aligned}
&\int_a^b\int_c^d \frac{F(x)G(y)}{L(f(x),g(y))}dxdy \\
\leq\ & K\left(\int_a^b M^p(f(t))F^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_c^d N^q(g(t))G^q(t)\,dt\right)^{\frac{1}{q}},
\end{aligned} \tag{2.1}$$

where K is a constant, and

$$\int_c^d N^{-p}(g(y))\left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)^p dy \leq K^p\int_a^b M^p(f(t))F^p(t)\,dt, \tag{2.2}$$

are equivalent.

Proof. Suppose that (2.2) is satisfied, then we have

$$\begin{aligned}
&\int_a^b\int_c^d \frac{F(x)\,G(y)}{L(f(x),g(y))}dxdy\\
=\ &\int_c^d N(g(y))\,G(y)\left(N^{-1}(g(y))\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)dy\\
\le\ &\left(\int_c^d N^q(g(y))\,G^q(y)\,dy\right)^{\frac{1}{q}}\left(\int_c^d N^{-p}(g(y))\left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)^p dy\right)^{\frac{1}{p}}\\
\le\ &K^p\left(\int_a^b M^p(f(t))\,F^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_c^d N^q(g(t))\,G^q(t)\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, suppose that (2.1) is satisfied , then

$$\begin{aligned}
&\int_c^d N^{-p}(g(y))\left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)^p dy\\
=\ &\int_c^d \left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)N^{-p}(g(y))\left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)^{\frac{p}{q}}dy\\
\le\ &K\left(\int_a^b M^p(f(x))\,F^p(x)\,dx\right)^{\frac{1}{p}}\\
\times\ &\left(\int_c^d N^q(g(y))\,N^{-pq}(g(y))\left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)^p dy\right)^{\frac{1}{q}}\\
=\ &K\left(\int_a^b M^p(f(x))\,F^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_c^d N^{-p}(g(y))\left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)^p dy\right)^{\frac{1}{q}},
\end{aligned}$$

which implies

$$\int_c^d N^{-p}(g(y))\left(\int_a^b \frac{F(x)}{L(f(x),g(y))}dx\right)^p dy \le K^p\int_a^b M^p(f(t))\,F^p(t)\,dt.$$

∎

The following is our main result:

Theorem 2.2. *Let* F, G, f, g, f', g' *be positive functions,* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < c < \lambda$ $(c = a+1, b+1)$ *and*

$$\begin{aligned} 0 &< \int_0^\infty \frac{[f(t)]^{\frac{(aq-bp)}{q}+(1-\lambda)} F^p(t)}{[f'(t)]^{\frac{p}{q}}} dt < \infty, \\ 0 &< \int_0^\infty \frac{[g(t)]^{\frac{(bp-aq)}{p}+(1-\lambda)} G^q(t)}{[g'(t)]^{\frac{q}{p}}} dt < \infty, \end{aligned}$$

then we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{F(x)G(y)}{(f(x)+g(y))^\lambda} dxdy \\ &\le B^{\frac{1}{p}}(a+1, \lambda-a-1) B^{\frac{1}{q}}(b+1, \lambda-b-1) \\ &\times \left(\int_0^\infty \frac{[f(t)]^{\frac{(aq-bp)}{q}+(1-\lambda)} F^p(t)}{[f'(t)]^{\frac{p}{q}}} dt \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{[g(t)]^{\frac{(bp-aq)}{p}+(1-\lambda)} G^q(t)}{[g'(t)]^{\frac{q}{p}}} \right)^{\frac{1}{q}}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\int_0^\infty [g(y)]^{p\left(\frac{a}{p}-\frac{b}{q}\right)+(\lambda-1)} \left(\int_0^\infty \frac{F(x)}{(f(x)+g(y))^\lambda} dx \right)^p dy \\ &\le B(a+1, \lambda-a-1) B^{\frac{p}{q}}(b+1, \lambda-b-1) \\ &\times \int_0^\infty \frac{[f(t)]^{\frac{(aq-bp)}{q}+(1-\lambda)} F^p(t)}{[f'(t)]^{\frac{p}{q}}} dt. \end{aligned} \tag{2.4}$$

The inequalities (2.3) and (2.4) are equivalent.

Proof. Observe that

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{F(x)G(y)}{(f(x)+g(y))^\lambda} dxdy \\ &= \int_0^\infty \int_0^\infty \frac{F(x) \frac{[g(y)]^{\frac{a}{p}} [g'(y)]^{\frac{1}{p}}}{[f(x)]^{\frac{b}{q}} [f'(x)]^{\frac{1}{q}}}}{(f(x)+g(y))^{\frac{\lambda}{p}}} \times \frac{G(y) \frac{[f(x)]^{\frac{b}{q}} [f'(x)]^{\frac{1}{q}}}{[g(y)]^{\frac{a}{p}} [g'(y)]^{\frac{1}{p}}}}{(f(x)+g(y))^{\frac{\lambda}{q}}} dxdy \\ &\le \left(\int_0^\infty \int_0^\infty \frac{F^p(x) \frac{[g(y)]^a}{[f(x)]^{\frac{bp}{q}}} \frac{g'(y)}{[f'(x)]^{\frac{p}{q}}}}{(f(x)+g(y))^\lambda} dxdy \right)^{\frac{1}{p}} \times \left(\int_0^\infty \int_0^\infty \frac{G^q(y) \frac{[f(x)]^b}{[g(y)]^{\frac{aq}{p}}} \frac{f'(x)}{[g'(y)]^{\frac{q}{p}}}}{(f(x)+g(y))^\lambda} dxdy \right)^{\frac{1}{q}} \\ &= M^{\frac{1}{p}} \times N^{\frac{1}{q}}, \end{aligned}$$

say. Then

$$M = \int_0^\infty \frac{[f(x)]^{\frac{aq-bp}{q}+(1-\lambda)} F^p(x)}{[f'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{\left[\frac{g(y)}{f(x)}\right]^a \frac{g'(y)}{f(x)}}{\left(1+\frac{g(y)}{f(x)}\right)^\lambda} dy$$

$$= \int_0^\infty \frac{[f(x)]^{\frac{aq-bp}{q}+(1-\lambda)} F^p(x)}{[f'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{u^a}{(1+u)^\lambda} du$$

$$= B(a+1, \lambda-a-1) \int_0^\infty \frac{[f(x)]^{\frac{aq-bp}{q}+(1-\lambda)} F^p(x)}{[f'(x)]^{\frac{p}{q}}} dx.$$

Similarly , we can show that

$$N = B(b+1, \lambda-b-1) \int_0^\infty \frac{[g(y)]^{\frac{bp-aq}{p}+(1-\lambda)} G^q(y)}{[g'(y)]^{\frac{q}{p}}} dy.$$

The equivalence of (2.3) and (2.4) follows from Lemma 2.1, by putting:

$$M(f(t)) = \frac{[f(t)]^{\left(\frac{a}{p}-\frac{b}{q}\right)+(1-\lambda)}}{[f'(t)]^{\frac{1}{q}}},$$

$$N(g(t)) = \frac{[g(t)]^{\left(\frac{b}{q}-\frac{a}{p}\right)+(1-\lambda)}}{[g'(t)]^{\frac{1}{p}}}.$$

∎

Corollary 2.3. *Theorem 2.2 implies the inequality (1.4), in which the constant coefficient is the best possible.*

Proof. Follows by putting $a = p$, $b = q$, $f(x) = Ax$, $g(y) = By$.

It remains to show that $B^{\frac{1}{p}}(p+1, \lambda-p-1)\, B^{\frac{1}{q}}(q+1, \lambda-q-1)$ has the value $B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ as best possible.

For this purpose, let us consider $a + b = k$. Then

$$\begin{aligned} & B^{\frac{1}{p}}(a, \lambda-a)\, B^{\frac{1}{q}}(b, \lambda-b) \\ = & \frac{1}{\Gamma\lambda} \Gamma^{\frac{1}{p}} a \Gamma^{\frac{1}{q}} b \Gamma^{\frac{1}{p}}(\lambda-a)\, \Gamma^{\frac{1}{q}}(\lambda-b) \\ \geq & \frac{1}{\Gamma\lambda} \Gamma\left(\frac{a}{p}+\frac{b}{q}\right) \Gamma\left(\frac{\lambda-a}{p}+\frac{\lambda-b}{q}\right) \quad \left(\log\Gamma \text{ being convex, } \frac{1}{p}+\frac{1}{q}=1\right). \end{aligned}$$

Now, let

$$f(a) = \frac{a}{p}+\frac{b}{q} = \frac{a}{p}+\frac{k-b}{q}$$

$$f'(a) = 0 = \frac{1}{p}-\frac{1}{q} \text{ which implies } p = q = 2.$$

Therefore, $\min f(a) = \frac{k}{2}$ which is realised for $a = b$. This implies

$$\min B^{\frac{1}{p}}(a, \lambda-a)\, B^{\frac{1}{q}}(b, \lambda-b) = B(a, \lambda-a).$$

In order to find $\min B^{\frac{1}{p}}(a+1,\lambda-a-1)\,B^{\frac{1}{q}}(b+1,\lambda-b-1)$, we can take $a+1=\lambda-b-1$, that is $a+b=\lambda-2$. If $a=p, b=q$, then we have

$$p+q=\lambda-2\Rightarrow pq=\lambda-2\Rightarrow a=\frac{\lambda-2}{q}\Rightarrow a+1=\frac{\lambda+q-2}{q},$$

therefore

$$\min B^{\frac{1}{p}}(a+1,\lambda-a-1)\,B^{\frac{1}{q}}(b+1,\lambda-b-1)=B\left(\frac{\lambda+p-2}{p},\frac{\lambda+q-2}{q}\right).$$

∎

3. SERIES ANALOGUES

In what follows, some results on the associated double series forms are also stated.

Theorem 3.1. *Let* F, G, f, g, f', g' *be positive functions,* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<c<\lambda$ $(c=a+1,b+1)$ *and*

$$\begin{aligned}0 &< \sum_{m=1}^{\infty}\frac{[f(m)]^{\frac{aq-bp}{q}+(1-\lambda)}F^p(m)}{[f'(m)]^{\frac{p}{q}}}<\infty,\\ 0 &< \sum_{n=1}^{\infty}\frac{[g(n)]^{\frac{bp-aq}{p}+(1-\lambda)}G^q(n)}{[g'(n)]^{\frac{q}{p}}}<\infty.\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{F(m)G(n)}{(f(m)+g(n))^{\lambda}}\\ \leq\ & B^{\frac{1}{p}}(a+1,\lambda-a-1)\,B^{\frac{1}{q}}(b+1,\lambda-b-1)\\ \times\ &\left(\sum_{m=1}^{\infty}\frac{[f(m)]^{\frac{aq-bp}{q}+(1-\lambda)}F^p(m)}{[f'(m)]^{\frac{p}{q}}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{[g(n)]^{\frac{bp-aq}{p}+(1-\lambda)}G^q(n)}{[g'(n)]^{\frac{q}{p}}}\right)^{\frac{1}{q}},\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\sum_{n=1}^{\infty}[g(n)]^{p\left(\frac{a}{p}-\frac{b}{q}\right)+(\lambda-1)}\left(\sum_{m=1}^{\infty}\frac{F(m)}{(f(m)+g(n))^{\lambda}}\right)^{p}\\ \leq\ & B(a+1,\lambda-a-1)\,B^{\frac{p}{q}}(b+1,\lambda-b-1)\\ \times\ &\sum_{m=1}^{\infty}\frac{[f(m)]^{\frac{aq-bp}{q}+(1-\lambda)}F^p(m)}{[f'(m)]^{\frac{p}{q}}}.\end{aligned}\tag{3.2}$$

Inequalities (3.1) and (3.2) are equivalent.

Proof. On replacing the integral by the sum and following the same steps done in the proof of Theorem 2.2, we can state that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{F(m)G(n)}{(f(m)+g(n))^{\lambda}}\leq L^{\frac{1}{p}}\times H^{\frac{1}{q}},$$

where

$$\begin{aligned} L &= \sum_{m=1}^{\infty} \frac{[f(m)]^{\frac{aq-bp}{q}+(1-\lambda)} F^p(m)}{[f'(m)]^{\frac{p}{q}}} \sum_{n=1}^{\infty} \frac{\left[\frac{g(n)}{f(m)}\right]^a \frac{g'(n)}{f(m)}}{\left(1+\frac{g(n)}{f(m)}\right)^{\lambda}} \\ &\le \sum_{m=1}^{\infty} \frac{[f(m)]^{\frac{aq-bp}{q}+(1-\lambda)} F^p(m)}{[f'(m)]^{\frac{p}{q}}} \int_0^{\infty} \frac{\left[\frac{g(y)}{f(m)}\right]^a \frac{g'(y)}{f(m)}}{\left(1+\frac{g(y)}{f(m)}\right)^{\lambda}} dy \\ &= \sum_{m=1}^{\infty} \frac{[f(m)]^{\frac{aq-bp}{q}+(1-\lambda)} F^p(m)}{[f'(m)]^{\frac{p}{q}}} \int_0^{\infty} \frac{u^a}{(1+u)^{\lambda}} du \\ &= B(a+1, \lambda-a-1) \sum_{m=1}^{\infty} \frac{[f(m)]^{\frac{aq-bp}{q}+(1-\lambda)} F^p(m)}{[f'(m)]^{\frac{p}{q}}}. \end{aligned}$$

Similarly,

$$H \le B(b+1, \lambda-b-1) \sum_{n=1}^{\infty} \frac{[g(n)]^{\frac{bp-aq}{p}+(1-\lambda)} G^q(n)}{[g'(n)]^{\frac{q}{p}}}.$$

The equivalence of (3.1) and (3.2) follows from Lemma 2.1, by replacing the integral with the sum, and putting

$$\begin{aligned} L(f(m)) &= \frac{[f(m)]^{\left(\frac{a}{p}-\frac{b}{q}\right)+(1-\lambda)}}{[f'(m)]^{\frac{1}{q}}}, \\ H(g(n)) &= \frac{[g(n)]^{\left(\frac{b}{q}-\frac{a}{p}\right)+(1-\lambda)}}{[g'(n)]^{\frac{1}{p}}}. \end{aligned}$$

∎

Corollary 3.2. *Theorem 3.1 implies the inequality (1.5), in which the constant coefficient is the best possible.*

REFERENCES

[1] G. H. HARDY, J. E. LITTLEWOOD, and G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

[2] H. KE, On Hilbert's inequality, *Chinese Ann. of Math.*, **13B**, No.1 (1992), 35-39.

[3] M. GAO, TAN LI, and L. DEBNATH, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.*, **229** (1999), 682-689.

[4] M. GAO, A note on Hilbert double series theorem, *Hunan Ann. Math.*, **12**, Nos. 1-2 (1992), 142-147.

[5] B. YANG, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220** (1998), 778-785.

[6] B. YANG and L. DEBNATH, On the extended Hardy-Hilbert's inequality, *J. Math. Anal. Appl.*, **272** (2002), 187-199.