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A NEW HARDY-HILBERT'S TYPE INEQUALITY FOR DOUBLE SERIES AND ITS APPLICATIONS

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ABSTRACT. In this paper, it is shown that a new Hardy-Hilbert's type inequality for double series can be established by introducing a parameter λ $(1-\frac{q}{p} < \lambda \leq 2,\ \frac{1}{p}+\frac{1}{q}=1,\ p \geq q > 1)$ and the weight function of the form $\omega_r(\lambda,\ n) = (n!)^{2-\lambda-r}(\ln(en)-c)^{1-r}$, where c is Euler constant and $r = p,\ q$. And the coefficient $B\left(\lambda - \left(1-\frac{2-\lambda}{p}\right),\ 1-\frac{2-\lambda}{p}\right)$ is proved to be the best possible. And as mathematics esthetics, several important constants $\pi,\ e$ and c appear simultaneously in the coefficient and the weight function when $\lambda = 1$. In particular, for case $p = 2$, some new Hilbert's type inequalities are obtained. As applications, some extensions of Hardy-Littlewood's inequality are given.

Key words and phrases: Hardy-Hilbert's type inequality, Double series, Weight function, Beta function, Gamma function, Psi function.

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1. INTRODUCTION

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. If $\sum\limits_{n=1}^{\infty} a_n^p < +\infty$ and $\sum\limits_{n=1}^{\infty} b_n^q < +\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \le \frac{\pi}{\sin\ \pi/p}\left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.1}$$

where the coefficients $\frac{\pi}{\sin \pi/p}$ contained in (1.1) is the best possible. And the equality in (1.1) holds if and only if $\{a_n\}$, or $\{b_n\}$ is a zero-sequence (see [1]). This is the famous Hardy-Hilbert theorem for double series. Recently, it has been studied in some papers, and some sharper results were obtained (such as ([2]–[4]) etc.). Lately, the inequality (1.1) is extended in some papers (such as ([5]-[7]) etc.). Now let us consider the following inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m!+n!)^{\lambda}} \le k(\lambda)\left(\sum_{n=1}^{\infty}\omega_p(\lambda,\ n)a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\omega_q(\lambda,\ n)b_n^q\right)^{1/q}, \tag{1.2}$$

where $1-\frac{q}{p}<\lambda\le 2\ (p\ge q>1)$. The purpose of the present paper is to decide the weight function $\omega_r(\lambda,\ n)\ \ (r=p,\ q)$ of (1.2) and by introducing a parameter s to find the best possible value of $k(\lambda)$ which the inequality (1.2) keeps valid. As applications, we will give some new extensions on Hardy-Littlewood's theorem. For convenience, the beta function $B\left(\lambda-\left(1-\frac{2-\lambda}{p}\right),\ 1-\frac{2-\lambda}{p}\right)$ is denoted by B^*. Throughout the paper we will frequently use these notations.

2. SOME LEMMAS

Lemma 2.1. *Let $\Gamma(x)$ be gamma function. Then $\Gamma'(x)>0$ when $x\ge 2$.*

Proof. According to the paper [8], define psi function by

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}. \tag{2.1}$$

This function can be written in form

$$\psi(x)=-c-\sum_{k=0}^{\infty}\left(\frac{1}{x+k}-\frac{1}{k+1}\right) \tag{2.2}$$

where c is Euler constant.

In particular,

$$\psi(1)=-c \text{ and } \psi(n)=\sum_{k=1}^{n-1}\frac{1}{k}-c \quad (2,\ 3,\ \dots) \tag{2.3}$$

It is obvious that $\psi'(x)>0,\quad x\in(0,\ +\infty)$. Hence $\psi(x)$ is increasing. It follows from (2.3) and (2.1) that $\Gamma'(1)<0$ and $\Gamma'(2)>0$. As a result, there exists $x_0\in(1,\ 2)$, such that $\Gamma'(x_0)=0$. Clearly $\Gamma'(x)>0$ when $x\ge 2$. ∎

Lemma 2.2. *Let $r>1$, $0\le rs<1$ and $\lambda>1-rs$. Then*

$$\int_0^{\infty}\frac{1}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{rs}dt=B(\lambda-(1-rs),\ 1-rs) \tag{2.4}$$

where $B(p,q)$ is the beta function.

Proof. According to the definition of the beta function we have

$$B(p,q) = \int_0^1 u^{p-1} (1-u)^{q-1} \, du.$$

Put $t = 1/u - 1$, then

$$\int_0^\infty \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{rs} dt = \int_0^1 u^{\lambda - 2 + rs} (1-u)^{-rs} \, du.$$

This shows that the equality (2.4) is true. ∎

Lemma 2.3. *Let* $0 \le ps < 1$ *and* $1 - qs < \lambda \le 2$*. Define a function* Φ *by*

$$\Phi(s) = \{B(\lambda - (1-ps), \, 1-ps)\}^{1/p} \{B(\lambda - (1-qs), \, 1-qs)\}^{1/q} \tag{2.5}$$

where $B(m,n)$ *is beta function. Then* $\Phi(s)$ *attains the minimum* B^**, when* $s = \frac{2-\lambda}{pq}$*.*

Proof. Based on the relation $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where $\Gamma(x)$ is the gamma function, we can write (2.5) as

$$\Phi(s) = \tfrac{1}{\Gamma(\lambda)} \left(I_p^{1/p} I_q^{1/q} \right)$$

where $\quad I_r = \Gamma(1-rs)\,\Gamma(\lambda - (1-rs)), \quad r = p, q.$

Taking the derivative of $\Phi(s)$ we have

$$\Phi'(s) = \Phi(s)\,\Omega(s),$$

where $\Omega(s) = -\psi(1-ps) + \psi(\lambda - (1-ps)) - \psi(1-qs) + \psi(\lambda - (1-qs))$, here $\psi(x)$ is the psi function defined by (2.1). We choose thus s such that $1 - ps = \lambda - (1-qs)$, so that $1 - qs = \lambda - (1-ps)$, hence $s = \frac{2-\lambda}{p+q}$. Since that $\frac{1}{p} + \frac{1}{q} = 1$, it follows that $s = \frac{2-\lambda}{pq}$. We therefore have $\Omega\left(\frac{2-\lambda}{pq}\right) = 0$. i.e. $\Phi'\left(\frac{2-\lambda}{pq}\right) = 0$. It is easy to deduce from (2.2) that $\psi'(x) > 0$, it follows that $\Omega'(s) > 0$, hence $\Omega(s)$ is strictly increasing. Owing to the fact that $\Omega\left(\frac{2-\lambda}{pq}\right) = 0, \Omega(s) > 0$ when $s > \frac{2-\lambda}{pq}$. This shows that $\Phi'(s) > 0$. Similarly, we have $\Phi'(s) < 0$ when $s < \frac{2-\lambda}{pq}$. Consequently, the minimum of $\Phi(s)$ is

$$\Phi\left(\tfrac{2-\lambda}{pq}\right) = \left(B\left(\lambda - \left(1 - \tfrac{2-\lambda}{q}\right), 1 - \tfrac{2-\lambda}{q}\right)\right)^{1/p} \left(B\left(\lambda - \left(1 - \tfrac{2-\lambda}{p}\right), 1 - \tfrac{2-\lambda}{p}\right)\right)^{1/q}.$$

Since $1 - \frac{2-\lambda}{q} = \lambda - \left(1 - \frac{2-\lambda}{p}\right), \quad 1 - \frac{2-\lambda}{p} = \lambda - \left(1 - \frac{2-\lambda}{q}\right)$ and $B(m,n) = B(n,m)$, we have the relation:

$$B\left(\lambda - \left(1 - \tfrac{2-\lambda}{q}\right), \; 1 - \tfrac{2-\lambda}{q}\right) = B\left(\lambda - \left(1 - \tfrac{2-\lambda}{p}\right), \; 1 - \tfrac{2-\lambda}{p}\right).$$

We therefore obtain $\Phi\left(\frac{2-\lambda}{pq}\right) = B^*$. The lemma is proved. ∎

3. MAIN RESULTS

In the section, we will apply the above lemmas to build some new inequalities.

In order to ensure that $\Gamma'(x) > 0$, we consider only $x \in [2, \; +\infty)$ (see 2.1), bellow.

Theorem 3.1. *Let* $a_n,\ \ b_n\ \ \geq\ \ 0\ (n=2,\ 3,\ \cdots)$. *If* $\sum_{n=2}^{\infty}(n!)^{2-\lambda-p}(\ln(en)-c)^{1-p}a_n^p<\ +\infty$ *and* $\sum_{n=2}^{\infty}(n!)^{2-\lambda-q}(\ln(en)-c)^{1-q}b_n^q<\ +\infty$*, where* c *is Euler constant, then*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(m!\ +\ n!)^{\lambda}}\ \leq\ B^*\left\{\sum_{n=2}^{\infty}(n!)^{2-\lambda-p}(\ln(en)\ -c)^{1-p}a_n^p\right\}^{1/p}\times\left\{\sum_{n=2}^{\infty}(n!)^{2-\lambda-q}(\ln(en)-c)^{1-q}b_n^q\right\}^{1/q}, \tag{3.1}$$

where the coefficient B^* *is the best possible. And the equality in (3.1) holds if and only if* $\{a_n\}$*, or* $\{b_n\}$*is a zero-sequence.*

Proof. Let's introduce a parameter s such that $0\ \leq\ ps\ <\ 1$. Consider gamma function $\Gamma(x), x\in[2,+\infty)$, and notice that $\Gamma(n+1)=n!$. For convenience, we denote that

$$a_m=A_m\left(\Gamma'(m+1)\right)^{1/q}\ \text{and}\ b_n=B_n\left(\Gamma'(n+1)\right)^{1/p}.$$

And then define two functions:

$$\alpha=\frac{A_m\left\{\Gamma'(n+1)\right\}^{1/p}}{(m!+\ n!)^{\lambda/p}}\left(\frac{m!}{n!}\right)^{s}\ \text{and}\ \beta=\frac{B_n\left\{\Gamma'(m+1)\right\}^{1/q}}{(m!+\ n!)^{\lambda/q}}\left(\frac{n!}{m!}\right)^{1/s}. \tag{3.2}$$

Apply Hölder's inequality to estimate the right-hand side of (3.1) as follows:

$$\begin{aligned}
&\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(m!\ +\ n!)^{\lambda}} \\
&= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{A_m\left(\Gamma'(n+1)\right)^{1/p}}{(m!\ +\ n!)^{\lambda/p}}\left(\frac{m!}{n!}\right)^{s}\frac{B_n\left(\Gamma'(m+1)\right)^{1/q}}{(m!\ +\ n!)^{\lambda/q}}\left(\frac{n!}{m!}\right)^{s} \\
&= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\alpha\beta\ \leq\ \left\{\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\alpha^p\right\}^{1/p}\left\{\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\beta^q\right\}^{1/q} \\
&= \left(\sum_{n=2}^{\infty}\omega_p(\lambda,n)A_n^p\right)^{1/p}\left(\sum_{n=2}^{\infty}\omega_q(\lambda,n)B_n^q\right)^{1/q},
\end{aligned} \tag{3.3}$$

where $\omega_p(\lambda,n)=\sum_{m=2}^{\infty}\frac{\Gamma'(m+1)}{(n!+m!)^{\lambda}}\left(\frac{n!}{m!}\right)^{ps}$ and $\omega_q(\lambda,n)=\sum_{m=2}^{\infty}\frac{\Gamma'(m+1)}{(m!+n!)^{\lambda}}\left(\frac{n!}{m!}\right)^{qs}$.

By Lemma 2.2, we have

$$\begin{aligned}
(3.4)\qquad & \omega_p(\lambda, n) \\
&= \sum_{m=2}^{\infty} \frac{\Gamma'(m+1)}{(\Gamma(n+1)+\Gamma(m+1))^{\lambda}} \left(\frac{\Gamma(n+1)}{\Gamma(m+1)}\right)^{ps} \\
&\le \int_1^{\infty} \frac{\Gamma'(x+1)}{(\Gamma(n+1)+\Gamma(x+1))^{\lambda}} \left(\frac{\Gamma(n+1)}{\Gamma(x+1)}\right)^{ps} dx \\
&= \int_1^{\infty} \frac{(\Gamma(n+1))^{-\lambda}(\Gamma'(x+1))}{(1+\Gamma(x+1)/\Gamma(n+1))^{\lambda}} \left(\frac{\Gamma(n+1)}{\Gamma(x+1)}\right)^{ps} dx \\
&= \int_{1/\Gamma(n+1)}^{\infty} \frac{(\Gamma(n+1))^{1-\lambda}}{(1+t)^{\lambda}} \left(\frac{1}{t}\right)^{ps} dt \\
&= (\Gamma(n+1))^{1-\lambda} \left\{ \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} \left(\frac{1}{t}\right)^{ps} dt - \int_0^{1/\Gamma(n+1)} \frac{1}{(1+t)^{\lambda}} \left(\frac{1}{t}\right)^{ps} \right\} \\
&\le (\Gamma(n+1))^{1-\lambda} B(\lambda-(1-ps), 1-ps).
\end{aligned}$$

Similarly,

$$(3.5)\qquad \omega_q(\lambda, n) \le (\Gamma(n+1))^{1-\lambda} B(\lambda-(1-qs), 1-qs).$$

It follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned}
(3.6)\qquad & \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(m!+n!)^{\lambda}} \\
&\le \Phi(s) \left(\sum_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} A_n^p\right)^{1/p} \left(\sum_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} B_n^q\right)^{1/q}
\end{aligned}$$

where $\Phi(s)$ is defined by (2.2).

It follows from Lemma 2.3 that the minimum of $\Phi(s)$ is B^* when $s = \frac{2-\lambda}{pq}$, where λ satisfies the constraint $1 - \frac{q}{p} < \lambda \le 2$. Notice that $A_n^p = (\Gamma'(n+1))^{1-p} a_n^p$ and $B_n^q = (\Gamma'(n+1))^{1-q} b_n^q$. Therefore we obtain from (3.6) that

$$\begin{aligned}
(3.7)\qquad \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(m!+n!)^{\lambda}} &\le B^* \left(\sum_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} (\Gamma'(n+1))^{1-p} a_n^p\right)^{1/p} \\
&\quad \times \left(\sum_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} (\Gamma'(n+1))^{1-q} b_n^q\right)^{1/q}.
\end{aligned}$$

And it is obvious that the equality in (3.7) if and only if $\{a_n\}$, or $\{b_n\}$is a zero-sequence.

It remains to show that the constant factor B^* in (3.7) is the best possible.

Let $\tilde{a}_m = (\Gamma(m+1))^{-(2-\lambda+\varepsilon)/p} (\Gamma'(m+1))$ and $\tilde{b}_n = (\Gamma(n+1))^{-(2-\lambda+\varepsilon)/q} (\Gamma'(n+1))$. Assume that $0 < \varepsilon < (\lambda-1) + \frac{q}{2p}$, since the functions $\Gamma(x)$ is strictly increasing in $[2, +\infty)$, we have

$$\begin{aligned}\frac{2}{\varepsilon} &= \int_2^{+\infty} (\Gamma(x+1))^{-1-\varepsilon} d\Gamma(x+1) < \sum_{m=2}^{\infty} (\Gamma(m+1))^{-1-\varepsilon} (\Gamma'(m+1)) \\ &= \sum_{m=2}^{\infty} (\Gamma(m+1))^{1-\lambda} (\Gamma'(m+1))^{1-p} \tilde{a}_m^p \\ &= (\Gamma(3))^{-1-\varepsilon} \Gamma'(3) + \sum_{m=3}^{\infty} (\Gamma(m+1))^{-1-\varepsilon} (\Gamma'(m+1)) \\ &< (\Gamma(3))^{-1-\varepsilon} \Gamma'(3) + \int_2^{+\infty} (\Gamma(x))^{-1-\varepsilon} d\Gamma(x) = 2^{-1-\varepsilon}(3-2c) + \frac{2}{\varepsilon},\end{aligned}$$

where c is Euler constant.

Similarly, we have

$$\frac{2}{\varepsilon} < \sum_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} (\Gamma'(n+1))^{1-q} \tilde{b}_n^q < 2^{-1-\varepsilon}(3-2c) + \frac{2}{\varepsilon}.$$

Hence $\sum\limits_{m=2}^{\infty} (\Gamma(m+1))^{1-\lambda} (\Gamma'(m+1))^{1-p} \tilde{a}_m^p = \frac{2}{\varepsilon} + O(1). \quad (\varepsilon \to 0).$

Similarly, $\sum\limits_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} (\Gamma'(n+1))^{1-q} \tilde{b}_n^q = \frac{2}{\varepsilon} + O(1). \quad (\varepsilon \to 0).$

If B^* is not best possible, then there exists $k > 0$ and k less than B^* such that

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(m!+n!)^{\lambda}} &< k \left(\sum_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} (\Gamma'(n+1))^{1-p} \tilde{a}_n^p \right)^{1/p} \\ &\quad \times \left(\sum_{n=2}^{\infty} (\Gamma(n+1))^{1-\lambda} (\Gamma'(n+1))^{1-q} \tilde{b}_n^q \right)^{1/q} \\ &= \frac{2}{\varepsilon}(k + \circ(1)) \quad (\varepsilon \to 0).\end{aligned} \tag{3.8}$$

On the other hand, let $u = \Gamma(x+1)$ and $v = \Gamma(y+1)$, then we have

$$\begin{aligned}&\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(m!+n!)^{\lambda}} \\ &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{(\Gamma(m+1))^{-(2-\lambda+\varepsilon)/p} (\Gamma'(m+1)) (\Gamma(n+1))^{-(2-\lambda+\varepsilon)/q} (\Gamma'(n+1))}{(m!+n!)^{\lambda}} \\ &> \int_2^{\infty}\int_2^{\infty} \frac{u^{-(2-\lambda+\varepsilon)/p} v^{-(2-\lambda+\varepsilon)/q}}{(u+v)^{\lambda}} du dv \\ &= \int_2^{\infty} \left\{ \int_2^{\infty} \frac{v^{-(2-\lambda+\varepsilon)/q}}{(u+v)^{\lambda}} dv \right\} u^{-(2-\lambda+\varepsilon)/p} du\end{aligned}$$

$$= \int_2^\infty \left\{ \int_{2/\Gamma(x+1)}^\infty \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt \right\} u^{-1-\varepsilon} du$$
$$= \frac{2}{\varepsilon} \int_{2/\Gamma(x+1)}^\infty \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt.$$

If the lower limit $\frac{2}{\Gamma(x+1)}$ of the integral is replaced by zero, then the resulting error is smaller than $(\Gamma(x+1))^{-\alpha}/\alpha$, where α is positive and it is independent of ε. In fact, we have

$$\int_0^{2/\Gamma(x+1)} \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt < \int_0^{2/\Gamma(x+1)} t^{-(2-\lambda+\varepsilon)/q} dt = \frac{(\Gamma(x+1))^{-\beta}}{\beta},$$

where $\beta = 1 - (2-\lambda+\varepsilon)/q$. If $0 < \varepsilon < (\lambda - 1) + \frac{q}{2p}$, then we may take α such that

$$\alpha = 1 - \frac{(2-\lambda) + ((\lambda-1) + q/2p)}{q} = \frac{1}{2p}.$$

Consequently, we get

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{\tilde{a}_m \tilde{b}_n}{(m! + n!)^\lambda} > \frac{1}{\varepsilon} \{B^* + \circ(1)\} \quad (\varepsilon \to 0). \tag{3.9}$$

Clearly, when ε is small enough, the inequality (3.8) is in contradiction with (3.9). Hence the coefficient B^* is the best possible value of which the inequality (3.7) keeps valid.

At last, let us consider the weight function of (3.7). Notice that

$$\sum_{k=1}^n \frac{1}{k} = \ln n + c + \varepsilon_n,$$

where c is Euler constant and $\varepsilon_n \downarrow 0 \quad (n \to \infty)$. It is known from (2.1) and (2.3) that

$$\begin{aligned} \omega_r(\lambda, n) &= (\Gamma(n+1))^{1-\lambda} (\Gamma'(n+1))^{1-r} \\ &= (n!)^{1-\lambda} \left\{ n! \left(\sum_{k=1}^n \tfrac{1}{k} - c \right) \right\}^{1-r} \\ &= (n!)^{2-\lambda-r} \{\ln n + \varepsilon_n\}^{1-r} \le (n!)^{2-\lambda-r} \{\ln n + \varepsilon_1\}^{1-r} \\ &= (n!)^{2-\lambda-r} \{\ln(en) - c\}^{1-r}. \qquad r = p,\ q. \end{aligned} \tag{3.10}$$

It shows from (3.7) and (3.10) that the inequality (3.1) is valid. The proof of the Theorem ?3.1? is completed. ∎

We can attain some interesting results based on Theorem 3.1.

In particular, for $\lambda = 1$, B^* is reduced to $\frac{\pi}{\sin \pi/p}$. Hence other extension on (1.1) is attained.

Corollary 3.2. *With the assumptions as Theorem 3.1, if* $\sum_{n=2}^{\infty}\{n!(\ln(en)-c)\}^{1-p}a_n^p < +\infty$ *and* $\sum_{n=2}^{\infty}\{n!(\ln(en)-c)\}^{1-q}b_n^q < +\infty$*, where* c *is Euler constant, then*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{m!+n!} \le \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=2}^{\infty}\{n!(\ln(en)-c)\}^{1-p}a_n^p\right\}^{1/p} \times \left\{\sum_{n=2}^{\infty}\{n!(\ln(en)-c)\}^{1-q}b_n^q\right\}^{1/q}, \tag{3.11}$$

where the coefficient $\frac{\pi}{\sin(\pi/p)}$ *is the best possible. And the equality in (3.11) holds if and only if* $\{a_n\}$*, or* $\{b_n\}$ *is a zero-sequence.*

When $p = 2$*, based on Theorem 3.1, Hilbert's double series type inequality with parameter* λ *can be built.*

Corollary 3.3. *If* $\sum_{n=2}^{\infty}\frac{a_n^2}{(n!)^\lambda(\ln(en)-c)} < +\infty$ *and* $\sum_{n=2}^{\infty}\frac{b_n^2}{(n!)^\lambda(\ln(en)-c)} < +\infty$*, where* c *is Euler constant, then*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(m!+n!)^\lambda} \le B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=2}^{\infty}\frac{a_n^2}{(n!)^\lambda(\ln(en)-c)}\right\}^{1/2}\left\{\sum_{n=2}^{\infty}\frac{b_n^2}{(n!)^\lambda(\ln(en)-c)}\right\}^{1/2}, \tag{3.12}$$

where the coefficient $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ *is beta function and it is the best possible value which the inequality (3.12) keeps valid. And the equality in (3.12) holds if and only if* $\{a_n\}$*, or* $\{b_n\}$ *is a zero-sequence.*

In particular, for $\lambda = 1$ *we obtain from (3.12) the following result immediately.*

Corollary 3.4. *If* $\sum_{n=2}^{\infty}\frac{a_n^2}{n!(\ln(en)-c)} < +\infty$ *and* $\sum_{n=2}^{\infty}\frac{b_n^2}{n!(\ln(en)-c)} < +\infty$*, where* c *is Euler constant, then*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{m!+n!} \le \pi\left\{\sum_{n=2}^{\infty}\frac{a_n^2}{n!(\ln(en)-c)}\right\}^{1/2}\left\{\sum_{n=2}^{\infty}\frac{b_n^2}{n!(\ln(en)-c)}\right\}^{1/2} \tag{3.13}$$

where the coefficient π *is best possible. And the equality in (3.13) holds if and only if* $\{a_n\}$*, or* $\{b_n\}$ *is a zero-sequence.*

4. SOME APPLICATION

In this section we shall give some extensions on Hardy-Littlewood's theorem.

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$ for all x. Define a sequence $\{a_n\}$ by

$$a_n = \int_0^1 x^n f(x)dx \quad n = 0,\ 1,\ 2,\ \dots .$$

Hardy-Littlewood [1] proved that

$$\sum_{n=0}^{\infty} a_n^2 < \pi\int_0^1 f^2(x)dx \tag{4.1}$$

where π is the best constant that the inequality (4.1) keeps valid.

Some extensions of (4.1) will be given by means of *english/references* Corollaries 3.4, bel- low.

Theorem 4.1. *Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$ for all x. Define a sequence $\{a_n\}$by*

$$a_n = \int_0^1 x^{\Gamma(n+1)-1/2} f(x)dx \quad n = 2, 3, \ldots,$$

then

$$\left(\sum_{n=2}^{\infty} a_n^2\right)^2 < \frac{\pi}{\sin \pi/p} \left\{\sum_{m=2}^{\infty} (m!(\ln(em)-c))^{1-p} a_m^p\right\}^{1/p} \times \left\{\sum_{n=2}^{\infty} (n!(\ln(en)-c))^{1-q} a_n^q\right\}^{1/q} \int_0^1 f^2(x)\,dx, \tag{4.2}$$

where the constant factor $\frac{\pi}{\sin \pi/p}$ is best possible.

Proof. By our assumptions, we may write a_n^2 in the form:

$$a_n^2 = \int_0^1 a_n x^{\Gamma(n+1)-1/2} f(x)dx.$$

Applying Cauchy-Schwarz's inequality and Corollary 3.4 to estimate the right hand side of (4.2) as follows:

$$\begin{aligned}
\left(\sum_{n=2}^{\infty} a_n^2\right)^2 &= \left(\sum_{n=2}^{\infty} \int_0^1 a_n x^{\Gamma(n+1)-1/2} f(x)dx\right)^2 \\
&= \left\{\int_0^1 \left(\sum_{n=2}^{\infty} a_n x^{\Gamma(n+1)-1/2}\right) f(x)dx\right\}^2 \\
&\leq \int_0^1 \left(\sum_{n=2}^{\infty} a_n x^{\Gamma(n+1)-1/2}\right)^2 dx \int_0^1 f^2(x)dx \\
&= \int_0^1 \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} a_m a_n x^{\Gamma(m+1)+\Gamma(n+1)-1} dx \int_0^1 f^2(x)dx \\
&= \left(\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m a_n}{m!+n!}\right) \int_0^1 f^2(x)dx \\
&\leq \frac{\pi}{\sin \pi/p} \left\{\sum_{m=2}^{\infty} (m!(\ln(em)-c))^{1-p} a_m^p\right\}^{1/p} \\
&\quad \times \left\{\sum_{n=2}^{\infty} (n!(\ln(en)-c))^{1-q} a_n^q\right\}^{1/q} \int_0^1 f^2(x)\,dx.
\end{aligned} \tag{4.3}$$

Since $f(x) \neq 0$ for all x, $a_n \neq 0$ for all $n \geq 2$. Consequently, it is impossible to take equality in (4.3). It follows that the inequality (4.1) is valid and the theorem is therefore proved. ∎

In particular, for case $p = 2$, based on (4.2), we have the following result:

Theorem 4.2. *With the assumptions as the Theorem 4.1, then*

$$\left(\sum_{n=2}^{\infty} a_n^2\right)^2 < \pi \left\{\sum_{n=2}^{\infty} \frac{a_n^2}{n!\left(\ln(en) - c\right)}\right\} \int_0^1 f^2(x)\, dx \tag{4.4}$$

where the constant factor π is best possible.

Obviously, the inequalities (4.2) and (4.4) are extensions of (4.1).

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