The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 3, Issue 1, Article 12, pp. 1-7, 2006

NOTES ON SAKAGUCHI FUNCTIONS

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Received 13 April, 2005; accepted 12 December, 2005; published 15 May, 2006.

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ABSTRACT. By using the definition for certain univalent functions $f(z)$ in the open unit disk $\mathbb{U}$ given by K. Sakaguchi [2], two classes $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$ of analytic functions in $\mathbb{U}$ are introduced. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$.

Key words and phrases: Sakaguchi function, coefficient inequality, distortion inequality.

2004 *Mathematics Subject Classification.* Primary 30C45.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > \alpha \tag{1.2}$$

for some $\alpha (0 \leq \alpha < \frac{1}{2})$ and for all $z \in \mathbb{U}$. The class $\mathcal{S}(0)$ when $\alpha = 0$ was introduced by Sakaguchi [2]. Therefore, a function $f(z) \in \mathcal{S}(\alpha)$ is called Sakaguchi function of order α. We also denote by $\mathcal{T}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha)$. For $f(z)$ belonging to $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$, Cho, Kwon and Owa [1] have given

Lemma 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{2(n-1)|a_{2n-2}| + (2n-1-2\alpha)|a_{2n-1}|\} \leq 1 - 2\alpha \tag{1.3}$$

for some $\alpha (0 \leq \alpha < \frac{1}{2})$, then $f(z) \in \mathcal{S}(\alpha)$.

Lemma 1.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{4(n-1)^2|a_{2n-2}| + (2n-1)(2n-1-2\alpha)|a_{2n-1}|\} \leq 1 - 2\alpha \tag{1.4}$$

for some $\alpha (0 \leq \alpha < \frac{1}{2})$, then $f(z) \in \mathcal{T}(\alpha)$.

In view of the above lemmas, we see

Example 1.1. *Let us consider a function $f(z)$ given by*

$$f(z) = z + \frac{1}{3}\delta_2 z^2 + \left(1 - \frac{8}{3(3-2\alpha)}\right)\delta_3 z^3 \tag{1.5}$$

with $|\delta_2| = |\delta_3| = 1$. Then, since

$$\sum_{n=2}^{\infty} \{2(n-1)|a_{2n-2}| + (2n-1-2\alpha)|a_{2n-1}|\} < 1 - 2\alpha,$$

we see that $f(z) \in \mathcal{S}(\alpha)$.

Example 1.2. *Let us consider a function $f(z)$ given by*

$$f(z) = z + \frac{1}{6}\delta_2 z^2 + \frac{1}{3}\left(1 - \frac{8}{3(3-2\alpha)}\right)\delta_3 z^3 \tag{1.6}$$

with $|\delta_2| = |\delta_3| = 1$. Then, since

$$zf'(z) = z + \frac{1}{3}\delta_2 z^2 + \left(1 - \frac{8}{3(3-2\alpha)}\right)\delta_3 z^3 \in \mathcal{S}(\alpha),$$

we have that $f(z) \in \mathcal{T}(\alpha)$.

2. COEFFICIENT INEQUALITIES

Applying Carathéodory function

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{2.1}$$

in $\mathbb{U}$, we first discuss the coefficient inequalities for functions $f(z)$ in $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$.

Theorem 2.1. *If* $f(z) \in \mathcal{S}(\alpha)$, *then*

$$|a_{2n}| \leq \frac{\prod_{j=1}^{n+1}(j-2\alpha)}{n(n!)} \quad (n \geq 1) \tag{2.2}$$

and

$$|a_{2n+1}| \leq \frac{\prod_{j=1}^{n}(j-2\alpha)}{n!} \quad (n \geq 1). \tag{2.3}$$

Proof. We define the function $p(z)$ by

$$p(z) = \frac{1}{1-2\alpha}\left(\frac{2zf'(z)}{f(z)-f(-z)} - 2\alpha\right) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{2.4}$$

for $f(z) \in \mathcal{S}(\alpha)$. Then $p(z)$ is a Carathéodory function and satisfies $|p_n| \leq 2 \ (n \geq 1)$. Since

$$2zf'(z) = (f(z)-f(-z))\left((1-2\alpha)p(z) + 2\alpha\right),$$

we obtain that

$$a_{2n} = \frac{1-2\alpha}{2n}\left(p_{2n+1} + a_3 p_{2n-1} + \cdots + a_{2n+1} p_1\right) \tag{2.5}$$

and

$$a_{2n+1} = \frac{1-2\alpha}{2n}\left(p_{2n} + a_3 p_{2n-2} + \cdots + a_{2n-1} p_2\right). \tag{2.6}$$

Taking $n = 1$, we see that

$$|a_3| \leq 1 - 2\alpha \tag{2.7}$$

and

$$|a_2| \leq (1-2\alpha)(1 + |a_3|). \tag{2.8}$$

Thus, using the mathematical induction, we complete the proof of the theorem. ∎

Remark 2.1. Equalities in Theorem 2.1 are attended for $f(z)$ given by

$$\frac{zf'(z)}{f(z)-f(-z)} = \frac{1+(1-4\alpha)z}{2(1-z)}.$$

Theorem 2.2. *If* $f(z) \in \mathcal{T}(\alpha)$, *then*

$$|a_{2n}| \leq \frac{\prod_{j=1}^{n+1}(j-2\alpha)}{2n^2(n!)} \quad (n \geq 1) \tag{2.9}$$

and

$$|a_{2n+1}| \leq \frac{\prod_{j=1}^{n}(j-2\alpha)}{(2n+1)(n!)} \quad (n \geq 1). \tag{2.10}$$

3. DISTORTION INEQUALITIES

In view of Lemma 1.1 and Lemma 1.2, we introduce the subclasses $\mathcal{S}_0(\alpha)$ and $\mathcal{T}_0(\alpha)$. If $f(z) \in \mathcal{S}(\alpha)$ satisfies the coefficient inequalities (1.3), then we say that $f(z) \in \mathcal{S}_0(\alpha)$. Also, if $f(z) \in \mathcal{T}(\alpha)$ satisfies the coefficient inequalities (1.4), then we say that $f(z) \in \mathcal{T}_0(\alpha)$. For $f(z)$ belonging to $\mathcal{S}_0(\alpha)$ and $\mathcal{T}_0(\alpha)$, Cho, Kwon and Owa [1] have shown that

Theorem A. *If $f(z) \in \mathcal{S}_0(\alpha)$, then*

$$|z| - \frac{1-2\alpha}{2}|z|^2 - \frac{1-2\alpha}{3-2\alpha}|z|^3 \leq |f(z)| \leq |z| + \frac{1-2\alpha}{2}|z|^2 + \frac{1-2\alpha}{3-2\alpha}|z|^3 \tag{3.1}$$

and

$$1 - (1-2\alpha)|z| - \frac{3(1-2\alpha)}{3-2\alpha}|z|^2 \leq |f'(z)| \leq 1 + (1-2\alpha)|z| + \frac{3(1-2\alpha)}{3-2\alpha}|z|^2 \tag{3.2}$$

for $z \in \mathbb{U}$.

Theorem B. *If $f(z) \in \mathcal{T}_0(\alpha)$, then*

$$|z| - \frac{1-2\alpha}{4}|z|^2 - \frac{1-2\alpha}{3(3-2\alpha)}|z|^3 \leq |f(z)| \leq |z| + \frac{1-2\alpha}{4}|z|^2 + \frac{1-2\alpha}{3(3-2\alpha)}|z|^3 \tag{3.3}$$

and

$$1 - \frac{1-2\alpha}{2}|z| - \frac{1-2\alpha}{3-2\alpha}|z|^2 \leq |f'(z)| \leq 1 + \frac{1-2\alpha}{2}|z| + \frac{1-2\alpha}{3-2\alpha}|z|^2 \tag{3.4}$$

for $z \in \mathbb{U}$.

Now, we show

Theorem 3.1. *If $f(z) \in \mathcal{S}_0(\alpha)$, then*

$$|z| - \sum_{n=2}^{j}|a_n||z|^n - A_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j}|a_n||z|^n + A_j|z|^{j+1} \tag{3.5}$$

and

$$1 - \sum_{n=2}^{2j-2} n|a_n||z|^{n-1} - B_j|z|^{2j-2} \leq |f'(z)| \leq 1 + \sum_{n=2}^{2j-2} n|a_n||z|^{n-1} + B_j|z|^{2j-2} \tag{3.6}$$

where

$$A_j = \frac{1-2\alpha - \sum_{n=2}^{j}\{n - (1+(-1)^{n+1})\,\alpha\}|a_n|}{j+1-(1+(-1)^j)^{\alpha}} \quad (j \geq 2) \tag{3.7}$$

and

$$B_j = (2j-1)\frac{1-2\alpha - \sum_{n=2}^{2j-2}\{n - (1+(-1)^{n+1})\,\alpha\}|a_n|}{2j-1-2\alpha} \quad (j \geq 2). \tag{3.8}$$

Proof. Note that the coefficient inequalities (1.3) can be written as

$$\sum_{n=2}^{\infty}\{n - \left(1+(-1)^{n+1}\right)\alpha\}|a_n| \leq 1-2\alpha. \tag{3.9}$$

This gives us that

$$\sum_{n=2}^{j}\{n - \left(1+(-1)^{n+1}\right)\alpha\}|a_n| + \{j+1-\left(1+(-1)^j\right)\alpha\}\sum_{n=j+1}^{\infty}|a_n| \leq 1-2\alpha \tag{3.10}$$

and

$$\sum_{n=2}^{2j-2}\{n-\left(1+(-1)^{n+1}\right)\alpha\}|a_n|+\left(1-\frac{2\alpha}{2j-1}\right)\sum_{n=2j-1}^{\infty}n|a_n|\leq 1-2\alpha. \tag{3.11}$$

Therefore, $f(z)\in\mathcal{S}_0(\alpha)$ satisfies

$$\sum_{n=j+1}^{\infty}|a_n|\leq A_j \tag{3.12}$$

and

$$\sum_{n=2j-1}^{\infty}n|a_n|\leq B_j. \tag{3.13}$$

Thus, the distortion inequalitity (3.5) follows from (3.12) and the distortion inequality (3.6) follows from (3.13). ∎

Remark 3.1. If we take $j=2$ in Theorem 3.1, then we have Theorem A due to Cho, Kwon and Owa [1].

Furthermore, we also have

Theorem 3.2. *If $f(z)\in\mathcal{T}_0(\alpha)$, then*

$$|z|-\sum_{n=2}^{j}|a_n||z|^n-C_j|z|^{j+1}\leq|f(z)|\leq|z|+\sum_{n=2}^{j}|a_n||z|^n+C_j|z|^{j+1} \tag{3.14}$$

and

$$1-\sum_{n=2}^{j}n|a_n||z|^{n-1}-D_j|z|^j\leq|f'(z)|\leq 1+\sum_{n=2}^{j}n|a_n||z|^{n-1}+D_j|z|^j \tag{3.15}$$

for $z\in\mathbb{U}$, where

$$C_j=\frac{1-2\alpha-\sum_{n=2}^{j}n\{n-(1+(-1)^{n+1})\alpha\}|a_n|}{(j+1)\{j+1-(1+(-1)^j)\alpha\}}\quad(j\geq 2) \tag{3.16}$$

and

$$Dj=\frac{1-2\alpha-\sum_{n=2}^{j}n\{n-(1+(-1)^{n+1})\alpha\}|a_n|}{j+1-(1+(-1)^j)\alpha}\quad(j\geq 2). \tag{3.17}$$

Proof. Noting that the coefficient inequalities (1.4) satisfy

$$\sum_{n=2}^{\infty}n\{n-\left(1+(-1)^{n+1}\right)\alpha\}|a_n|\leq 1-2\alpha, \tag{3.18}$$

we have that

$$\begin{aligned}&\sum_{n=2}^{j}n\{n-\left(1+(-1)^{n+1}\right)\alpha\}|a_n|\\ +\quad&(j+1)\{j+1-\left(1+(-1)^{j+2}\right)\alpha\}\sum_{n=j+1}^{\infty}|a_n|\leq 1-2\alpha,\end{aligned} \tag{3.19}$$

which implies that

$$\sum_{n=j+1}^{\infty} |a_n| \leq C_j. \tag{3.20}$$

Further, by virtue of (3.18), we see that

$$\sum_{n=2}^{j} n\{n - \left(1 + (-1)^{n+1}\right) \alpha\}|a_n| + \{j + 1 - \left(1 + (-1)^{j+2}\right) \alpha\} \sum_{n=j+1}^{\infty} |a_n| \leq 1 - 2\alpha, \tag{3.21}$$

which derives

$$\sum_{n=j+1}^{\infty} |a_n| \leq D_j. \tag{3.22}$$

Therefore, the proof of the theorem follows from (3.21) and (3.22). ∎

Remark 3.2. If we let $j = 2$ in Theorem 3.2, then we have Theorem B by Cho, Kwon and Owa [1].

4. RELATION BETWEEN THE CLASSES

By the definitions for the classes $\mathcal{S}_0(\alpha)$, and $\mathcal{T}_0(\alpha)$, we know that

$$\mathcal{S}_0(\alpha) \subset \mathcal{S}_0(\beta) \subset \mathcal{S}_0(0) \quad \left(0 \leq \beta \leq \alpha < \frac{1}{2}\right)$$

and

$$\mathcal{T}_0(\alpha) \subset \mathcal{T}_0(\beta) \subset \mathcal{T}_0(0) \quad \left(0 \leq \beta \leq \alpha < \frac{1}{2}\right).$$

Let us discuss a relation between $\mathcal{S}_0(\beta)$ and $\mathcal{T}_0(\alpha)$.

Theorem 4.1. *If $f(z) \in \mathcal{T}_0(\alpha)$, then $f(z) \in \mathcal{S}_0\left(\frac{1+2\alpha}{4}\right)$.*

Proof. Let $f(z) \in \mathcal{T}_0(\alpha)$. Then, if $f(z)$ satisfies

$$\frac{n - \left(1 + (-1)^{n+1}\right) \beta}{1 - 2\beta} \leq n \, \frac{n - \left(1 + (-1)^{n+1}\right) \alpha}{1 - 2\alpha} \tag{4.1}$$

for all $n \geq 2$, we have that $f(z) \in \mathcal{S}_0(\beta)$ which satisfies the inequality (4.1). After calculation (4.1), we have that

$$\beta \leq n \, \frac{n - 1 + \left(3 + (-1)^{n+1}\right) \alpha}{2n^2 - \left(1 + (-1)^{n+1}\right)(2n\alpha - 2\alpha + 1)}. \tag{4.2}$$

If n is even, then (4.2) becomes

$$\beta \leq \frac{n - 1 + 2\alpha}{2n}. \tag{4.3}$$

This implies that

$$\beta \leq \frac{1 + 2\alpha}{4} \quad (\text{for even } n). \tag{4.4}$$

On the other hand, if n is odd, then (4.2) becomes

$$\beta \leq \frac{n^2 - (1 - 4\alpha)n}{2n^2 - 4n\alpha + 4\alpha - 2}. \tag{4.5}$$

Since, for odd n and $0 \leq \alpha < \frac{1}{2}$,

$$\frac{n^2 - (1-4\alpha)n}{2n^2 - 4n\alpha + 4\alpha - 2} - \frac{1+2\alpha}{4} = \frac{(1-2\alpha)(n-1)(n-1-2\alpha)}{4(n^2 - 2n\alpha + 2\alpha - 1)} > 0, \tag{4.6}$$

we conclude that: $\beta \leq \dfrac{1+2\alpha}{4}$ for all n. Thus we conclude that $\mathcal{T}_0(\alpha) \subset \mathcal{S}_0\left(\dfrac{1+2\alpha}{4}\right)$. ∎

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