PARAMETER DEPENDENCE OF THE SOLUTION OF SECOND ORDER NONLINEAR ODE’S VIA PEROV’S FIXED POINT THEOREM
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ABSTRACT. Using the Perov’s fixed point theorem, the smooth dependence by parameter of the solution of a two point boundary value problem corresponding to nonlinear second order ODE’s is obtained.

Key words and phrases: Two point boundary value problem, Nonlinear second order differential equations, Perov’s fixed point theorem, Smooth dependence by parameter.

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1. Introduction

Consider the following two point boundary value problem corresponding to the nonlinear second order ODE:

\[
\begin{align*}
&y''(x) = f(x, y(x), y'(x)), \quad x \in [a, b] \\
y(a) = \alpha, \quad y(b) = \beta, \quad \alpha, \beta \in \mathbb{R}.
\end{align*}
\]

The existence and uniqueness in \(C^2[a, b]\) of the solution of (1.1) is studied in [2] using a fixed point theorem on vector valued generalized metric spaces, which have an equivalent enunciation with Theorem 2.1.

It is known that the problem (1.1) is equivalent with the following integro-differential equation (see [1], [2] and [5]):

\[
\begin{align*}
y(x) &= \frac{x-a}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_{a}^{b} G(x, s) f(s, y(s), y'(s)) \, ds,
\end{align*}
\]

for \(x \in [a, b]\).

If we are interested by the parameter dependence of the solution of equation (1.2), then this equation becomes,

\[
\begin{align*}
y(x, \lambda) &= \frac{x-a}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_{a}^{b} G(x, s) f(s, y(s), \lambda, y'(s), \lambda) \, ds,
\end{align*}
\]

for \(x \in [a, b], \lambda \in [c, d]\), where \(G(x, s)\) is the well known Green’s function.

Using the Perov’s fixed point theorem (see [3], [4] and [8]) and a result of I. A. Rus (see [8]), we obtain here the smooth dependence, of the solution of (1.3) and of his derivative, by the parameter \(\lambda\). A similar result of smooth dependence by the end points \(a\) and \(b\) of the solution of fredholm integral equations which use an idea of Sotomayor (see [9]) was obtained by I. A. Rus in [8].

2. Preliminaries

Let \(X\) be a nonempty set and \(A : X \to X\) an operator. The fixed points set of \(A\) will be

\[F_A = \{x \in X : A(x) = x\}.\]

**Definition 2.1.** (Rus, [6] or [7]) Let \((X, d)\) be a metric space. An operator \(A : X \to X\) is Picard operator if there exists \(x^* \in X\) such that:

(a) \(F_A = \{x^*\}\),

(b) the sequence \((A^n(x_0))_{n \in \mathbb{N}}\) converges to \(x^*\), for all \(x_0 \in X\), where \(A^0 = 1_X, A^1 = A, A^n = A \circ A^{n-1}, \forall n \in \mathbb{N}^*\).

**Definition 2.2.** (Rus, [6] or [7]) Let \((X, d)\) be a metric space. An operator \(A : X \to X\) is weakly Picard operator if the sequence \((A^n(x_0))_{n \in \mathbb{N}}\) converges for all \(x_0 \in X\) and the limit (which may depend on \(x_0\)) is a fixed point of \(A\).

For the following notion, let \(X \neq \emptyset, n \in \mathbb{N}\) and \(d : X \times X \to \mathbb{R}^n_+\) where,

\[\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \quad \forall i = 1, n\}\]

**Definition 2.3.** (in [4] or [2]) The pair \((X, d)\) is generalized metric space iff the function \(d\) have the following properties:

\[(gm1) \quad d(x, y) \geq 0, \forall x, y \in X \text{ and } d(x, y) = 0 \iff x = y\]
The function \( d \) is called generalized metric.

The euclidean space \( \mathbb{R}^n \) is ordered by the relation:

\[
x \leq y \iff x_i \leq y_i, \quad \forall i = 1, n,
\]

for \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \).

A generalized metric space is complet if any fundamental sequence in \( X \) is convergent. Let \( M_n(\mathbb{R}^+) \) the set of matrices with all elements positive.

**Definition 2.4.** (in [4]) Let \((X, d)\) be a generalized metric space. A map \( T : X \to X \) satisfy a generalized Lipschitz inequality if there exists a matrix \( A \in M_n(\mathbb{R}^+) \) such that:

\[
d(T(x), T(y)) \leq Ad(x, y), \quad \forall x, y \in X.
\]

**Theorem 2.1.** (Perov, [3, 4]) Let \((X, d)\) be a generalized metric space and \( A : X \to X \) a mapping which have the generalized Lipschitz inequality property with a matrix \( Q \in M_n(\mathbb{R}^+) \). If all eigenvalues of \( Q \) lies in the open unit ball from the complex plane, then:

(i) the operator \( A \) has a unique fixed point \( x^* \in X \)

(ii) for any \( x_0 \in X \), the sequence \((x_m)_{m \in \mathbb{N}} \subseteq X \) defined by \( x_m = A(x_{m-1}) \), \( \forall m \in \mathbb{N}^* \), is convergent to \( x^* \)

(iii) the following inequality holds:

\[
d(x_m, x^*) \leq Q^m(I_n - Q)^{-1} \cdot d(x_0, x_1), \forall m \in \mathbb{N}^*.
\]

Using this Perov’s fixed point theorem, I. A. Rus obtains the result:

**Theorem 2.2.** (of fiber generalized contractions, Rus [8]) Let \((X, d)\) be a metric space (generalized or not) and \((Y, \rho)\) be a complete generalized metric space (\( \rho(x, y) \in \mathbb{R}^+_n \)). Let \( A : X \times Y \to X \times Y \) be a continuous operator and \( C : X \times Y \to Y \) an operator. Suppose that:

(i) \( B : X \to X \) is a weakly Picard operator

(ii) \( A(x, y) = (B(x), C(x, y)) \), for all \( x \in X, y \in Y \)

(iii) there exists a matrix \( Q \in M_n(\mathbb{R}^+) \), with \( Q^m \to 0 \) as \( m \to \infty \), such that

\[
\rho(C(x, y_1), C(x, y_2)) \leq Q \cdot \rho(y_1, y_2),
\]

for all \( x \in X, y_1 \) and \( y_2 \in Y \).

Then, the operator \( A \) is weakly Picard operator. Moreover, if \( B \) is Picard operator, then \( A \) is Picard operator.

**Remark 2.1.** (see [4]) For a matrix \( Q \in M_n(\mathbb{R}^+) \), the following properties are equivalent:

(i) \( Q^m \to 0 \) as \( m \to \infty \),

(ii) all eigenvalues of \( Q \) lies in the open unit ball from the complex plane.

3. **The main result**

Deriving the equation (1.3) in respect with \( x \) we obtain,

\[
y'_x(x, \lambda) = \frac{\beta - \alpha}{b - a} - \int_a^b \frac{\partial G(\lambda)}{\partial x}(s, y(s, \lambda), y'(s, \lambda), \lambda) \, ds,
\]

\( x \in [a, b], \lambda \in [c, d] \).
Denoting \( z = y' \), from (1.3) and (3.1), we can take in consideration the system of integral equations,

\[
\begin{align*}
    y(x, \lambda) &= \frac{a-x}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_a^b G(x, s) f(s, y(s, \lambda), z(s, \lambda), \lambda) \, ds \\
    z(x, \lambda) &= \frac{\beta-a}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x, s) f(s, y(s, \lambda), z(s, \lambda), \lambda) \, ds, \quad x \in [a, b], \lambda \in [c, d],
\end{align*}
\]

(3.2)

where

\[
G(x, s) = \begin{cases} 
    \frac{(s-a)(b-x)}{(x-a)(b-s)}, & \text{if } s \leq x \\
    \frac{(s-a)(b-x)}{b-a}, & \text{if } s \geq x
\end{cases}
\]

and

\[
\frac{\partial G}{\partial x}(x, s) = \begin{cases} 
    \frac{-(s-a)}{b-a}, & \text{if } s < x \\
    \frac{(b-s)}{b-a}, & \text{if } s > x.
\end{cases}
\]

Let

\[
C([a, b] \times [c, d]) = \{ y : [a, b] \times [c, d] \to \mathbb{R} \mid y \ \text{continuous} \}
\]

and on this set we define the Chebyshev's norm,

\[
\|u\|_C = \max\{|u(x, y)| : x \in [a, b], \ y \in [c, d]\}.
\]

Let

\[
X = Y = C([a, b] \times [c, d]) \times C([a, b] \times [c, d])
\]

and on this product space we consider the generalized metric \( d_C : X \times X \to \mathbb{R}^2 \), defined by

\[
d_C((y_1, z_1), (y_2, z_2)) = (\|y_1 - y_2\|_C, \ \|z_1 - z_2\|_C), \quad \forall (y_1, z_1), (y_2, z_2) \in X.
\]

We define the operators,

\[
B : X \to X, \quad C : X \times X \to X
\]

and \( A : X \times X \to X \times X \), by

\[
B(y, z) = (B_1(y, z), B_2(y, z))
\]

(3.3)

\[
B_1(y, z)(x, \lambda) = \frac{x-a}{b-a} \cdot \beta + \frac{b-x}{b-a} \cdot \alpha - \int_a^b G(x, s) f(s, y(s, \lambda), z(s, \lambda), \lambda) \, ds
\]

(3.4)

\[
B_2(y, z)(x, \lambda) = \frac{\beta - \alpha}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x, s) f(s, y(s, \lambda), z(s, \lambda), \lambda) \, ds
\]

C((y, z), (u, v)) = (C_1((y, z), (u, v)), C_2((y, z), (u, v)))

\[
C_1((y, z), (u, v))(x, \lambda) = -\int_a^b G(x, s) \cdot \frac{\partial f}{\partial \lambda}(s, y(s, \lambda), z(s, \lambda), \lambda) + \frac{\partial f}{\partial y}(s, y(s, \lambda), z(s, \lambda), \lambda) \cdot u(s, \lambda) \, ds
\]

(3.5)

\[
+ \frac{\partial f}{\partial x}(s, y(s, \lambda), z(s, \lambda), \lambda) \cdot u(s, \lambda) + \frac{\partial f}{\partial y}(s, y(s, \lambda), z(s, \lambda), \lambda) \cdot v(s, \lambda) \, ds
\]

C_2((y, z), (u, v))(x, \lambda) = -\int_a^b \frac{\partial G}{\partial x}(x, s) \cdot \frac{\partial f}{\partial \lambda}(s, y(s, \lambda), z(s, \lambda), \lambda) +
Theorem 3.1.

For \((y, z) \in X\) fixed, consider the system,

\[
\begin{align*}
\frac{\partial f}{\partial x}(s, y(s, \lambda), z(s, \lambda), \lambda) \cdot u(s, \lambda) & + \frac{\partial f}{\partial y}(s, y(s, \lambda), z(s, \lambda), \lambda) \cdot v(s, \lambda) ds \\
A((y, z), (u, v)) &= (B(y, z), C((y, z), (u, v))).
\end{align*}
\]

We will suppose that the following conditions hold:

(C) (boundedness): there exist \(M > 0\) such that

\[
|f(x, u, v, \lambda)| \leq M, \quad \forall (x, u, v, \lambda) \in [a, b] \times \mathbb{R}^2 \times [c, d].
\]

(C) (Lipschitz): there exist \(L_1 > 0\), \(L_2 > 0\) such that

\[
\frac{|\partial f(x, u, v, \lambda)|}{\partial u} \leq L_1, \quad \frac{|\partial f(x, u, v, \lambda)|}{\partial v} \leq L_2, \quad \forall (x, u, v, \lambda) \in [a, b] \times \mathbb{R}^2 \times [c, d].
\]

(C) (smoothness): \(f(x, \cdot, \cdot, \cdot) \in C^1(\mathbb{R}^2 \times [c, d]), \quad \forall x \in [a, b].
\]

Theorem 3.1.

(a) With the conditions (C), (C), (C), if \(L_1(b-a)^2 < 1\) and \(L_2(b-a) < \frac{3}{4}\), then the system (3.2) of integral equations has in \(X\) an unique solution \((y^*, z^*)\) such that \(y^*, z^* \in C^1([a, b] \times [c, d]), y^*(\cdot, \lambda) \in C^2[a, b], \quad \forall \lambda \in [c, d]\) and \(\frac{\partial}{\partial \lambda} y^* = z^*\).

(b) With the conditions (C) (C), if \(L_1(b-a)^2 < 1\) and \(L_2(b-a) < \frac{3}{4}\), then the pair \((\frac{\partial}{\partial x} y^*, \frac{\partial}{\partial x} z^*)\) is the unique solution in \(C([a, b] \times [c, d]) \subset Y\) of the system (3.7) for the fixed pair \((y, z) = (y^*, z^*)\).

Proof. By the conditions (C) and (C) follows that \(B(X) \subset C([a, b] \times [c, d]).\) Elementary calculus lead to:

\[
\begin{align*}
\|B_1(y_1, z_1)(x, \lambda) - B_1(y_2, z_2)(x, \lambda)\| & \leq (b-a) \|G\|_C \cdot (L_1 \|y_1 - y_2\|_C + L_2 \|z_1 - z_2\|_C) \\
& \leq \frac{(b-a)^2}{4} \cdot (L_1 \|y_1 - y_2\|_C + L_2 \|z_1 - z_2\|_C), \quad \forall (x, \lambda) \in [a, b] \times [c, d]
\end{align*}
\]

and

\[
\begin{align*}
\|B_2(y_1, z_1)(x, \lambda) - B_2(y_2, z_2)(x, \lambda)\| & \leq (b-a) \|\frac{\partial G}{\partial x}\|_C \cdot (L_1 \|y_1 - y_2\|_C + L_2 \|z_1 - z_2\|_C) \\
& \leq (b-a)(L_1 \|y_1 - y_2\|_C + L_2 \|z_1 - z_2\|_C), \quad \forall (x, \lambda) \in [a, b] \times [c, d].
\end{align*}
\]

Then,

\[
d_C(B(y_1, z_1), B(y_2, z_2)) \leq \left( \frac{1}{4} L_1(b-a)^2 \frac{1}{4} L_2(b-a)^2 \right) \cdot d_C((y_1, z_1), (y_2, z_2)),
\]

\( \forall (y_1, z_1), (y_2, z_2) \in X. \)

Since the eigenvalues of the matrix,

\[
Q = \begin{pmatrix}
\frac{1}{4}L_1(b - a)^2 & \frac{1}{4}L_2(b - a)^2 \\
L_1(b - a) & L_2(b - a)
\end{pmatrix}
\]

are \( \lambda_1 = 0 \) and \( \lambda_2 = \frac{1}{4}L_1(b - a)^2 + L_2(b - a) \), from the conditions \( L_1(b - a)^2 < 1 \) and \( L_2(b - a) < \frac{3}{4} \) we infer that \( |\lambda_2| < 1 \) and so, \( Q^m \to 0 \) when \( m \to \infty \).

Applying the Perov’s fixed point Theorem 2.1, we conclude that the operator \( B \) has in \( C([a, b] \times [c, d]) \) a unique fixed point \( (y^*, z^*) \) and the sequence \( (y_m, z_m) \) converges uniformly to \( (y^*, z^*) \) in \( C([a, b] \times [c, d]) \) for any \((y_0, z_0) \in X\), where

\[
y_m(x, \lambda) = \frac{x - a}{b - a} \cdot \beta + \frac{b - x}{b - a} \cdot \alpha - \int_{a}^{b} G(x, s) f(s, y_{m-1}(s, \lambda), z_{m-1}(s, \lambda), \lambda) ds,
\]

(3.8)

\[
z_m(x, \lambda) = \frac{\beta - \alpha}{b - a} - \int_{a}^{b} \frac{\partial G}{\partial x}(x, s) f(s, y_{m-1}(s, \lambda), z_{m-1}(s, \lambda), \lambda) ds,
\]

\( \forall m \in \mathbb{N^*}, \ \forall (x, \lambda) \in [a, b] \times [c, d]. \)

Consequently,

\[
y^*(x, \lambda) = \frac{x - a}{b - a} \cdot \beta + \frac{b - x}{b - a} \cdot \alpha - \int_{a}^{b} G(x, s) f(s, y^*(s, \lambda), z^*(s, \lambda), \lambda) ds,
\]

(3.9)

and

\[
z^*(x, \lambda) = \frac{\beta - \alpha}{b - a} + \int_{a}^{x} \frac{s - a}{b - a} \cdot f(s, y^*(s, \lambda), z^*(s, \lambda), \lambda) ds - \int_{x}^{b} \frac{b - s}{b - a} \cdot f(s, y^*(s, \lambda), z^*(s, \lambda), \lambda) ds,
\]

(3.10)

\( \forall (x, \lambda) \in [a, b] \times [c, d]. \)

With the condition \((C_1)\) we see that

\[
\frac{\partial}{\partial x} y^*(x, \lambda) = \frac{\beta - \alpha}{b - a} + \int_{a}^{x} \frac{s - a}{b - a} \cdot f(s, y^*(s, \lambda), z^*(s, \lambda), \lambda) ds - \int_{x}^{b} \frac{b - s}{b - a} \cdot f(s, y^*(s, \lambda), z^*(s, \lambda), \lambda) ds
\]

and therefore \( \frac{\partial}{\partial x} y^* = z^* \). With the same condition we infer that \( y^*, z^* \in C^1([a, b] \times [c, d]). \)

Because \( \frac{\partial}{\partial x} y^* = z^* \) we conclude that \( y^*(\cdot, \lambda) \in C^2[a, b], \ \forall \lambda \in [c, d]. \)

It is easy to see that \( y^*(a, \lambda) = \alpha, \ y^*(b, \lambda) = \beta, \ \forall \lambda \in [c, d] \) and

\[
\frac{\partial}{\partial x} z^*(x, \lambda) = f(x, y^*(x, \lambda), z^*(x, \lambda), \lambda), \ \forall (x, \lambda) \in [a, b] \times [c, d],
\]

that is,

\[
\frac{\partial^2 y^*}{\partial x^2}(x, \lambda) = f(x, y^*(x, \lambda), \frac{\partial y^*}{\partial x}(x, \lambda), \lambda), \ \forall (x, \lambda) \in [a, b] \times [c, d].
\]
(b) Consider the operator

\[ C((y^*, z^*), \cdot) : Y \to Y, \]

which in the conditions (C1), (C2) and (C4), is well defined. The condition (C4) permits to consider the Lipschitz constants from the condition (C3) as

\[ L_1 = \left\| \frac{\partial f}{\partial y} \right\| \text{ and } L_2 = \left\| \frac{\partial f}{\partial z} \right\|. \]

From elementary calculus we obtain,

\[ |C_1((y^*, z^*), (u_1, v_1)) (x, \lambda) - C_1((y^*, z^*), (u_2, v_2)) (x, \lambda)| \leq \]

\[ \leq \int_a^b |G(x, s) (L_1 |u_1(s, \lambda) - u_2(s, \lambda)| + L_2 |v_1(s, \lambda) - v_2(s, \lambda)|)ds \leq \]

\[ \leq \frac{(b-a)^2}{4} (L_1 \|u_1 - u_2\|_C + L_2 \|v_1 - v_2\|_C), \quad \forall (x, \lambda) \in [a, b] \times [c, d] \]

and

\[ |C_2((y^*, z^*), (u_1, v_1)) (x, \lambda) - C_2((y^*, z^*), (u_2, v_2)) (x, \lambda)| \leq \]

\[ \leq (b-a)(L_1 \|u_1 - u_2\|_C + L_2 \|v_1 - v_2\|_C), \quad \forall (x, \lambda) \in [a, b] \times [c, d]. \]

Then,

\[ dC_c(C((y^*, z^*), (u_1, v_1)), C((y^*, z^*), (u_2, v_2))) \leq QdC((u_1, v_1), (u_2, v_2)), \]

\[ \forall (u_1, v_1), (u_2, v_2) \in Y \text{ and } Q^m \to 0 \text{ for } m \to \infty. \]

From Theorem 2.2 we infer that the operator \( A \) has in \( X \times Y \) a unique fixed point \( ((y^*, z^*), (u^*, v^*)) \in X \times Y \), that is

\[ C((y^*, z^*), (u^*, v^*)) = (u^*, v^*). \]

Moreover, for \( y_0 \in C^2([a, b] \times [c, d]), z_0 = \frac{\partial}{\partial \lambda} y_0, u_0 = \frac{\partial}{\partial \lambda} y_0, v_0 = \frac{\partial}{\partial \lambda} z_0 \) the sequence defined by

\[ ((x_m, y_m), (u_m, v_m))_m = (A^m((y_0, z_0), (u_0, v_0)))_m \]

converges uniformly to \( ((y^*, z^*), (u^*, v^*)) \) in \( X \times Y \).

From the conditions (C1), (C2) and (C3) we infer that \( y_m \in C^2([a, b] \times [c, d]), z_m \in C^1([a, b] \times [c, d]), u_m, v_m \in C([a, b] \times [c, d]), \quad \forall m \in \mathbb{N} \) and

\[ y_m \Rightarrow y^*, \quad z_m = \frac{\partial y_m}{\partial x} \Rightarrow z^* \]

\[ u_m = \frac{\partial y_m}{\partial \lambda} \Rightarrow u^* \text{ and } v_m = \frac{\partial z_m}{\partial \lambda} \Rightarrow v^*, \]

that is,

\[ z^* = \frac{\partial y^*}{\partial x}, \quad u^* = \frac{\partial y^*}{\partial \lambda}, \quad v^* = \frac{\partial z^*}{\partial \lambda}. \]

\[ \square \]

**Corollary 3.2.** Under the conditions of Theorem 3.1 the two point boundary value problem

\[ \begin{cases} \frac{\partial^2}{\partial x^2} y(x, \lambda) = f(x, y(x, \lambda), \frac{\partial}{\partial x} y(x, \lambda), \lambda), & (x, \lambda) \in [a, b] \times [c, d], \\ y(a, \lambda) = \alpha, & y(b, \lambda) = \beta, & \forall \lambda \in [c, d] \end{cases} \]

has in

\[ C^{2,1}([a, b] \times [c, d]) = \{ y : [a, b] \times [c, d] \to \mathbb{R} | y \in C^1([a, b] \times [c, d]) \}

and \( y(\cdot, \lambda) \in C^2([a, b], \forall \lambda \in [c, d]) \)

a unique solution \( y^* \) such that \( y^* \) and his partial derivative in respect by \( x \) are smooth dependent by the parameter \( \lambda \).
Proof. Follows directly by the proof of Theorem 3.1.

REFERENCES


