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# ON PERTURBED REFLECTION COEFFICIENTS

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ABSTRACT. Many control and signal processing applications require testing stability of polynomials. Classical tests for locating zeros of polynomials are recursive, but they must be stopped whenever the so called "singular polynomials" appear. These "singular cases" are often avoided by perturbing the "singular polynomial". Perturbation techniques although always successful are not proven to be well-founded. Our aim is to give a mathematical foundation to a perturbation method in order to overcome "singular cases" when using Levinson recursion as a testing method. The non–singular polynomials are proven to be dense in the set of all polynomials respect the $L^2-$norm on the unit circle . The proof is constructive and can be used algorithmically.

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## 1. INTRODUCTION

The problem of root distribution of a polynomial has been long treated, and it is discussed in many textbooks on engineering applications such as system theory or automatic control. The first systematic approach to investigate real root distribution of a polynomial was presented by Sturm [15] in 1829. Concerning the asymptotical stability of a linear difference equation, the number of roots in the unit circle was determined by Schur and Cohn [4] and further obtained by Marden [11], Jury [9] and Riable [14].

A common problem that arises when we apply classical tests for locating the zeros of polynomials is the appearance of the so called "singular cases" in which backward recursion stops ([9], [8], [2]).

In order to overcome this situation, procedures involving coefficients of polynomials are usual in the literature [5]. For instance, a perturbation of the coefficients is often carried out claiming for continuity arguments. However, continuity of zeros with respect to coefficients does not mean that a small perturbation of coefficients always allows us to attain a "non–singular" polynomial. Beyond continuity, density statements are required to assure that non–singular polynomials are always close to the singular ones.

As far as we know, density statements involving polynomials and reflection coefficients have never been used to justify these perturbation techniques. Our concern in this paper is to present one of the above mentioned statements when dealing with reflection coefficients and backward Levinson recursion [10]. To this end, we give a density theorem assuring that, for any polynomial, there exists a non–singular (Levinson's recursion sense) one as close to it, in the $L^2$–norm sense, as desired.

## 2. NOTATIONS AND BASIC RESULTS

Backward Levinson recursion is important when checking the position of the zeros of a polynomial with respect to unit circle ([13], [6]). Given a $n$-degree monic polynomial

$$A_n(z) = z^n + \sum_{k=0}^{n-1} a_{nk} z^k \ ,$$

the backward Levinson recursion [10] assigns a $(n-1)$-degree monic polynomial $A_{n-1}(z)$, being its coefficients

$$a_{n-1,j} = \frac{1}{1-|\alpha_n|^2}(a_{n,j+1} - \alpha_n a_{n,n-1-j}) \ , \tag{2.1}$$

for $j = 0, 1, \ldots, n-1$, where $a_{nn} = 1$ and $\alpha_n = a_{n0}$. Here $\overline{a}$ denotes the complex conjugate of $a$. From (2.1), the polynomial $A_{n-1}(z)$ is readily expressed as

$$zA_{n-1}(z) = \frac{1}{1-|\alpha_n|^2}\Big[A_n(z) - \alpha_n A_n^*(z)\Big], \tag{2.2}$$

where $A_n^*(z) = z^n\overline{A_n(1/\overline{z})}$. Backward Levinson recursion is not defined in cases such that $|\alpha_n| = 1$. These are the so-called singular cases.

From (2.1) and (2.2), forward Levinson recursion can be easily obtained, namely,

$$a_{nj} = a_{n-1,j-1} + \alpha_n \overline{a}_{n-1,n-1-j} \ , \ j = 0, 1, \ldots, n \ , \tag{2.3}$$

or in polynomial notation

$$A_n(z) = zA_{n-1}(z) + \alpha_n A_{n-1}^*(z) \ . \tag{2.4}$$

Levinson recursions detach the role of $\alpha_n = a_{n0}$ which is called reflection coefficient of $A_n(z)$.

From a given monic polynomial $A_j(z)$, and a sequence of reflection coefficients $\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n$, forward Levinson recursion (2.3),(2.4), generates a sequence of polynomials of increasing degree. The last term of the sequence is denoted $A_n(z) = [A_j(z); \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n]$. Frequently, polynomial $A_j(z)$ is assumed to be the constant polynomial $A_0(z) = 1$.

The main classical result on location of zeros using Levinson recursions is the next theorem based on the Rouchè theorem ([1], [3], [12], [16]).

**Theorem 2.1.** *Let $A_n(z)$ be a monic polynomial with complex coefficients. Then, the zeros of $A_n(z)$ lie in the unit circle if and only if $A_n(z) = [1; \alpha_1, \alpha_2, \ldots, \alpha_n]$ with $|\alpha_k| < 1$ for $k = 1, 2, \ldots, n$.*

Theorem 2.1 motivates the question of which polynomials have an expression in reflection coefficients as $A_n(z) = [1; \alpha_1, \alpha_2, \ldots, \alpha_n]$, i.e. generated from the constant polynomial 1. This question is directly related with the cases in which the polynomial $A_n(z)$ has a unitary reflection coefficient. Then backward Levinson recursion (2.2) does not work. These polynomials are called unitary.

**Definition 2.1.** A monic complex polynomial $A_n(z) = z^n + \sum_{k=0}^{n-1} a_{nk} z^k$, is unitary if its reflection coefficient $\alpha_n = a_{n0}$ satisfies $|\alpha_n| = 1$.

Unitary polynomials can be divided into two classes depending on the following property.

**Definition 2.2.** A unitary polynomial $A_n(z) = z^n + \sum_{k=0}^{n-1} a_{nk} z^k$, is self–inversive if there exists a unitary complex number $u$ such that $A_n(z) = u A_n^*(z)$ or, equivalently, $a_{nk} = u a_{n,n-k}$ for $k = 0, 1, \ldots, n$.

Self–inversive polynomials can be obtained from the constant polynomial 1 and an appropriate sequence of reflection coefficients by using forward Levinson recursion (2.4). This can be summarized in the following theorem proven in [7].

**Theorem 2.2.** *Let $A_n(z)$ be a unitary polynomial. Then, there exist a sequence of reflection coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $A_n(z) = [1; \alpha_1, \alpha_2, \ldots, \alpha_n]$ if and only if $A_n(z)$ is self–inversive. Moreover, for $n \geq 3$, if the sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ exists, it is not unique.*

A consequence of theorem 2.2 is that non self–inversive, unitary polynomials can not be obtained from 1 using forward Levinson recursion and this motivates a characterization of polynomials by reflection coefficients.

**Definition 2.3.** The reflection coefficient characterization of a complex monic polynomial $A_n(z)$ is

$$A_n(z) = [A_j(z); \alpha_{j+1}, \alpha_{j+2}, \cdots, \alpha_n] \,, \tag{2.5}$$

where $0 \leq j \leq n$ and $A_j(z)$, called base polynomial, is either $A_j(z) = A_0(z) = 1$ or $A_j(z)$ is a non self-inversive, unitary polynomial. The constants $\alpha_k \in \mathbb{C}$, $k = j+1, j+2, \cdots, n$, are reflection coefficients.

If all reflection coefficients in the characterization of $A_n(z)$ are not unitary, then, the characterization is unique. Polynomials which base polynomial is $A_0(z) = 1$, with one or more unitary reflection coefficients have multiple characterization.

The preceding characterization provides a classification of the set of polynomials of any degree into non-overlapping subsets defined as follows [7]:

(a)
$$\mathcal{C}[A_j] = \left\{ A_k(z) = [A_j(z); \alpha_{j+1}, \cdots, \alpha_k], |\alpha_s| \neq 1, j+1 \leq s \leq k \right\}$$
with $j \leq k$, $\alpha_s \in \mathbb{C}$, and $A_j(z)$ a non self-inversive, unitary polynomial.

(b)
$$\mathcal{C}[1] = \left\{ A_k(z) = [1; \alpha_1, \alpha_2, \cdots, \alpha_k],\ 1 \le k \right\} .$$
In the literature concerning reflection coefficients, the polynomials usually considered belong to $\mathcal{C}[1]$ class, but they are often restricted to those which reflection coefficients have modulus less than 1. After definition 2.3, $\mathcal{C}[1]$ contains both polynomials for which backward Levinson recursion (2.2) applies normally until obtaining the constant base polynomial and polynomials that, in some step of the backward recursion, produce a self–inversive polynomial. Self–inversive polynomials have been considered singular because backward Levinson recursion (2.2) can not be applied. However, there are polynomials that are transformed into them by the forward Levinson recursion. For instance, if $A_n(z)$ is self–inversive, then its monic derivative $A_{n-1}(z) = A'_n(z)/n$ is such that $A_n(z) = [A_{n-1}(z); \alpha_n]$, where $\alpha_n$ is the reflection coefficient of $A_n(z)$. Therefore, self–inversive polynomials, being unitary, cannot be considered singular for they belong to $\mathcal{C}[1]$ and perturbation procedures are not necessary. Polynomials which are not in $\mathcal{C}[1]$ may be called singular and they have to be perturbed to carry out backward recursion.

## 3. DENSITY THEOREM

Attention is now centered on the density of the $\mathcal{C}[1]$ class in the set of all monic polynomials with complex coefficients. In this case, density means that every polynomial is as close to one in $\mathcal{C}[1]$ class as desired, with respect to the $L^2$–distance.

The $L^2(|z| = 1)$–norm and distance on the unit circle are defined as follows.

**Definition 3.1.** Let $A(z)$ and $B(z)$ be two monic polynomials with complex coefficients. The $L^2(|z| = 1)$–norm on the unit circle is
$$\|A(z)\|^2 = \oint |A(z)|^2 \, dz ,$$
and the corresponding distance is
$$\mathrm{d}(A(z), B(z)) = \oint |A(z) - B(z)|^2 \, dz , \tag{3.1}$$
where the path of the integrals is the unit circle.

The main result is the following theorem.

**Theorem 3.1.** *Let $A_n(z)$ be a monic complex polynomial. Then, for all $\epsilon > 0$, there exists a polynomial $A_n^m(z) = [1; \alpha_1^m, \alpha_2^m, \cdots, \alpha_n^m]$, such that $\mathrm{d}(A_n, A_n^m) < \epsilon$.*

Before proving this theorem, a lemma concerning to distances of polynomials under Levinson recursion is needed. It relates the distance between two polynomials $A_n(z), B_n(z)$, and the distance between the two polynomials $A_{n+1}(z), B_{n+1}(z)$, obtained from the formers by applying forward recursion (2.4) with the same reflection coefficient $\alpha_{n+1}$.

**Lemma 3.2.** *Let $A_n(z)$ and $B_n(z)$ be two monic polynomials of degree $n$ and let $A_{n+1}(z) = [A_n(z); \alpha_{n+1}]$ and $B_{n+1}(z) = [B_n(z); \alpha_{n+1}]$, then*
$$\mathrm{d}(A_{n+1}, B_{n+1}) \le (1 + |\alpha_{n+1}|)\mathrm{d}(A_n, B_n) \tag{3.2}$$
*where $\mathrm{d}(\,,\,)$ is the distance (3.1).*

*Proof.* The inequality claimed is a direct consequence of the properties of $L^2(|z| = 1)$ norm. In fact, applying forward recursion (2.4) to polynomials $A_n(z)$ and $B_n(z)$, we have
$$\begin{aligned} A_{n+1}(z) &= zA_n(z) + \alpha_{n+1}A_n^*(z) = zA_n(z) + \alpha_{n+1}z^n\overline{A_n(1/\overline{z})}, \\ B_{n+1}(z) &= zB_n(z) + \alpha_{n+1}B_n^*(z) = zB_n(z) + \alpha_{n+1}z^n\overline{B_n(1/\overline{z})}, \end{aligned}$$

and

$$\mathrm{d}(A_{n+1}, B_{n+1}) = \|A_{n+1} - B_{n+1}\| \leq \|z(A_n - B_n)\| + \|\alpha_{n+1}(A_n^* - B_n^*)\| \ .$$

Taking into account that $\|z(A_n - B_n)\| = \|A_n - B_n\| = \|A_n^* - B_n^*\|$ inequality (3.2) immediately follows and the proof of the lemma is complete. ∎

The proof of theorem 3.1 has a constructive character and it can be used to obtain polynomials in $\mathcal{C}[1]$ class by perturbing singular polynomials. The proof follows.

*Proof.* The result is clearly true if $A_n(z)$ belongs to $\mathcal{C}[1]$. Otherwise, $A_n(z)$ is a polynomial for which backward recursion (2.2) does not work, (i.e.,$|a_{j0}| = 1$ for some $j \leq n$, and $A_j(z)$ is non self–inversive). Without loss of generality we can assume that $j = n$. Let $A_n(z) = A_n^0(z)$ with $a_{n0} = \alpha_n^0$, such that $|\alpha_n^0| = 1$. In order to get $A_n^m(z)$, we will build up a finite sequence of polynomials

$$A_n^0(z), A_n^1(z), A_n^2(z), \cdots, A_n^m(z)\ ,\ 0 \leq m \leq n\ ,$$

which constant term are denoted by $a_{n0}^k = \alpha_n^k$.
Assume that $A_n^k(z)$ has been obtained. If $A_n^k(z) \in \mathcal{C}[1]$, then set $m = k$. Otherwise, backward recursion (2.2) can be carried out in order to obtain

$$A_n^k(z), A_{n-1}^k(z), \cdots, A_{n-j}^k(z),\ 0 \leq j \leq n-1\ ,$$

where $A_{n-j}^k(z)$ is non self–inversive, $|\alpha_{n-j}^k| = 1$ and $|\alpha_{n-s}^k| \neq 1, n-s > n-j$. Define a perturbed $\alpha_{n-j}^{k+1}$ such that $|\alpha_{n-j}^{k+1}| = 1 + \delta_{k+1}, \delta_{k+1} > 0$ and $\arg\ \alpha_{n-j}^{k+1} = \arg\ \alpha_{n-j}^k = \theta_k$, and the corresponding perturbation of $A_n^k(z)$ by

$$A_n^{k+1}(z) = \left[A_{n-j}^k(z) + \delta_{k+1}e^{i\theta_k}; \alpha_{n-j+1}^k, \alpha_{n-j+2}^k, \cdots, \alpha_n^k\right],$$

so that $\alpha_\ell^k = \alpha_\ell^{k+1}$ for $\ell = n-j+1, n-j+2, \cdots, n$.
Hence,

$$A_{n-j}^{k+1}(z) - A_{n-j}^k(z) = \alpha_{n-j}^{k+1} - \alpha_{n-j}^k = \delta_{k+1}e^{i\theta_k},$$

and using theorem 3.2,

$$\mathrm{d}^2(A_n^k, A_n^{k+1}) \leq \mathrm{d}^2(A_{n-j}^k, A_{n-j}^{k+1}) \prod_{s=0}^{j-1}(1 + |\alpha_{n-s}^k|)^2 = \delta_{k+1}^2 \prod_{s=0}^{j-1}(1 + |\alpha_{n-s}^{k+1}|)^2\ .$$

This procedure is finite because, for $A_n^{k+1}(z)$, we can carry out backward recursion (2.2) at least one more time than for $A_n^k(z)$, i.e., $m \leq n$.

The distance between $A_n^0(z)$ and $A_n^m(z)$ can be bounded as follows

$$\mathrm{d}^2(A_n^0, A_n^m) \leq \sum_{r=1}^{m} \mathrm{d}^2(A_n^{r-1}, A_n^r) \leq \sum_{r=1}^{m}\left[\delta_r^2 \prod_{s=0}^{j-1}(1 + |\alpha_{n-s}^r|)^2\right] \tag{3.3}$$

Since the considered $\delta_r{}'s$ can be taken as small as desired, we can choose these $\delta_r{}'s$ such that

$$\delta_r^2 \prod_{s=0}^{j-1}(1 + |\alpha_{n-s}^r|)^2 < \frac{\epsilon}{m}.$$

Therefore, $\mathrm{d}^2(A_n^0, A_n^m) < \epsilon$ and the proof of theorem 3.1 is complete. ∎

Observe that a simple perturbation of the last reflection coefficient (the independent term of the polynomial) allows us to step down backward Levinson recursion at least once for each perturbation, being the change of the $L^2(|z| = 1)$–norm of the polynomial explicitly bounded.

Schematically, the proof of theorem 3.1, can be summarized as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\alpha_n$ | $A_n^0(z)$ | $\alpha_n$ | $A_n^1(z)$ | $\alpha_n$ | $A_n^2(z)$ | $\cdots$ |
| $\alpha_{n-1}$ | $A_{n-1}^0(z)$ | $\alpha_{n-1}$ | $A_{n-1}^1(z)$ | $\alpha_{n-1}$ | $A_{n-1}^2(z)$ | $\cdots$ |
| $\alpha_{n-2}$ | $A_{n-2}^0(z)$ | $\alpha_{n-2}$ | $A_{n-2}^1(z)$ | $\alpha_{n-2}$ | $A_{n-2}^2(z)$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\alpha_{n-k}$ | $A_{n-k}^0(z)$ | $\alpha_{n-k}^1$ | $A_{n-k}^1(z)$ | $\alpha_{n-k}^1$ | $A_{n-k}^2(z)$ | $\cdots$ |
| Backward | | $\alpha_{n-k-1}^1$ | $A_{n-k-1}^1(z)$ | $\alpha_{n-k-1}^1$ | $A_{n-k-1}^2(z)$ | $\cdots$ |
| recursion | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| does not | | $\alpha_{n-j}^1$ | $A_{n-j}^1(z)$ | $\alpha_{n-j}^2$ | $A_{n-j}^2(z)$ | $\cdots$ |
| work | | | | $\vdots$ | $\vdots$ | $\vdots$ |
| | | | | | | $\vdots$ |
| | | | | | | $1$ |

where $A_n^0(z) = A_n(z)$, $|\alpha_{n-k}| = |\alpha_{n-j}^1| = 1$ and $\alpha_{n-k}^1 = \alpha_{n-k} + \delta_{n-k}e^{i\theta_{n-k}}$, $\alpha_{n-j}^2 = \alpha_{n-j}^1 + \delta_{n-j}e^{i\theta_{n-j}}$, and so on. That is to say, we get $A_n^m(z)$ by carrying out the following steps:

*Step 1::* Set $A_n^0(z) = A_n(z)$. If $|\alpha_n| = 1$ and $A_n^0(z)$ is non self-inversive, then $k = 0$, and go to Step 2. Otherwise, apply backward recursion until an unitary $\alpha_{n-k}, (0 < k \le n)$ appears and $A_{n-k}^n(z)$ be non self-inversive. If $k = n$, $A_n(z) \in \mathcal{C}[1]$, and we are done.

*Step 2::* Set $A_{n-k}^1(z) = A_{n-k}^0(z) + \delta_{n-k}e^{i\theta_{n-k}}$. Build up $A_n^1(z)$ from $A_{n-k}^1(z)$ by applying forward recursion with reflection coefficients $\alpha_{n-k+1}, \dots, \alpha_{n-1}, \alpha_n$. On the other hand, apply to $A_{n-k}^1(z)$ Step 1, increasing the superscript by one, until an unitary $\alpha_{n-j}^1, (k + 1 < j \le n)$ appears and $A_{n-j}^1(z)$ be non self-inversive. If $j < n$, then carry out Step 2 again increasing the superscript by one.

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