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# BOUNDEDNESS FOR VECTOR-VALUED MULTILINEAR SINGULAR INTEGRAL OPERATORS ON TRIEBEL-LIZORKIN SPACES

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ABSTRACT. In this paper, the boundedness for some vector-valued multilinear operators associated to certain fractional singular integral operators on Triebel-Lizorkin space are obtained. The operators include Calderón-Zygmund singular integral operator and fractional integral operator.

*Key words and phrases:* Vector-valued multilinear operators, Triebel-Lizorkin space, Lipschitz space, Calderón-Zygmund singular integral operator, Fractional integral operator.

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## 1. INTRODUCTION

Let $T$ be a Calderón-Zygmund singular integral operator, a well-known result of Coifman, Rochberg and Weiss (see [7]) states that the commutator $[b, T]f = T(bf) - bTf$ (where $b \in BMO$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when $T$ is replaced by the Riesz potentials. In [9], [10], these results on the Triebel-Lizorkin spaces and the case $b \in Lip\beta$ (where $Lip\beta$ is the homogeneous Lipschitz space) are obtained. The purpose of this paper is to prove the boundedness for some vector-valued multilinear operators associated to certain fractional singular integral operators on the Triebel-Lizorkin spaces. In fact, we shall establish the boundedness on the Triebel-Lizorkin spaces for the vector-valued multilinear operators only under certain conditions on the size of the operators. As applications, the boundedness for the vector-valued multilinear operators associated to the Calderón-Zygmund singular integral operator and fractional integral operator on the Triebel-Lizorkin spaces are obtained.

## 2. NOTATIONS AND RESULTS

Throughout this paper, $Q$ will denote a cube of $R^n$ with side parallel to the axes, and for a cube $Q$, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup\limits_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. For $1 \le p < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-p\delta/n}}\int_Q |f(y)|^p dy\right)^{1/p},$$

we denote $M_{\delta,r}(f) = M_r(f)$ if $\delta = 0$, which is the Hardy-Littlewood maximal function when $r = 1$. For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space, the Lipschitz space $\dot{\wedge}_\beta$ is the space of functions $f$ such that

$$||f||_{\dot{\wedge}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where $\Delta_h^k$ denotes the $k$-th difference operator (see [10]).

We are going to consider the fractional singular integral operator as follows.

**Definition 2.1.** Let $T : S \to S'$ be a linear operator and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$Tg(x) = \int_{R^n} K(x, y)g(y)dy$$

for every bounded and compactly supported function $g$. Let $m_j$ be the positive integers ($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $R^n$ ($j = 1, \cdots, l$). For $1 < r < \infty$, the vector-valued multilinear operator associated to $T$ is defined by

$$|T_A(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T_A(f_i)(x)|^r\right)^{1/r},$$

where

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x - y|^m} K(x, y)f_i(y)dy$$

and

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha.$$

We also denote that

$$|T(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^r\right)^{1/r} \quad \text{and} \quad |f|_r = \left(\sum_{i=1}^{\infty} |f_i(x)|^r\right)^{1/r}.$$

Note that when $m = 0$, $T_A$ is just the vector-valued commutator of $T$ and $A$ (see[11]). While when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2]-[6], [8]). The purpose of this paper is to study the boundedness for the multilinear operator on Triebel-Lizorkin spaces. We shall prove the following theorem in Section 3.

**Theorem 2.1.** *Let $1 < r < \infty$, $0 < \beta < 1/l$ and $D^\alpha A_j \in \dot{\Lambda}_\beta$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Suppose $T$ is the fractional singular integral operator as Definition such that $|T|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $0 \le \delta < n$, $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$. If $T$ satisfies the following size condition:*

$$|T_A(f)(x) - T_A(f)(x_0)|_r \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m} ||D^{\alpha_j} A_j||_{\dot{\Lambda}_\beta}\right) |Q|^{l\beta/n} M_{\delta,s}(|f|_r)(x)$$

*for any cube $Q = Q(x_0, l)$ with $f = \{f_i\}$, $supp f_i \subset (2Q)^c$, $x \in Q$ and some $1 \le s < \infty$. Then*

*(a). $|T_A|_r$ is bounded from $L^p(R^n)$ to $\dot{F}_q^{l\beta,\infty}(R^n)$ for $0 \le \delta < n$, $1 < p < n/\delta$, $1/p - 1/q = \delta/n$;*

*(b). $|T_A|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $0 \le \delta < n - l\beta$, $1 < p < n/(\delta + l\beta)$ and $1/p - 1/q = (\delta + l\beta)/n$.*

From the theorem, we get the following

**Corollary 2.2.** *Let $1 < r < \infty$, $0 \le \delta < n$, $\varepsilon > 0$, $0 < \beta < \min(1/l, \varepsilon/l)$ and $D^\alpha A_j \in \dot{\Lambda}_\beta$ for all $\alpha$ with $|\alpha| = m$ and $j = 1, \cdots, l$. Suppose $K$ is a locally integrable function on $R^n \times R^n$ satisfies*

$$|K(x, y)| \le C|x - y|^{-n+\delta}$$

*and*

$$|K(y, x) - K(z, x)| \le C|y - z|^\varepsilon |x - z|^{-n-\varepsilon+\delta}$$

*if $2|y - z| \le |x - z|$. Denote*

$$T(f_i)(x) = \int_{R^n} K(x, y) f_i(y) dy$$

*and*

$$|T_A(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T_A(f_i)(x)|^r\right)^{1/r},$$

*where*

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x - y|^m} K(x, y) f_i(y) dy$$

*for every bounded and compactly supported function $f_i$ and $x \in (supp f_i)^c$. Suppose $|T|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $0 \le \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then*

*(a). $|T_A|_r$ is bounded from $L^p(R^n)$ to $\dot{F}_q^{l\beta,\infty}(R^n)$ for $0 \le \delta < n$, $1 < p < n/\delta$, $1/p - 1/q = \delta/n$;*

*(b). $|T_A|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $0 \le \delta < n - l\beta$, $1 < p < n/(\delta + l\beta)$ and $1/p - 1/q = (\delta + l\beta)/n$.*

## 3. PROOF OF THEOREM 2.1

To prove the theorem, we need the following lemmas.

**Lemma 3.1.** (see [10]) *For* $0 < \beta < 1$, $1 < p < \infty$, *we have*

$$||f||_{\dot{F}_p^{\beta,\infty}} \approx \left|\left|\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|dx\right|\right|_{L^p} \approx \left|\left|\sup_{\in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c|dx\right|\right|_{L^p}.$$

**Lemma 3.2.** (see [10]) *For* $0 < \beta < 1, 1 \le p \le \infty$, *we have*

$$||b||_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q|dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx\right)^{1/p}.$$

**Lemma 3.3.** (see [10]) *Let* $b \in \dot{\Lambda}_\beta$ *and* $1 < s < \infty$, *then*

$$||(b - b_Q)f\chi_Q||_{L^s} \le C||b||_{\dot{\Lambda}_\beta}|Q|^{1/s+\beta/n-\delta/n}M_{\delta,s}(f).$$

**Lemma 3.4.** (see [1]) *Suppose that* $1 < r < \infty$, $1 \le s < p < n/\delta$ *and* $1/q = 1/p - \delta/n$. *Then*

$$||M_{\delta,s}(|f|_r)||_{L^q} \le C|||f|_r||_{L^p}.$$

**Lemma 3.5.** (see [5]) *Let* $A$ *be a function on* $R^n$ *and* $D^\alpha A \in L^q(R^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

*where* $\tilde{Q}(x, y)$ *is the cube centered at* $x$ *and having side length* $5\sqrt{n}|x - y|$.

*Proof of Theorem 2.1.* We first prove a sharp estimate for $|T_A(f)|_r$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ with $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
T_A(f_i)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x - y|^m} K(x, y)f_i(y)dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x - y|^m} K(x, y)h_i(y)dy \\
&+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x - y|^m} K(x, y)g_i(y)dy \\
&- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\alpha_1}}{|x - y|^m} D^{\alpha_1}\tilde{A}_1(y)K(x, y)g_i(y)dy \\
&- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x - y)^{\alpha_2}}{|x - y|^m} D^{\alpha_2}\tilde{A}_2(y)K(x, y)g_i(y)dy \\
&+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x - y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x - y|^m} K(x, y)g_i(y)dy,
\end{aligned}$$

then, by Minkowski' inequality,

$$\begin{aligned}
& \frac{1}{|Q|^{1+2\beta/n}}\int_Q \left||T_A(f)(x)|_r - |T_{\tilde{A}}(h)(x_0)|_r\right| dx \\
\leq & \frac{1}{|Q|^{1+2\beta/n}}\int_Q \left(\sum_{i=1}^{\infty}|T_A(f_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^r\right)^{1/r} dx \\
\leq & \frac{1}{|Q|^{1+2\beta/n}}\int_Q \left(\sum_{i=1}^{\infty}\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy\right|^r\right)^{1/r} dx \\
+ & \frac{C}{|Q|^{1+2\beta/n}} \\
\times & \int_Q \left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy\right|^r\right)^{1/r} dx \\
+ & \frac{C}{|Q|^{1+2\beta/n}} \\
\times & \int_Q \left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy\right|^r\right)^{1/r} dx \\
+ & \frac{C}{|Q|^{1+2\beta/n}} \\
\times & \int_Q \left(\sum_{i=1}^{\infty}\left|\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy\right|^r\right)^{1/r} dx \\
+ & \frac{1}{|Q|^{1+2\beta/n}}\int_Q \left(\sum_{i=1}^{\infty}|T_{\tilde{A}}(h_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^r\right)^{1/r} dx \\
:= & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now, let us estimate $I_1$, $I_2$, $I_3$, $I_4$ and $I_5$, respectively. For $x \in Q$ and $y \in \tilde{Q}$, using Lemma 3.5 and Lemma 3.2, we get

$$\begin{aligned}
& |R_m(\tilde{A}_j;x,y)| \\
\leq & C|x-y|^m \sum_{|\alpha|=m}\sup_{x\in\tilde{Q}}|D^\alpha A_j(x) - (D^\alpha A_j)_{\tilde{Q}}| \\
\leq & C|x-y|^m|Q|^{\beta/n}\sum_{|\alpha|=m}||D^\alpha A_j||_{\dot{\Lambda}_\beta},
\end{aligned}$$

thus, by the $(L^s, L^q)$-boundedness of $|T|_r$ with $1 < s < n/\delta$ and $1/q = 1/s - \delta/n$, we obtain

$$I_1 \leq C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\frac{1}{|Q|}\int_Q |T(g)(x)|_r dx$$

$$
\begin{aligned}
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\left(\frac{1}{|Q|}\int_Q|T(g)(x)|_r^q dx\right)^{1/q}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{-1/q}\left(\int_{\tilde{Q}}|f(x)|_r^s dx\right)^{1/s}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\left(\frac{1}{|\tilde{Q}|^{1-s\delta/n}}\int_{\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}
$$

For $I_2$, using Lemma 3.3, we get, for $1 < s < n/\delta$ and $1/q = 1/s - \delta/n$,

$$
\begin{aligned}
I_2 &\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{\dot{\Lambda}_\beta}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|^{1+\beta/n}}\int_Q|T(D^{\alpha_1}\tilde{A}_1 g)(x)|_r dx\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{\dot{\Lambda}_\beta}\sum_{|\alpha_1|=m_1}|||T((D^{\alpha_1}A-(D^{\alpha_1}A)_{\tilde{Q}})g)|_r||_{L^q}|Q|^{-\beta/n-1/q}\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{\dot{\Lambda}_\beta}|Q|^{-\beta/n-1/q}\sum_{|\alpha_1|=m}||(D^{\alpha_1}A-(D^{\alpha_1}A)_{\tilde{Q}})|g|_r||_{L^s}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}
$$

For $I_3$, similar to the proof of $I_2$, we get

$$
I_3 \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{\delta,s}(|f|_r)(\tilde{x}).
$$

Similarly, for $I_4$, denoting $s = pq_3$ for $1 < p < n/\delta$, $q_1, q_2, q_3 > 1$, $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $1/t = 1/p - \delta/n$, we obtain

$$
\begin{aligned}
I_4 &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{|Q|^{1+2\beta/n}}\int_Q|T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r dx\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-2\beta/n-1/t}\left(\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r^t dx\right)^{1/t}\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-2\beta/n-1/t}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)||g(x)|_r^p dx\right)^{1/p}
\end{aligned}
$$

$$\begin{aligned}
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{-2\beta/n}\left(\frac{1}{|\tilde{Q}|^{1-s\delta/n}}\int_{\tilde{Q}}|f(x)|_r^{pq_3}dx\right)^{1/pq_3} \\
&\times \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(x)|^{pq_1}dx\right)^{1/pq_1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(x)|^{pq_2}dx\right)^{1/pq_2} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

For $I_5$, using the size condition of $T$, we have

$$I_5 \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{\delta,s}(f)(\tilde{x}).$$

We now put these estimates together, and taking the supremum over all $Q$ such that $\tilde{x}\in Q$, and using Lemma 3.1 and 3.4, we obtain

$$|||T_A(f)|_r||_{\dot{F}_q^{2\beta,\infty}} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|||f|_r||_{L^p}.$$

This completes the proof of $(a)$.

For $(b)$, by the same argument as in proof of $(a)$, we have

$$\frac{1}{|Q|}\int_Q ||T_A(f)(x)|_r - |T_{\tilde{A}}(h)(x_0)|_r|\,dx \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{\delta+2\beta,s}(|f|_r),$$

thus, we get the sharp estimate of $T_A$ as following

$$(|T_A(f)|_r)^{\#} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)M_{\delta+2\beta,s}(|f|_r).$$

Now, using Lemma 3.4, we get

$$\begin{aligned}
|||T_A(f)|_r||_{L^q} &\leq C||(|T_A(f)|_r)^{\#}||_{L^q} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)||M_{\delta+2\beta,r}(|f|_r)||_{L^q} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|||f|_r||_{L^p}.
\end{aligned}$$

This completes the proof of $(b)$ and the theorem. ∎

*Proof of Corollary 2.2.* It suffices to verify that $T$ satisfies the size condition in Theorem 2.1.

Suppose $f = \{f_i\}$ with $suppf_i \subset (\tilde{Q})^c$ and $x \in Q = Q(x_0, l)$. We write

$$\begin{aligned}
& T_{\tilde{A}}(f_i)(x) - T_{\tilde{A}}(f_i)(x_0) \\
= & \int_{R^n} \left( \frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) f_i(y) dy \\
+ & \int_{R^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^m} K(x_0,y) f_i(y) dy \\
+ & \int_{R^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^m} K(x_0,y) f_i(y) dy \\
- & \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[ \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,y) - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0,y) \right] \\
\times & D^{\alpha_1} \tilde{A}_1(y) f_i(y) dy \\
- & \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[ \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y) - \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0,y) \right] \\
\times & D^{\alpha_2} \tilde{A}_2(y) f_i(y) dy \\
+ & \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[ \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0,y) \right] \\
\times & D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) f_i(y) dy \\
= & J_i^{(1)} + J_i^{(2)} + J_i^{(3)} + J_i^{(4)} + J_i^{(5)} + J_i^{(6)}.
\end{aligned}$$

By Lemma 3.5 and the following inequality, for $b \in \dot{\wedge}_\beta$,

$$|b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q ||b||_{\dot{\wedge}_\beta} |x-y|^\beta dy \leq ||b||_{\dot{\wedge}_\beta} (|x-x_0| + d)^\beta,$$

we get

$$|R_{m_j}(\tilde{A}_j; x, y)| \leq \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{\dot{\wedge}_\beta} (|x-y| + d)^{m_j+\beta}.$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition of $K$,

$$\begin{aligned}
|J_i^{(1)}| \leq & C \int_{R^n \setminus \tilde{Q}} \left( \frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j; x, y)| |f_i(y)| dy \\
\leq & C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\dot{\wedge}_\beta} \right) \\
\times & \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \left( \frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta-2\beta}} \right) |f_i(y)| dy
\end{aligned}$$

$$\begin{aligned} &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}(2^{k(2\beta-1)}+2^{k(2\beta-\varepsilon)}) \\ &\times \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f_i(y)|dy, \end{aligned}$$

thus, by Minkowski' inequality, for $1 \leq s < \infty$,

$$\begin{aligned} &\left(\sum_{i=1}^{\infty}|J_i^{(1)}|^r\right)^{1/r} \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}(2^{k(2\beta-1)}+2^{k(2\beta-\varepsilon)}) \\ &\times \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f(y)|_r dy \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}(2^{k(2\beta-1)}+2^{k(2\beta-\varepsilon)}) \\ &\times \left(\frac{1}{|2^k\tilde{Q}|^{1-s\delta/n}}\int_{2^k\tilde{Q}}|f(y)|_r^s dy\right)^{1/s} \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,s}(|f|_r)(x). \end{aligned}$$

For $J_i^{(2)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{A}_j;x,y)-R_{m_j}(\tilde{A}_j;x_0,y)=\sum_{|\eta|<m_j}\frac{1}{\eta!}R_{m_j-|\eta|}(D^\eta\tilde{A}_j;x,x_0)(x-y)^\eta$$

and Lemma 3.5, we get, for $1 \leq s < \infty$,

$$\begin{aligned} \left(\sum_{i=1}^{\infty}\left|J_i^{(2)}\right|^r\right)^{1/r} &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}}|f(y)|_r dy \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,s}(|f|_r)(\tilde{x}). \end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|J_i^{(3)}\right|^r\right)^{1/r} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,s}(|f|_r)(\tilde{x}).$$

For $J_i^{(4)}$, similar to the estimates of $J_i^{(1)}$ and $J_i^{(2)}$, we obtain, for $1 \leq s < \infty$,

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}\left|J_i^{(4)}\right|^r\right)^{1/r}\\
\leq\ & C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|\\
\times\ & |R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f(y)|_r dy\\
+\ & C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\\
\times\ & \frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_r dy\\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}(2^{k(2\beta-1)}+2^{k(2\beta-\varepsilon)})\\
\times\ & \left(\frac{1}{|2^k\tilde{Q}|^{1-s\delta/n}}\int_{2^k\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|J_i^{(5)}\right|^r\right)^{1/r}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,s}(|f|_r)(\tilde{x}).$$

For $J_i^{(6)}$, we get

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}\left|J_i^{(6)}\right|^r\right)^{1/r}\\
\leq\ & C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\right|\\
\times\ & |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_r dy\\
\leq\ & C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)\\
\times\ & |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_r dy\\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)\\
\times\ & \sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta-2\beta}}\right)|f(y)|_r dy
\end{aligned}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,s}(|f|_r)(x).$$

Thus

$$|T_A(f)(x)-T_A(f)(x_0)|_r\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{2\beta/n}M_{\delta,s}(|f|_r)(x).$$

This completes the proof of the corollary. ∎

## 4. APPLICATIONS

In this section we shall apply the Theorem and Corollary to some particular operators such as the Calderón-Zygmund singular integral operator and fractional integral operator. Let $1<r<\infty$, $D^\alpha A_j\in\dot{\Lambda}_\beta$ for $|\alpha|=m$ and $j=1,\cdots,l$.

**Application 1.** *Calderón-Zygmund singular integral operator.*

Let $T$ be the Calderón-Zygmund operator(see[6], [12]), the vector-valued multilinear operator related to $T$ is defined by

$$|T_A(f)(x)|_r=\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)|^r\right)^{1/r},$$

where

$$T_A(f_i)(x)=\int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy.$$

Then it is easily to see that $T$ satisfies the conditions in Corollary 2.2 with $\delta=0$, thus $T_A$ is bounded from $L^p(R^n)$ to $\dot{F}_p^{l\beta,\infty}(R^n)$ for $0<\beta<1/l$, $1<p<\infty$ and from $L^p(R^n)$ to $L^q(R^n)$ for $0<\beta<1/l$, $1<p<n/(l\beta)$ and $1/p-1/q=l\beta/n$.

**Application 2.** *Fractional integral operator with rough kernel.*

For $0<\delta<n$, let $T_\delta$ be the fractional integral operator with rough kernel defined by(see[8], [12])

$$T^\delta g(x)=\int_{R^n}\frac{\Omega(x-y)}{|x-y|^{n-\delta}}g(y)dy,$$

the vector-valued multilinear operator related to $T_\delta$ is defined by

$$|T_A^\delta(f)(x)|_r=\left(\sum_{i=1}^{\infty}|T_A^\delta(f_i)(x)|^r\right)^{1/r},$$

where

$$T_A^\delta(f_i)(x)=\int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^{m+n-\delta}}\Omega(x-y)f_i(y)dy$$

and $\Omega$ is homogeneous of degree zero on $R^n$, $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$ and $\Omega\in Lip_\varepsilon(S^{n-1})$ for some $0<\varepsilon\leq 1$, that is there exists a constant $L>0$ such that for any $x,y\in S^{n-1}$, $|\Omega(x)-\Omega(y)|\leq L|x-y|^\varepsilon$. When $\Omega\equiv 1$, $T^\delta$ is the Riesz potentials. Then $T_\delta$ satisfies the conditions in Corollary. Thus $T_\delta^A$ is bounded from $L^p(R^n)$ to $\dot{F}_q^{l\beta,\infty}(R^n)$ for $0<\beta<\min(1/l,\varepsilon/l)$, $1<p<n/\delta$, $1/p-1/q=\delta/n$ and from $L^p(R^n)$ to $L^q(R^n)$ for $0<\delta<n-l\beta$, $0<\beta<\min(1/l,\varepsilon/l)$, $1<p<n/(\delta+l\beta)$ and $1/p-1/q=(\delta+l\beta)/n$.

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