ABSTRACT. In this paper, the boundedness for some vector-valued multilinear operators associated to certain fractional singular integral operators on Triebel-Lizorkin space are obtained. The operators include Calderón-Zygmund singular integral operator and fractional integral operator.

Key words and phrases: Vector-valued multilinear operators, Triebel-Lizorkin space, Lipschitz space, Calderón-Zygmund singular integral operator, Fractional integral operator.

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1. Introduction

Let $T$ be a Calderón-Zygmund singular integral operator, a well-known result of Coifman, Rochberg and Weiss (see [7]) states that the commutator $[b,T]f = T(bf) - bTf$ (where $b \in BMO$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when $T$ is replaced by the Riesz potentials. In [9], [10], these results on the Triebel-Lizorkin spaces and the case $b \in Lip(\beta)$ (where $Lip(\beta)$ is the homogeneous Lipschitz space) are obtained. The purpose of this paper is to prove the boundedness for some vector-valued multilinear operators associated to certain fractional singular integral operators on the Triebel-Lizorkin spaces. In fact, we shall establish the boundedness on the Triebel-Lizorkin spaces for the vector-valued multilinear operators only under certain conditions on the size of the operators. As applications, the boundedness for the vector-valued multilinear operators associated to the Calderón-Zygmund singular integral operator and fractional integral operator on the Triebel-Lizorkin spaces are obtained.

2. Notations and Results

Throughout this paper, $Q$ will denote a cube of $\mathbb{R}^n$ with side parallel to the axes, and for a cube $Q$, let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|dy$. For $1 \leq p < \infty$ and $0 \leq \delta < n$, let $$M_{\delta,p}(f)(x) = \sup_{x \in Q} \left( \frac{1}{|Q|^{1 - \delta_p/n}} \int_Q |f(y)|^p dy \right)^{1/p},$$ we denote $M_{\delta,r}(f) = M_{r}(f)$ if $\delta = 0$, which is the Hardy-Littlewood maximal function when $r = 1$. For $\beta > 0$ and $p > 1$, let $F_{\beta,\infty}^p$ be the homogeneous Triebel-Lizorkin space, the Lipschitz space $\Lambda_\beta$ is the space of functions $f$ such that $$||f||_{\Lambda_\beta} = \sup_{x,h \in \mathbb{R}^n \atop h \neq 0} \left| \Delta_h^{[\beta]+1} f(x) \right| / |h|^\beta < \infty,$$ where $\Delta_h^k$ denotes the $k$-th difference operator (see [10]).

We are going to consider the fractional singular integral operator as follows.

Definition 2.1. Let $T : S \to S'$ be a linear operator and there exists a locally integrable function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ such that $$Tg(x) = \int_{\mathbb{R}^n} K(x,y)g(y)dy$$ for every bounded and compactly supported function $g$. Let $m_j$ be the positive integers ($j = 1, \ldots, l$), $m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $\mathbb{R}^m$ ($j = 1, \ldots, l$). For $1 < r < \infty$, the vector-valued multilinear operator associated to $T$ is defined by $$|T_A(f)(x)|_r = \left( \sum_{i=1}^{\infty} |T_A(f_i)(x)|_r^r \right)^{1/r},$$ where $$T_A(f_i)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l R_{m_j+1}(A_j; x, y) K(x,y)f_i(y)dy$$ and $$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$
We also denote that
\[
|T(f)(x)|_r = \left( \sum_{i=1}^{\infty} |T(f_i)(x)|^r \right)^{1/r}
\]
and
\[
|f|_r = \left( \sum_{i=1}^{\infty} |f_i(x)|^r \right)^{1/r}.
\]

Note that when \(m = 0\), \(T_A\) is just the vector-valued commutator of \(T\) and \(A\) (see [11]). While when \(m > 0\), it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2]-[6], [8]). The purpose of this paper is to study the boundedness for the multilinear operator on Triebel-Lizorkin spaces. We shall prove the following theorem in Section 3.

**Theorem 2.1.** Let \(1 < r < \infty, 0 < \beta < 1/l\) and \(D^\alpha A_j \in \mathring{\Lambda}_\beta\) for all \(\alpha\) with \(|\alpha| = m_j\) and \(j = 1, \cdots, l\). Suppose \(T\) is the fractional singular integral operator as Definition such that \(|T|_r\) is bounded from \(L^p(R^n)\) to \(L^q(R^n)\) for \(0 \leq \delta < n, 1 < p < n/\delta\) and \(1/p - 1/q = \delta/n\). If \(T\) satisfies the following size condition:

\[
|T_A(f)(x) - T_A(f)(x_0)| \leq C \prod_{j=1}^{l} \left( \sum_{|\alpha_j| = m} ||D^{\alpha_j} A_j||_{\Lambda_\beta} \right) |Q|^{\beta/n} M_{\alpha, \delta}(|f|_r)(x)
\]

for any cube \(Q = Q(x_0, l)\) with \(f = \{f_i\}, \text{supp} f_i \subset (2Q)^c, x \in Q\) and some \(1 \leq s < \infty\). Then

(\(a\). \(T_A\) is bounded from \(L_r\) to \(F^{1,\infty}_q(R^n)\) for \(0 \leq \delta < n, 1 < p < n/\delta, 1/p - 1/q = \delta/n\);

(\(b\). \(T_A\) is bounded from \(L^p(R^n)\) to \(L^q(R^n)\) for \(0 \leq \delta < n - l\beta, 1 < p < n/(\delta + l\beta)\) and \(1/p - 1/q = (\delta + l\beta)/n\).

From the theorem, we get the following

**Corollary 2.2.** Let \(1 < r < \infty, 0 \leq \delta < n, \varepsilon > 0, 0 < \beta < \min(1/l, \varepsilon/l)\) and \(D^\alpha A_j \in \mathring{\Lambda}_\beta\) for all \(\alpha\) with \(|\alpha| = m\) and \(j = 1, \cdots, l\). Suppose \(K\) is a locally integrable function on \(R^n \times R^n\) satisfies

\[
|K(x, y)| \leq C|x - y|^{-n+\delta}
\]

and

\[
|K(y, x) - K(z, x)| \leq C|y - z|^{\varepsilon}|x - z|^{-n-\varepsilon+\delta}
\]

if \(2|y - z| \leq |x - z|\). Denote

\[
T(f_i)(x) = \int_{R^n} K(x, y)f_i(y)dy
\]

and

\[
|T_A(f)(x)| = \left( \sum_{i=1}^{\infty} |T_A(f_i)(x)|^r \right)^{1/r},
\]

where

\[
T_A(f_i)(x) = \int_{R^n} \prod_{j=1}^{l} \frac{R_{m_j+1}(A_j; x, y)}{|x - y|^m} K(x, y)f_i(y)dy
\]

for every bounded and compactly supported function \(f_i\), \(x \in (\text{supp} f_i)^c\). Suppose \(|T|_r\) is bounded from \(L^p(R^n)\) to \(L^q(R^n)\) for \(0 \leq \delta < n, 1 < p < n/\delta\) and \(1/q = 1/p - \delta/n\). Then

(\(a\). \(T_A\) is bounded from \(L^p(R^n)\) to \(F^{1,\infty}_q(R^n)\) for \(0 \leq \delta < n, 1 < p < n/\delta, 1/p - 1/q = \delta/n\);

(\(b\). \(T_A\) is bounded from \(L^p(R^n)\) to \(L^q(R^n)\) for \(0 \leq \delta < n - l\beta, 1 < p < n/(\delta + l\beta)\) and \(1/p - 1/q = (\delta + l\beta)/n\).

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3. **Proof of Theorem 2.1**

To prove the theorem, we need the following lemmas. 

**Lemma 3.1.** (see [10]) For $0 < \beta < 1$, $1 < p < \infty$, we have 

$$
\|f\|_{L^p} \approx \left\| \frac{1}{|Q|^{\frac{1}{1+\beta/n}}} \int_Q |f(x) - f_Q|dx \right\| \approx \left\| \sup_{c \in Q} \frac{1}{|Q|^{\frac{1}{1+\beta/n}}} \int_Q |f(x) - c|dx \right\|_{L^p}.
$$

**Lemma 3.2.** (see [10]) For $0 < \beta < 1$, $1 \leq p \leq \infty$, we have 

$$
\|b\|_{\delta, \beta} \approx \sup_Q \frac{1}{|Q|^{\frac{1}{1+\beta/n}}} \int_Q |b(x) - b_Q|dx \approx \sup_Q \frac{1}{|Q|^{\frac{1}{1+\beta/n}}} \left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p}.
$$

**Lemma 3.3.** (see [10]) Let $b \in \lambda_\beta$ and $1 < s < \infty$, then 

$$
\|(b - b_Q)f\|_{L^s} \leq C\|b\|_{\lambda_\beta} |Q|^{1/s+\beta/n-\delta/n} M_{\delta,s}(f).
$$

**Lemma 3.4.** (see [1]) Suppose that $1 < r < \infty$, $1 \leq s < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then 

$$
\|M_{\delta,s}(f)\|_{L^q} \leq C\|f\|_{L^p}.
$$

**Lemma 3.5.** (see [5]) Let $A$ be a function on $R^n$ and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then 

$$
|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{|Q(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},
$$

where $\tilde{Q}(x, y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$. 

**Proof of Theorem 2.1** We first prove a sharp estimate for $|T_A(f)|$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\bar{x} \in Q$. Let $\bar{Q} = 5\sqrt{n}Q$ and $\bar{A}_j(x) = A_j(x) - \frac{1}{\alpha_1} (D^\alpha A_j)_{\bar{Q}} x^\alpha$, then $R_m(A_j; x, y) = R_m(\bar{A}_j; x, y)$ and $D^\alpha A_j(\bar{Q})$. Write $f = g + h = \{g_i\} + \{h_i\}$ with $g_i = f_i{\chi}_{\bar{Q}}$ and $h_i = f_i{\chi}_{R^n \setminus \bar{Q}}$. Write 

$$
T_A(f_i)(x) = \int_{R^n} \prod_{j=1}^{m_2} R_{m_2 + 1}(\bar{A}_j; x, y) K(x, y)f_i(y)dy
$$

$$
= \int_{R^n} \prod_{j=1}^{m_2} R_{m_2 + 1}(\bar{A}_j; x, y) K(x, y)h_i(y)dy
$$

$$
+ \int_{R^n} \prod_{j=1}^{m_1} R_{m_1}(\bar{A}_j; x, y) K(x, y)g_i(y)dy
$$

$$
- \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \int_{R^n} R_{m_2 + 1}(\bar{A}_2; x, y)(x - y)^{\alpha_1} D^{\alpha_1} \bar{A}_1(y) K(x, y)g_i(y)dy
$$

$$
- \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \int_{R^n} R_{m_1}(\bar{A}_1; x, y)(x - y)^{\alpha_2} D^{\alpha_2} \bar{A}_2(y) K(x, y)g_i(y)dy
$$

$$
+ \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} (x - y)^{\alpha_1 + \alpha_2} D^{\alpha_1} \bar{A}_1(y) D^{\alpha_2} \bar{A}_2(y) K(x, y)g_i(y)dy.
$$
then, by Minkowski’ inequality,
\[ \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left| |T_\beta(f)(x)|_r - |T_\beta(h)(x_0)|_r \right| dx \]
\[ \leq \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left( \sum_{i=1}^\infty |T_\beta(f_i)(x) - T_\beta(h_i)(x)|^r \right)^{1/r} dx \]
\[ \leq \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left( \sum_{i=1}^\infty \left( \int_{R^n} \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) \right) \frac{1}{|x - y|^m} K(x, y) g_i(y) dy \right)^{1/r} dx \]
\[ + \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left( \sum_{i=1}^\infty \sum_{|\alpha_1| = m_1} R_{m_2}(\tilde{A}_2; x, y) (x - y)^{\alpha_1} D^{\alpha_1} \tilde{A}_1(y) K(x, y) g_i(y) dy \right)^{1/r} dx \]
\[ + \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left( \sum_{i=1}^\infty \sum_{|\alpha_2| = m_2} R_{m_3}(\tilde{A}_3; x, y) (x - y)^{\alpha_2} D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) dy \right)^{1/r} dx \]
\[ + \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left( \sum_{|\alpha_1| = m_1} \sum_{|\alpha_2| = m_2} \int_{R^n} (x - y)^{\alpha_1 + \alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) dy \right)^{1/r} dx \]
\[ + \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left( \sum_{i=1}^\infty |T_\beta(h_i)(x) - T_\beta(h_i)(x_0)|^r \right)^{1/r} dx \]
\[ := I_1 + I_2 + I_3 + I_4 + I_5. \]

Now, let us estimate \( I_1, I_2, I_3, I_4 \) and \( I_5 \), respectively. For \( x \in Q \) and \( y \in \tilde{Q} \), using Lemma 3.5 and Lemma 3.2, we get
\[ |R_m(\tilde{A}_j; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \sup_{x \in Q} |D^\alpha A_j(x) - (D^\alpha A_j)|_Q \]
\[ \leq C|x - y|^m Q^{\beta/n} \sum_{|\alpha| = m} ||D^\alpha A_j||_{L_\beta}, \]
thus, by the \((L^s, L^q)\)-boundedness of \(|T|_r\) with \( 1 < s < n/\delta \) and \( 1/q = 1/s - \delta/n \), we obtain
\[ I_1 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{L_\beta} \right) \frac{1}{|Q|} \int_Q |T(g)(x)|_r dx \]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\lambda_\beta} \right) \left( \frac{1}{|Q|} \int_Q |T(g)(x)|^q dx \right)^{1/q}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\lambda_\beta} \right) |Q|^{-1/q} \left( \int_Q |f(x)|^s dx \right)^{1/s}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\lambda_\beta} \right) \left( \frac{1}{|Q|^{1-s\delta/n}} \int_Q |f(y)|^s dy \right)^{1/s}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\lambda_\beta} \right) M_{\delta,s}(\|f\|_r)(\mathfrak{x}).
\]

For \(I_2\), using Lemma 3.3, we get, for \(1 < s < n/\delta\) and \(1/q = 1/s - \delta/n\),
\[
I_2 \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\lambda_\beta} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T(D^{\alpha_1} \check{A}_1 g)(x)|_r dx
\]
\[
\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\lambda_\beta} \sum_{|\alpha_1|=m_1} \|T((D^{\alpha_1} A - (D^{\alpha_1} A)_{\check{Q}}) g)|_r \|_{L^s} |Q|^{-\beta/n-1/q}
\]
\[
\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\lambda_\beta} |Q|^{-\beta/n-1/q} \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} A - (D^{\alpha_1} A)_{\check{Q}}|g|_r \|_{L^s}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\lambda_\beta} \right) M_{\delta,s}(\|f\|_r)(\mathfrak{x}).
\]

For \(I_3\), similar to the proof of \(I_2\), we get
\[
I_3 \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\lambda_\beta} \right) M_{\delta,s}(\|f\|_r)(\mathfrak{x}).
\]

Similarly, for \(I_4\), denoting \(s = pq_3\) for \(1 < p < n/\delta, q_1, q_2, q_3 > 1, 1/q_1 + 1/q_2 + 1/q_3 = 1\) and \(1/t = 1/p - \delta/n\), we obtain
\[
I_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|^{1+2\beta/n}} \int_Q |T(D^{\alpha_1} \check{A}_1 D^{\alpha_2} \check{A}_2 g)(x)|_r dx
\]
\[
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n-1/t} \left( \int_{R^n} |T(D^{\alpha_1} \check{A}_1 D^{\alpha_2} \check{A}_2 g)(x)|_r^t dx \right)^{1/t}
\]
\[
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n-1/t} \left( \int_{R^n} |D^{\alpha_1} \check{A}_1(x)D^{\alpha_2} \check{A}_2(x)||g(x)||^t_p dx \right)^{1/p}
\]
\[
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n} \left( \frac{1}{|Q|^{1-s\delta/n}} \int_Q |f(x)|^{pq_1} \, dx \right)^{1/pq_1} \times \left( \frac{1}{|Q|} \int_Q |D^{\alpha_1} \hat{A}_1(x)|^{pq_1} \, dx \right)^{1/pq_1} \left( \frac{1}{|Q|} \int_Q |D^{\alpha_2} \hat{A}_2(x)|^{pq_2} \, dx \right)^{1/pq_2} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\Lambda_{\beta}} \right) M_{\delta,s}(|f|_r)(\hat{x}).
\]

For \( I_5 \), using the size condition of \( T \), we have
\[
I_5 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\Lambda_{\beta}} \right) M_{\delta,s}(f)(\hat{x}).
\]

We now put these estimates together, and taking the supremum over all \( Q \) such that \( \hat{x} \in Q \), and using Lemma 3.7 and 3.4, we obtain
\[
|||T_A(f)|_r||_{F_{q,\infty}^{2\beta}} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\Lambda_{\beta}} \right) |||f|||_{L^p}.
\]

This completes the proof of (a).

For (b), by the same argument as in proof of (a), we have
\[
\frac{1}{|Q|} \int_Q ||T_A(f)(x)|_r - |T_A(h)(x_0)|_r| \, dx \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\Lambda_{\beta}} \right) M_{\delta+2\beta,s}(|f|_r),
\]

thus, we get the sharp estimate of \( T_A \) as following
\[
(|T_A(f)|_r)^\# \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\Lambda_{\beta}} \right) M_{\delta+2\beta,s}(|f|_r).
\]

Now, using Lemma 3.4 we get
\[
|||T_A(f)|_r||_{L^s} \leq C ||(||T_A(f)|_r)||^\#_{L^s} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\Lambda_{\beta}} \right) ||M_{\delta+2\beta,r}(|f|_r)||_{L^s} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{\Lambda_{\beta}} \right) ||f||_{L^p}.
\]

This completes the proof of (b) and the theorem. □

**Proof of Corollary 2.2** It suffices to verify that \( T \) satisfies the size condition in Theorem 2.1.
Suppose \( f = \{ f_i \} \) with \( \text{supp} f_i \subset \tilde{Q}^c \) and \( x \in Q = Q(x_0, l) \). We write

\[
T_\beta(f_i)(x) - T_\beta(f_i)(x_0) = \int_{\mathbb{R}^n} \left( K(x, y) - K(x_0, y) \right) \prod_{j=1}^{2} \left[ R_{m_j}(\tilde{A}_j; x, y) f_i(y) dy \right] \\
+ \int_{\mathbb{R}^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y) K(x_0, y) f_i(y) dy}{|x_0 - y|^m} \\
+ \int_{\mathbb{R}^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y) K(x_j, y) f_i(y) dy}{|x_0 - y|^m}
\]

\[
- \sum_{|\alpha|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \left[ \frac{R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\alpha_1}}{|x - y|^m} K(x, y) - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x - y)^{\alpha_1}}{|x_0 - y|^m} K(x_0, y) \right] \\
\times D^{\alpha_1} \tilde{A}_1(y) f_i(y) dy \\
- \sum_{|\alpha|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \left[ \frac{R_{m_1}(\tilde{A}_1; x, y)(x - y)^{\alpha_2}}{|x - y|^m} K(x, y) - \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x - y)^{\alpha_2}}{|x_0 - y|^m} K(x_0, y) \right] \\
\times D^{\alpha_2} \tilde{A}_2(y) f_i(y) dy \\
+ \sum_{|\alpha|=m_1, |\beta|=m_2} \frac{1}{\alpha_1! \beta_1!} \int_{\mathbb{R}^n} \left[ (x - y)^{\alpha_1 + \alpha_2} K(x, y) - \frac{(x_0 - y)^{\alpha_1 + \alpha_2}}{|x_0 - y|^m} K(x_0, y) \right] \\
\times D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) f_i(y) dy
\]

\[
= J^{(1)}_i + J^{(2)}_i + J^{(3)}_i + J^{(4)}_i + J^{(5)}_i + J^{(6)}_i.
\]

By Lemma 3.5 and the following inequality, for \( b \in \Lambda_{\beta} \),

\[
|b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q ||b||_{\Lambda_{\beta}} |x - y|^\beta dy \leq ||b||_{\Lambda_{\beta}} (|x - x_0| + d)^\beta,
\]

we get

\[
|R_{m_j}(\tilde{A}_j; x, y)| \leq \sum_{|\alpha|=m_j} ||D^{\alpha} A_j||_{\Lambda_{\beta}} (|x - y| + d)^{m_j + \beta}.
\]

Note that \( |x - y| \sim |x_0 - y| \) for \( x \in Q \) and \( y \in \mathbb{R}^n \setminus \tilde{Q} \), we obtain, by the condition of \( K \),

\[
|J^{(1)}_i| \leq C \int_{R^n \setminus \tilde{Q}} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1-\delta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{m+n+\epsilon-\delta}} \right) \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j; x, y)| f_i(y) dy
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} ||D^{\alpha} A_j||_{\Lambda_{\beta}} \right)
\]

\[
\times \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta-2\beta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{n+\epsilon-\delta-2\beta}} \right) f_i(y) dy
\]

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\[
\sum_{j=1}^{\infty} \left\| D^{\alpha_j} A_j \right\|_{L^\beta} \right) |Q|^{2\beta/n} \sum_{k=0}^{\infty} \left( 2^{k(2\beta-1)} + 2^{k(2\beta-\varepsilon)} \right) \\
\times \frac{1}{|2^{k}Q|^{1-\delta/n}} \int_{2^{k}Q} |f(y)| dy
\]

thus, by Minkowski’ inequality, for \( 1 \leq s < \infty \),

\[
\left( \sum_{i=1}^{\infty} |J_i^{(1)}|^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left| D^{\alpha_j} A_j \right|_{L^\beta} \right) |Q|^{2\beta/n} \sum_{k=0}^{\infty} \left( 2^{k(2\beta-1)} + 2^{k(2\beta-\varepsilon)} \right) \\
\times \frac{1}{|2^{k}Q|^{1-\delta/n}} \int_{2^{k}Q} |f(y)| dy
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left| D^{\alpha_j} A_j \right|_{L^\beta} \right) |Q|^{2\beta/n} M_{\delta,s}(\|f\|_r)(x).
\]

For \( J_i^{(2)} \), by the formula (see [5]):

\[
R_{m_j} (\tilde{A}_j; x, y) - R_{m_j} (\tilde{A}_j; x_0, y) = \sum_{|\eta|<m_j} \frac{1}{\eta!} R_{m_j-|\eta|} (D^{\eta} \tilde{A}_j; x, x_0)(x - y)^\eta
\]

and Lemma [3.5] we get, for \( 1 \leq s < \infty \),

\[
\left( \sum_{i=1}^{\infty} |J_i^{(2)}|^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left| D^{\alpha_j} A_j \right|_{L^\beta} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta-2\beta}} |f(y)| dy
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left| D^{\alpha_j} A_j \right|_{L^\beta} \right) |Q|^{2\beta/n} M_{\delta,s}(\|f\|_r)(\tilde{x}).
\]

Similarly,

\[
\left( \sum_{i=1}^{\infty} |J_i^{(3)}|^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left| D^{\alpha_j} A_j \right|_{L^\beta} \right) |Q|^{2\beta/n} M_{\delta,s}(\|f\|_r)(\tilde{x}).
\]

For \( J_i^{(4)} \), similar to the estimates of \( J_i^{(1)} \) and \( J_i^{(2)} \), we obtain, for \( 1 \leq s < \infty \),
Similarly,

\[
\left( \sum_{i=1}^{\infty} \left| J_i^{(4)} \right|^r \right)^{1/r} \leq C \sum_{|\alpha|=m} \int_{R^n \setminus \hat{Q}} \frac{|x - y|^{\alpha_1} K(x, y) - (x_0 - y)^{\alpha_1} K(x_0, y)}{|x - y|^m} \left| D^{\alpha_1} \hat{A}_1(y) \right| \left| f(y) \right|_r dy
\]

\[
\times \left| R_{m_2} (\hat{A}_2; x, y) \right| \left| D^{\alpha_1} \hat{A}_1(y) \right| \left| f(y) \right|_r dy
\]

\[
+ C \sum_{|\alpha|=m_1} \int_{R^n \setminus \hat{Q}} \left| R_{m_2} (\hat{A}_2; x, y) - R_{m_2} (\hat{A}_2; x_0, y) \right| \left| (x_0 - y)^{\alpha_1} K(x_0, y) \right| \left| D^{\alpha_1} \hat{A}_1(y) \right| \left| f(y) \right|_r dy
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} \left| D^{\alpha_j} A_j \right|_{\lambda_{\alpha_j}} \right) \left| Q \right|^{2\beta/n} \sum_{k=0}^{\infty} (2^{k(2\beta - 1)} + 2^{k(2\beta - \epsilon)})
\]

\[
\times \left( \frac{1}{|2^{\delta} Q|^{1-\delta/n}} \int_{2^{\delta} Q} |f(y)|^s dy \right)^{1/s}
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} \left| D^{\alpha_j} A_j \right|_{\lambda_{\alpha_j}} \right) \left| Q \right|^{2\beta/n} M_{\delta,s}(\left| f \right|_r(\hat{x}).
\]

Similarly,

\[
\left( \sum_{i=1}^{\infty} \left| J_i^{(5)} \right|^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} \left| D^{\alpha_j} A_j \right|_{\lambda_{\alpha_j}} \right) \left| Q \right|^{2\beta/n} M_{\delta,s}(\left| f \right|_r(\hat{x}).
\]

For \( J_i^{(6)} \), we get

\[
\left( \sum_{i=1}^{\infty} \left| J_i^{(6)} \right|^r \right)^{1/r} \leq C \sum_{|\alpha|=m_1, |\alpha_2|=m_2} \int_{R^n \setminus \hat{Q}} \frac{|x - y|^{\alpha_1 + \alpha_2} K(x, y) - (x_0 - y)^{\alpha_1 + \alpha_2} K(x_0, y)}{|x - y|^m} \left| D^{\alpha_1 + \alpha_2} \hat{A}_1(y) \right| \left| f(y) \right|_r dy
\]

\[
\times \left| D^{\alpha_1} \hat{A}_1(y) \right| \left| D^{\alpha_2} \hat{A}_2(y) \right| \left| f(y) \right|_r dy
\]

\[
\leq C \sum_{|\alpha|=m_1, |\alpha_2|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1} \hat{Q}} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{n+\epsilon-\delta}} \right) \left| D^{\alpha_1} \hat{A}_1(y) \right| \left| D^{\alpha_2} \hat{A}_2(y) \right| \left| f(y) \right|_r dy
\]

\[
\times \left| D^{\alpha_1} \hat{A}_1(y) \right| \left| D^{\alpha_2} \hat{A}_2(y) \right| \left| f(y) \right|_r dy
\]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} |D^{\alpha_j} A_j|_{L^\beta} \right) |Q|^{2\beta/n} M_{\delta,s}(|f|_r)(x). \]

Thus
\[ |T_A(f)(x) - T_A(f)(x_0)|_r \leq C \sum_{|\alpha|=m} |D^\alpha A|_{L^\beta} |Q|^{2\beta/n} M_{\delta,s}(|f|_r)(x). \]

This completes the proof of the corollary. \( \square \)

4. Applications

In this section we shall apply the Theorem and Corollary to some particular operators such as the Calderón-Zygmund singular integral operator and fractional integral operator. Let \( 1 < r < \infty, D^\alpha A_j \in \check{A}_{\beta} \) for \( |\alpha| = m \) and \( j = 1, \ldots, l \).

**Application 1.** Calderón-Zygmund singular integral operator.

Let \( T \) be the Calderón-Zygmund operator (see [6], [12]), the vector-valued multilinear operator related to \( T \) is defined by
\[ |T_A(f_i)(x)|_r = \left( \sum_{i=1}^{\infty} |T_A(f_i)(x)|_r \right)^{1/r}, \]
where
\[ T_A(f_i)(x) = \int_{R^n} \prod_{j=1}^{l} R_{m_j+1}^{i} A_j(x, y) K(x, y) f_i(y) dy. \]

Then it is easily to see that \( T \) satisfies the conditions in Corollary 2.2 with \( \delta = 0 \), thus \( T_A \) is bounded from \( L^p(R^n) \) to \( L^{q,\infty}(R^n) \) for \( 0 < \beta < 1/l, 1 < p < \infty \) and from \( L^p(R^n) \) to \( L^q(R^n) \) for \( 0 < \beta < 1/l, 1 < p < n/(l\beta) \) and \( 1/p - 1/q = l\beta/n \).

**Application 2.** Fractional integral operator with rough kernel.

For \( 0 < \delta < n \), let \( T_\delta \) be the fractional integral operator with rough kernel defined by (see [8], [12])
\[ T_\delta g(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\delta}} g(y) dy, \]
the vector-valued multilinear operator related to \( T_\delta \) is defined by
\[ |T_\delta^A(f_i)(x)|_r = \left( \sum_{i=1}^{\infty} |T_\delta^A(f_i)(x)|_r \right)^{1/r}, \]
where
\[ T_\delta^A(f_i)(x) = \int_{R^n} \prod_{j=1}^{l} R_{m_j+1}^{i} A_j(x, y) \frac{\Omega(x-y) f_i(y) dy}{|x-y|^{m+n-\delta}}. \]

and \( \Omega \) is homogeneous of degree zero on \( R^n \), \( \int_{S^{n-1}} \Omega(x') d\sigma(x') = 0 \) and \( \Omega \in Lip_{\varepsilon}(S^{n-1}) \) for some \( 0 < \varepsilon \leq 1 \), that is there exists a constant \( L > 0 \) such that for any \( x, y \in S^{n-1}, |\Omega(x) - \Omega(y)| \leq L|x-y|^{\varepsilon} \). When \( \Omega \equiv 1 \), \( T_\delta^A \) is the Riesz potentials. Then \( T_\delta \) satisfies the conditions in Corollary. Thus \( T_\delta^A \) is bounded from \( L^p(R^n) \) to \( L^{q,\infty}(R^n) \) for \( 0 < \beta < \min(1/l, \varepsilon/l), 1 < p < n/\delta, 1/p - 1/q = \delta/n \) and from \( L^p(R^n) \) to \( L^q(R^n) \) for \( 0 < \delta < n-l\beta, 0 < \beta < \min(1/l, \varepsilon/l), 1 < p < n/(\delta + l\beta) \) and \( 1/p - 1/q = (\delta + l\beta)/n \).
REFERENCES


