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LOGARITHMICALLY COMPLETE MONOTONICITY PROPERTIES FOR THE GAMMA FUNCTIONS

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ABSTRACT. Some logarithmically completely monotonic functions involving the gamma functions are presented. As a consequence, some known results are proved and refined.

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1. INTRODUCTION

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \ldots). \tag{1.1}$$

If the inequality (1.1) is strict, then f is said to be strictly completely monotonic on I. Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [3], probability theory [5, 9, 15], physics [8], numerical and asymptotic analysis [11, 21], and combinatorics [2]. A detailed collection of the most important properties of completely monotonic functions can be found in [20, Chapter IV], and in an abstract in [4].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \tag{1.2}$$

for $x \in I$ and $n \in \mathbb{N} := \{1, 2, \ldots\}$. If inequality (1.2) is strict, then f is said to be strictly logarithmically completely monotonic. The terminology "(strictly) logarithmically completely monotonic function" was introduced in [18]. It was also shown in this paper that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t \quad (x > 0)$$

is one of the most important functions in analysis and its applications. The history and the development of this function are described in detail in [7]. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [16, p. 16] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, \mathrm{d}t,$$
$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} \, \mathrm{d}t$$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant.

When studying a problem on upper bound for permanents of $(0, 1)$–matrices, in 1964 H. Minc and L. Sathre [17] discovered several noteworthy inequalities involving $(n!)^{1/n}$. One of them is the following: If n is a positive integer, then

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1. \tag{1.3}$$

In 1985, D. Kershaw and A. Laforgia [14] showed that the function $x[\Gamma(1 + \frac{1}{x})]^x$ is strictly increasing on $(0, \infty)$, which is equivalent to the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ being strictly decreasing on $(0, \infty)$, and then, the left-hand inequality of (1.3) can be obtained as a consequence. Our Theorem 1.1 extends the result given by Kershaw and Laforgia as follows

Theorem 1.1. *The function* $1 - \ln x + \frac{1}{x} \ln \Gamma(x + 1)$ *is strictly completely monotonic on* $(0, \infty)$. *Moreover, the function* $\frac{[\Gamma(x+1)]^{1/x}}{x}$ *is strictly logarithmically completely monotonic on* $(0, \infty)$.

Motivated by the right-hand inequality of (1.3), we prove the following

Theorem 1.2. *Let $s > 0$ be a real number, then the function $f_s(x) = \frac{[\Gamma(x+s+1)]^{1/(x+s)}}{[\Gamma(x+1)]^{1/x}}$ is strictly logarithmically completely monotonic on $(0, \infty)$.*

In 1971, J. D. Kečkić and P. M. Vasić [13] proved that

$$\frac{b^{b-1}}{a^{a-1}} e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} e^{a-b} \quad \text{for} \quad b > a \geq 1. \tag{1.4}$$

The inequality (1.4) can also be found in [1]. It is easy to see that, in order to prove the inequality (1.4), it suffices to show that the functions

$$f_1(x) = \frac{(x/e)^x}{x\Gamma(x)} \quad \text{and} \quad f_2(x) = x^{1/2}(e/x)^x \Gamma(x)$$

are both strictly decreasing on $[1, \infty)$. Our Theorem 1.3 establish a more general result; we prove that the functions f_1 and f_2 are both strictly logarithmically completely monotonic on $(0, \infty)$.

Theorem 1.3. *The functions*

$$f_1(x) = \frac{(x/e)^x}{x\Gamma(x)} \quad \textit{and} \quad f_2(x) = x^{1/2}(e/x)^x \Gamma(x)$$

are both strictly logarithmically completely monotonic on $(0, \infty)$.

Theorem 1.4. *The function*

$$f_3(x) = \frac{(x/e)^x e^{-1/(12x)}}{\sqrt{x}\,\Gamma(x)}$$

is strictly logarithmically completely monotonic on $(0, \infty)$.

2. LEMMA

If $\varphi'' > 0$ for all x, $\varphi(0) = 0$, and φ/x is interpreted as $\varphi'(0)$ for $x = 0$, then φ/x increases for all x [12, p. 99].

We extend this result as follows

Lemma 2.1. *Let the function φ has derivatives of all orders on $(-\infty, \infty)$ and $\varphi(0) = 0$. Define the function f by*

$$f(x) = \begin{cases} \dfrac{\varphi(x)}{x}, & x \neq 0; \\[2ex] \varphi'(0), & x = 0, \end{cases}$$

then

$$f^{(n)}(x) = \begin{cases} \frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x), & x \neq 0; \\[2ex] \frac{1}{n+1} \varphi^{(n+1)}(0), & x = 0. \end{cases} \tag{2.1}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x) = x^n \varphi^{(n+1)}(x). \tag{2.2}$$

Remark 2.2. Write (2.1) for $x \neq 0$ as

$$x^{n+1}\left(\frac{\varphi(x)}{x}\right)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\varphi^{(n-k)}(x).$$

Differentiation and application (2.2) yields

$$(n+1)\left(\frac{\varphi(x)}{x}\right)^{(n)} + x\left(\frac{\varphi(x)}{x}\right)^{(n+1)} = \varphi^{(n+1)}(x). \tag{2.3}$$

Clearly, (2.3) is also valid for $x = 0$.

Proof. Using Leibniz' rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(k)}(x)v^{(n-k)}(x),$$

we have for $x \neq 0$,

$$\begin{aligned} f^{(n)}(x) &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(k)}\varphi^{(n-k)}(x) \\ &= \frac{1}{x^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\varphi^{(n-k)}(x). \end{aligned}$$

By direct computation, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\varphi^{(n-k)}(x) \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k k!(n-k)x^{n-k-1}\varphi^{(n-k)}(x) \\ &\quad + \sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\varphi^{(n-k+1)}(x) \\ &= \sum_{k=0}^{n-1}\binom{n}{k}(-1)^k k!(n-k)x^{n-k-1}\varphi^{(n-k)}(x) \\ &\quad + x^n\varphi^{(n+1)}(x) + \sum_{k=1}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\varphi^{(n-k+1)}(x) \\ &= \sum_{k=0}^{n-1}\binom{n}{k}(-1)^k k!(n-k)x^{n-k-1}\varphi^{(n-k)}(x) \\ &\quad + x^n\varphi^{(n+1)}(x) + \sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^{k+1}(k+1)! x^{n-k-1}\varphi^{(n-k)}(x) \\ &= \sum_{k=0}^{n-1}\left[\binom{n}{k}(n-k) - \binom{n}{k+1}(k+1)\right](-1)^k k! x^{n-k-1}\varphi^{(n-k)}(x) \\ &\quad + x^n\varphi^{(n+1)}(x) \\ &= x^n\varphi^{(n+1)}(x) \end{aligned} \tag{2.4}$$

since the term in square brackets is equal to 0.

Using L' Hospital rule and (2.4),

$$f^{(n)}(0) = \lim_{x\to 0} \frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x)$$
$$= \lim_{x\to 0} \frac{\varphi^{(n+1)}(x)}{n+1} = \frac{1}{n+1}\varphi^{(n+1)}(0).$$

The proof of is complete. ∎

3. PROOFS OF THEOREMS

Proof of Theorem 1.1. It has been shown in [14] that the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ is strictly decreasing on $(0,\infty)$, then $f(x) = 1 - \ln x + \frac{1}{x}\ln\Gamma(x+1)$ is also strictly decreasing on $(0,\infty)$. From the asymptotic expansion [11]

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12x} + O(x^{-3}) \quad \text{as } x \to \infty,$$

we conclude that $\lim_{x\to\infty} f(x) = 0$. This implies $f(x) > 0$ for $x > 0$.

Define the function φ by

$$\varphi(x) = x - x\ln x + \ln\Gamma(x+1), x \neq 0,$$
$$\varphi(0) = \lim_{x\to 0}\varphi(x) = 0.$$

By Lemma, we obtain

$$x^{n+1} f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x) \triangleq \Phi(x),$$
$$\Phi'(x) = x^n \varphi^{(n+1)}(x).$$

Using the representations [19, p. 153] (also see [10, p. 824])

$$\psi(x) = \frac{1}{2x} + \ln x - \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) e^{-xt}\,\mathrm{d}t \quad (x > 0), \tag{3.1}$$

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{3.2}$$

$$\frac{1}{(x+s)^n} = \frac{1}{(n-1)!}\int_0^\infty e^{-(x+s)t}\,\mathrm{d}t \quad (x > 0, s \geq 0, n = 1, 2, \ldots), \tag{3.3}$$

we conclude that

$$\varphi'(x) = \psi(x+1) - \ln x = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt}\,\mathrm{d}t$$

and therefore

$$\varphi^{(n+1)}(x) = (-1)^n \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) t^n e^{-xt}\,\mathrm{d}t.$$

If n is even, then we have for $x > 0$,

$$\varphi^{(n+1)}(x) > 0 \Longrightarrow \Phi'(x) > 0 \Longrightarrow \Phi(x) > \Phi(0) = 0 \Longrightarrow f^{(n)}(x) > 0$$
$$\Longrightarrow (-1)^n f^{(n)}(x) > 0.$$

If n is odd, then we have for $x > 0$,

$$\varphi^{(n+1)}(x) < 0 \Longrightarrow \Phi'(x) < 0 \Longrightarrow \Phi(x) < \Phi(0) = 0 \Longrightarrow f^{(n)}(x) < 0$$
$$\Longrightarrow (-1)^n f^{(n)}(x) > 0.$$

Hence,

$$(-1)^n f^{(n)}(x) > 0$$

for all real $x \in (0, \infty)$ and all integers $n \geq 0$.

Clearly, the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ is strictly logarithmically completely monotonic on $(0, \infty)$. The proof is complete. ∎

Remark 3.1. Theorem 1.1 has been shown in [18], we here provide another proof using Lemma above.

Proof of Theorem 1.2. Clearly,

$$\ln f_s(x) = \frac{\ln[\Gamma(x+s+1)]}{x+s} - \frac{\ln[\Gamma(x+1)]}{x} \triangleq g(x+s) - g(x).$$

By Lemma,

$$x^{n+1} g^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} [\ln \Gamma(x+1)]^{(n-k)} \triangleq \Phi(x),$$

$$\begin{aligned} \Phi'(x) &= x^n (\ln \Gamma(x+1))^{(n+1)} = x^n \psi^{(n)}(x+1) \\ &= (-1)^{n+1} x^n \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-(x+1)t} \, \mathrm{d}t. \end{aligned}$$

If n is odd, then we have for $x > 0$,

$$\Phi'(x) > 0 \Longrightarrow \Phi(x) > \Phi(0) = 0 \Longrightarrow g^{(n)}(x) > 0 \Longrightarrow (-1)^{n+1} g^{(n)}(x) > 0.$$

If n is even, then we have for $x > 0$,

$$\Phi'(x) < 0 \Longrightarrow \Phi(x) < \Phi(0) = 0 \Longrightarrow g^{(n)}(x) < 0 \Longrightarrow (-1)^{n+1} g^{(n)}(x) > 0.$$

Hence,

$$(-1)^{n+1} g^{(n)}(x) > 0 \tag{3.4}$$

for all real $x \in (0, \infty)$ and all integers $n \geq 1$. This implies that $g^{(2k)}(x)$ increases strictly and $g^{(2k-1)}(x)$ decreases strictly, and then

$$(-1)^n (\ln f_s(x))^{(n)} > 0$$

for all real $x \in (0, \infty)$ and all integers $n \geq 1$. The proof is complete. ∎

Remark 3.2. It was shown in [6, p. 83] that let f and g be functions such that $f(g(x))$ is defined for $x > 0$. If f and g' are completely monotonic, then $x \mapsto f(g(x))$ is also completely monotonic. From (3.4) it follows that $g'(x) = -\frac{\ln \Gamma(x+1)}{x^2} + \frac{\psi(x+1)}{x}$ is completely monotonic on $(0, \infty)$, this implies that $\exp(-g(x)) = \ln \frac{1}{[\Gamma(x+1)]^{1/x}}$ is completely monotonic on $(0, \infty)$.

Proof of Theorem 1.3. Using the representations (3.1) and (3.3), we have

$$(\ln f_1(x))' = \ln x - \psi(x) - \frac{1}{x} = -\int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} \, \mathrm{d}t$$

and therefore

$$(-1)^n (\ln f_1(x))^{(n)} = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) t^{n-1} e^{-xt} \, \mathrm{d}t > 0$$

for $x > 0$ and $n \in \mathbb{N}$.

Using (3.1) and (3.3), we obtain

$$(\ln f_2(x))' = \frac{1}{2x} + \psi(x) - \ln x = -\int_0^\infty \delta(t) e^{-xt} \, \mathrm{d}t,$$

where

$$\delta(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}.$$

Defficientation yields

$$\delta'(t) = \frac{1}{t^2} - \frac{1}{(e^{t/2} - e^{-t/2})^2} = \frac{1}{t^2} - \frac{1}{t^2 + 2\sum_{n=2}^{\infty} t^{2n}/(2n)!} > 0,$$

therefore, the function δ is strictly increasing on $(0, \infty)$, and then

$$\delta(t) > \lim_{t \to 0} \delta(t) = 0.$$

Hence,

$$(-1)^n (\ln f_2(x))^{(n)} = \int_0^\infty \delta(t) t^{n-1} e^{-xt} \, \mathrm{d}t > 0$$

for $x > 0$ and $n \in \mathbb{N}$. ∎

Remark 3.3. From the monotonicity of the functions f_1 and f_2, we conclude that

$$\frac{b^{b-1}}{a^{a-1}} e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} e^{a-b} \quad \text{for} \quad b > a > 0, \tag{3.5}$$

which extents the inequality (1.4) to all real $b > a > 0$.

If we denote by

$$I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, \quad a > 0, b > 0, a \neq b,$$

the so-called identric mean, then, (3.5) yields the following bounds for the $(b-a)th$ power of $I(a, b)$:

$$\left(\frac{b}{a} \right)^{1/2} \frac{\Gamma(b)}{\Gamma(a)} < I(a, b)^{b-a} < \frac{b}{a} \frac{\Gamma(b)}{\Gamma(a)} \quad \text{for} \quad b > a > 0. \tag{3.6}$$

It was proved in [1] that the inequality

$$\left(\frac{b}{a} \right)^r \frac{\Gamma(b)}{\Gamma(a)} < I(a, b)^{b-a} < \left(\frac{b}{a} \right)^s \frac{\Gamma(b)}{\Gamma(a)} \tag{3.7}$$

are valid for all real numbers $b > a \geq 1$ if and only if $r \leq 1/2$ and $s \geq \gamma$ ($\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant).

Proof of Theorem 1.4. Using the representations (3.1) and (3.3), we have

$$(\ln f_3(x))' = \ln x - \psi(x) - \frac{1}{2x} - \frac{1}{12x^2} = -\int_0^\infty \omega(t) e^{-xt} \, \mathrm{d}t,$$

where

$$\omega(t) = \frac{t}{12} - \frac{1}{e^t - 1} + \frac{1}{t} - \frac{1}{2}.$$

Defficientation yields

$$\omega'(t) = \frac{1}{12} + \frac{1}{(e^{t/2} - e^{-t/2})^2} - \frac{1}{t^2}.$$

Using power series expansion, we have

$$t^2 (e^{t/2} - e^{-t/2})^2 \omega'(t) = \sum_{n=3}^{\infty} \left[\frac{1}{6} - \frac{1}{n(2n-1)} \right] \frac{t^{2n}}{(2n-2)!} > 0.$$

Therefore, the function ω is strictly increasing on $(0,\infty)$, and then

$$\omega(t) > \lim_{t\to 0}\omega(t) = 0$$

and therefore

$$(-1)^n(\ln f_3(x))^{(n)} = \int_0^\infty \omega(t)t^{n-1}e^{-xt}\,\mathrm{d}t > 0$$

for $x > 0$ and $n \in \mathbb{N}$. ∎

Remark 3.4. From the monotonicity of the function f_3 , we conclude that

$$I(a,b)^{b-a} < \left(\frac{b}{a}\right)^{1/2} \exp\left(-\frac{b-a}{12ab}\right)\frac{\Gamma(b)}{\Gamma(a)} \quad \text{for} \quad b > a > 0. \tag{3.8}$$

The upper in (3.8) is an improvement over the upper in (3.6).

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