LOGARITHMICALLY COMPLETE MONOTONICITY PROPERTIES FOR THE GAMMA FUNCTIONS

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ABSTRACT. Some logarithmically completely monotonic functions involving the gamma functions are presented. As a consequence, some known results are proved and refined.

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1. INTRODUCTION

A function $f$ is said to be completely monotonic on an interval $I$, if $f$ has derivatives of all orders on $I$ and satisfies

$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \ldots).$ \hfill (1.1)

If the inequality (1.1) is strict, then $f$ is said to be strictly completely monotonic on $I$. Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [3], probability theory [5, 9, 15], physics [8], numerical and asymptotic analysis [11, 21], and combinatorics [2]. A detailed collection of the most important properties of completely monotonic functions can be found in [20, Chapter IV], and in an abstract in [4].

A positive function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\ln f$ satisfies

$(-1)^n [\ln f(x)]^{(n)} \geq 0$ for $x \in I$ and $n \in \mathbb{N} := \{1, 2, \ldots\}$. If inequality (1.2) is strict, then $f$ is said to be strictly logarithmically completely monotonic. The terminology “(strictly) logarithmically completely monotonic function” was introduced in [18]. It was also shown in this paper that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

The classical gamma function

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad (x > 0)$

is one of the most important functions in analysis and its applications. The history and the development of this function are described in detail in [7]. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [16, p. 16] as

$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, dt,$

$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} \, dt$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286 \ldots$ is the Euler-Mascheroni constant.

When studying a problem on upper bound for permanents of $(0,1)$–matrices, in 1964 H. Minc and L. Sathre [17] discovered several noteworthy inequalities involving $(n!)^{1/n}$. One of them is the following: If $n$ is a positive integer, then

$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1.$ \hfill (1.3)

In 1985, D. Kershaw and A. Laforgia [14] showed that the function $x[\Gamma(1 + \frac{1}{x})]^x$ is strictly increasing on $(0, \infty)$, which is equivalent to the function $[\Gamma(x+1)]^{1/x}$ being strictly decreasing on $(0, \infty)$, and then, the left-hand inequality of (1.3) can be obtained as a consequence. Our Theorem 1.1 extends the result given by Kershaw and Laforgia as follows

**Theorem 1.1.** The function $1 - \ln x + \frac{1}{x} \ln \Gamma(x+1)$ is strictly completely monotonic on $(0, \infty)$. Moreover, the function $[\Gamma(x+1)]^{1/x}$ is strictly logarithmically completely monotonic on $(0, \infty)$.

Motivated by the right-hand inequality of (1.3), we prove the following
Logarithmically complete monotonicity properties for the gamma functions

**Theorem 1.2.** Let \( s > 0 \) be a real number, then the function \( f_s(x) = \frac{[\Gamma(x+s+1)]^{1/(x+s)}}{[\Gamma(x+1)]^{1/x}} \) is strictly logarithmically completely monotonic on \((0, \infty)\).

In 1971, J. D. Kečkić and P. M. Vasić [13] proved that

\[\frac{b^{b-1}}{a^{a-1}} e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} e^{a-b} \quad \text{for} \quad b > a \geq 1.\]

The inequality (1.4) can also be found in [1]. It is easy to see that, in order to prove the inequality (1.4), it suffices to show that the functions \( f_1(x) = (x/e)^x x \Gamma(x) \) and \( f_2(x) = x^{1/2} (e/x)^x \Gamma(x) \) are both strictly decreasing on \([1, \infty)\). Our Theorem 1.3 establish a more general result; we prove that the functions \( f_1 \) and \( f_2 \) are both strictly logarithmically completely monotonic on \((0, \infty)\).

**Theorem 1.3.** The functions

\[ f_1(x) = \frac{(x/e)^x}{x \Gamma(x)} \quad \text{and} \quad f_2(x) = x^{1/2} (e/x)^x \Gamma(x) \]

are both strictly logarithmically completely monotonic on \((0, \infty)\).

**Theorem 1.4.** The function

\[ f_3(x) = \frac{(x/e)^x e^{-1/(12x)}}{\sqrt{x} \Gamma(x)} \]

is strictly logarithmically completely monotonic on \((0, \infty)\).

2. **Lemma**

If \( \varphi'' > 0 \) for all \( x \), \( \varphi(0) = 0 \), and \( \varphi/x \) is interpreted as \( \varphi'(0) \) for \( x = 0 \), then \( \varphi/x \) increases for all \( x \) [12, p. 99].

We extend this result as follows

**Lemma 2.1.** Let the function \( \varphi \) has derivatives of all orders on \((-\infty, \infty)\) and \( \varphi(0) = 0 \). Define the function \( f \) by

\[ f(x) = \begin{cases} \frac{\varphi(x)}{x}, & x \neq 0; \\ \varphi'(0), & x = 0, \end{cases} \]

then

\[ f^{(n)}(x) = \begin{cases} \frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} k! x^{n-k} \varphi^{(n-k)}(x), & x \neq 0; \\ \frac{1}{n+1} \varphi^{(n+1)}(0), & x = 0. \end{cases} \]

Moreover,

\[ \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} k! x^{n-k} \varphi^{(n-k)}(x) = x^n \varphi^{(n+1)}(x). \]
Remark 2.2. Write (2.1) for \( x \neq 0 \) as

\[ x^{n+1} \left( \frac{\varphi(x)}{x} \right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x). \]

Differentiation and application (2.2) yields

\[ (n + 1) \left( \frac{\varphi(x)}{x} \right)^{(n)} + x \left( \frac{\varphi(x)}{x} \right)^{(n+1)} = \varphi^{(n+1)}(x). \]

Clearly, (2.3) is also valid for \( x = 0 \).

Proof. Using Leibniz’ rule

\[ [u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x), \]

we have for \( x \neq 0 \),

\[ f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{x} \right)^{(k)} \varphi^{(n-k)}(x) = \frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x). \]

By direct computation, we have

\begin{align*}
\frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x) \\
= \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! (n - k) x^{n-k-1} \varphi^{(n-k)}(x) \\
+ \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k+1)}(x) \\
= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k! (n - k) x^{n-k-1} \varphi^{(n-k)}(x) \\
+ x^n \varphi^{(n+1)}(x) + \sum_{k=1}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k+1)}(x) \\
= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k! (n - k) x^{n-k-1} \varphi^{(n-k)}(x) \\
+ x^n \varphi^{(n+1)}(x) + \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1} (k + 1)! x^{n-k-1} \varphi^{(n-k)}(x) \\
= \sum_{k=0}^{n-1} \left[ \binom{n}{k} (n - k) - \binom{n}{k+1} (k + 1) \right] (-1)^k k! x^{n-k-1} \varphi^{(n-k)}(x) \\
+ x^n \varphi^{(n+1)}(x) \\
= x^n \varphi^{(n+1)}(x)
\end{align*}

since the term in square brackets is equal to 0.
Using L’Hospital rule and (2.4),
\[ f^{(n)}(0) = \lim_{x \to 0} \frac{1}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x) \]
\[ = \lim_{x \to 0} \frac{\varphi^{(n+1)}(x)}{n+1} = \frac{1}{n+1} \varphi^{(n+1)}(0). \]
The proof of is complete. ■

3. PROOFS OF THEOREMS

Proof of Theorem 1.1 It has been shown in [14] that the function \( \frac{\Gamma(x+1)^{1/x}}{x} \) is strictly decreasing on \((0, \infty)\), then \( f(x) = 1 - \ln x + \frac{1}{x} \ln \Gamma(x+1) \) is also strictly decreasing on \((0, \infty)\). From the asymptotic expansion [11]
\[ \ln \Gamma(x) = \left( x - \frac{1}{2} \right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12x} + O(x^{-3}) \text{ as } x \to \infty, \]
we conclude that \( \lim_{x \to \infty} f(x) = 0 \). This implies \( f(x) > 0 \) for \( x > 0 \).

Define the function \( \varphi \) by
\[ \varphi(x) = x - x \ln x + \ln \Gamma(x+1), x \neq 0, \]
\[ \varphi(0) = \lim_{x \to 0} \varphi(x) = 0. \]

By Lemma, we obtain
\[ x^{n+1} f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \varphi^{(n-k)}(x) \triangleq \Phi(x), \]
\[ \Phi'(x) = x^n \varphi^{(n+1)}(x). \]

Using the representations [19, p. 153] (also see [10, p. 824])
\[ \psi(x) = \frac{1}{2x} + \ln x - \int_{0}^{\infty} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} \, dt \quad (x > 0), \]
\[ \psi(x+1) = \psi(x) + \frac{1}{x}, \]
\[ \frac{1}{(x+s)^n} = \frac{1}{(n-1)!} \int_{0}^{\infty} e^{-(x+s)t} \, dt \quad (x > 0, s \geq 0, n = 1, 2, \ldots), \]
we conclude that
\[ \varphi'(x) = \psi(x+1) - \ln x = \int_{0}^{\infty} \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} \, dt \]
and therefore
\[ \varphi^{(n+1)}(x) = (-1)^n \int_{0}^{\infty} \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^n e^{-xt} \, dt. \]
If \( n \) is even, then we have for \( x > 0 \),
\[ \varphi^{(n+1)}(x) > 0 \implies \Phi'(x) > 0 \implies \Phi(x) > \Phi(0) = 0 \implies f^{(n)}(x) > 0 \implies (-1)^n f^{(n)}(x) > 0. \]
If \( n \) is odd, then we have for \( x > 0 \),
\[ \varphi^{(n+1)}(x) < 0 \implies \Phi'(x) < 0 \implies \Phi(x) < \Phi(0) = 0 \implies f^{(n)}(x) < 0 \implies (-1)^n f^{(n)}(x) > 0. \]
Hence, \((-1)^nf^{(n)}(x) > 0\)
for all real \(x \in (0, \infty)\) and all integers \(n \geq 0\).

Clearly, the function \(\frac{\Gamma(x+1)}{x}\) is strictly logarithmically completely monotonic on \((0, \infty)\). The proof is complete. \(\blacksquare\)

**Remark 3.1.** Theorem 1.1 has been shown in [18], we here provide another proof using Lemma above.

**Proof of Theorem 1.2.** Clearly,

\[
\ln f_s(x) = \ln(\Gamma(x + s + 1)) - \ln(\Gamma(x + 1)) \equiv g(x + s) - g(x).
\]

By Lemma,

\[
x^{n+1}g^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} k! x^{n-k} [\ln(\Gamma(x + 1))]^{(n-k)} \equiv \Phi(x),
\]

\[
\Phi'(x) = x^n (\ln(\Gamma(x + 1)))^{(n+1)} = x^n \psi^{(n)}(x + 1) = (-1)^{n+1} x^n \int_{0}^{\infty} \frac{t^n}{1 - e^{-t}} e^{-(x+1)t} dt.
\]

If \(n\) is odd, then we have for \(x > 0\),

\[
\Phi'(x) > 0 \implies \Phi(x) > \Phi(0) = 0 \implies g^{(n)}(x) > 0 \implies (-1)^{n+1} g^{(n)}(x) > 0.
\]

If \(n\) is even, then we have for \(x > 0\),

\[
\Phi'(x) < 0 \implies \Phi(x) < \Phi(0) = 0 \implies g^{(n)}(x) < 0 \implies (-1)^{n+1} g^{(n)}(x) > 0.
\]

Hence, \((3.4)\)

\((-1)^{n+1} g^{(n)}(x) > 0\)
for all real \(x \in (0, \infty)\) and all integers \(n \geq 1\). This implies that \(g^{(2k)}(x)\) increases strictly and \(g^{(2k-1)}(x)\) decreases strictly, and then

\((-1)^{n}(\ln f_s(x))^{(n)} > 0\)
for all real \(x \in (0, \infty)\) and all integers \(n \geq 1\). The proof is complete. \(\blacksquare\)

**Remark 3.2.** It was shown in [6, p. 83] that let \(f\) and \(g\) be functions such that \(f(g(x))\) is defined for \(x > 0\). If \(f\) and \(g'\) are completely monotonic, then \(x \mapsto f(g(x))\) is also completely monotonic. From \((3.4)\) it follows that \(g'(x) = -\frac{\ln \Gamma(x+1)}{x^2} + \frac{\psi(x+1)}{x}\) is completely monotonic on \((0, \infty)\), this implies that \(\exp(-g(x)) = \ln \frac{1}{\Gamma(x+1)^{1/x}}\) is completely monotonic on \((0, \infty)\).

**Proof of Theorem 1.3.** Using the representations \((3.1)\) and \((3.3)\), we have

\[
(\ln f_{1}(x))' = \ln x - \psi(x) - \frac{1}{x} = - \int_{0}^{\infty} \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt
\]

and therefore

\[
(-1)^{n}(\ln f_{1}(x))^{(n)} = \int_{0}^{\infty} \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^{n-1} e^{-xt} dt > 0
\]
for \(x > 0\) and \(n \in \mathbb{N}\).

Using \((3.1)\) and \((3.3)\), we obtain

\[
(\ln f_{2}(x))' = \frac{1}{2x} + \psi(x) - \ln x = - \int_{0}^{\infty} \delta(t) e^{-xt} dt,
\]
where

\[ \delta(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}. \]

Defficientation yields

\[ \delta'(t) = \frac{1}{t^2} - \frac{1}{(e^{t/2} - e^{-t/2})^2} = \frac{1}{t^2} - \frac{1}{t^2 + 2\sum_{n=2}^{\infty} t^{2n}/(2n)!} > 0, \]

therefore, the function \( \delta \) is strictly increasing on \((0, \infty)\), and then

\[ \delta(t) > \lim_{t \to 0} \delta(t) = 0. \]

Hence,

\[ (-1)^n (\ln f_2(x))^{(n)} = \int_0^\infty \delta(t) t^{n-1} e^{-xt} \, dt > 0 \]

for \( x > 0 \) and \( n \in \mathbb{N} \).

**Remark 3.3.** From the monotonicity of the functions \( f_1 \) and \( f_2 \), we conclude that

\[ \frac{b^{b-1}}{a^{a-1}} e^{a-b} < \Gamma(b) \Gamma(a) < \frac{b^{b-1/2}}{a^{a-1/2}} e^{a-b} \quad \text{for} \quad b > a > 0, \]

which extents the inequality (1.4) to all real \( b > a > 0 \).

If we denote by

\[ I(a, b) = \frac{1}{e}\left( \frac{b}{a} \right)^{1/(b-a)}, \quad a > 0, b > 0, a \neq b, \]

the so-called identric mean, then, (3.5) yields the following bounds for the \((b - a)th\) power of \( I(a, b) \):

\[ \left( \frac{b}{a} \right)^{1/2} \frac{\Gamma(b)}{\Gamma(a)} < I(a, b)^{b-a} < \frac{b}{a} \frac{\Gamma(b)}{\Gamma(a)} \quad \text{for} \quad b > a > 0. \]

It was proved in [1] that the inequality

\[ \left( \frac{b}{a} \right)^{r} \frac{\Gamma(b)}{\Gamma(a)} < I(a, b)^{b-a} < \left( \frac{b}{a} \right)^{s} \frac{\Gamma(b)}{\Gamma(a)} \]

are valid for all real numbers \( b > a \geq 1 \) if and only if \( r \leq 1/2 \) and \( s \geq \gamma (\gamma = 0.57721566490153286 \ldots \)

is the Euler-Mascheroni constant).

**Proof of Theorem 1.4.** Using the representations (3.1) and (3.3), we have

\[ (\ln f_3(x))' = \ln x - \psi(x) - \frac{1}{2x} - \frac{1}{12x^2} = -\int_0^\infty \omega(t) e^{-xt} \, dt, \]

where

\[ \omega(t) = \frac{t}{12} - \frac{1}{e^t - 1} + \frac{1}{t} - \frac{1}{2}. \]

Defficientation yields

\[ \omega'(t) = \frac{1}{12} + \frac{1}{(e^{t/2} - e^{-t/2})^2} - \frac{1}{t^2}. \]

Using power series expansion, we have

\[ t^2 (e^{t/2} - e^{-t/2})^2 \omega'(t) = \sum_{n=3}^{\infty} \left[ \frac{1}{6} - \frac{1}{n(2n-1)} \right] \frac{t^{2n}}{(2n-2)!} > 0. \]
Therefore, the function $\omega$ is strictly increasing on $(0, \infty)$, and then
\[
\omega(t) > \lim_{t \to 0} \omega(t) = 0
\]
and therefore
\[
(-1)^n (\ln f_3(x))^{(n)} = \int_0^\infty \omega(t)t^{n-1}e^{-xt} \, dt > 0
\]
for $x > 0$ and $n \in \mathbb{N}$.

**Remark 3.4.** From the monotonicity of the function $f_3$, we conclude that
\[
I(a, b)^{b-a} < \left( \frac{b}{a} \right)^{1/2} \exp \left( -\frac{b-a}{12ab} \right) \frac{\Gamma(b)}{\Gamma(a)} \quad \text{for} \quad b > a > 0.
\]
The upper in (3.8) is an improvement over the upper in (3.6).

**REFERENCES**


