The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 2, Issue 2, Article 14, pp. 1-12, 2005

WEYL TRANSFORM ASSOCIATED WITH BESSEL AND LAGUERRE FUNCTIONS

E. JEBBARI AND M . SIFI

Received 22 Deecember 2004; accepted 29 April 2005; published 29 December 2005.
Communicated by: Carlo Bardaro

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCES OF TUNIS, 1060 TUNIS, TUNISIA
mohamed.sifi@fst.rnu.tn

ABSTRACT. We define and study the Wigner transform associated to Bessel and Laguerre transform and we prove an inversion formula for this transform. Next we consider a class of symbols which allows to define the Bessel-Laguerre Weyl transform. We establish a relation between the Wigner and Weyl transform. At last, we discuss criterion in term of symbols for the boundedness and compactness of the Bessel-Laguerre Weyl transform.

Key words and phrases: Bessel-Laguerre Fourier transform, Weyl transform, Wigner transform.

2000 *Mathematics Subject Classification.* 34K99, 44A05, 41A58.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Many authors have developed properties of pseudo-differential operators arising in quantum mechanics.

At first H. Weyl [5] has considered these operators as bounded operators on $L^2(\mathbb{R}^n)$.

In this paper we consider a system of partial differential operators D_1 and D_2 defined on $K = [0, +\infty[\times[0, +\infty[$, by

$$\begin{cases} D_1 = \dfrac{\partial^2}{\partial t^2} + \dfrac{2\alpha}{t}\dfrac{\partial}{\partial t}, & \alpha \geq 0, t > 0 \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2 D_1, & x > 0 \end{cases},$$

and we develop the harmonic analysis associated with these operators (Translation operators, Convolution product, Bessel-Laguerre Fourier transform,...).

Using these results we first define and study the Bessel-Laguerre Wigner transform on $\mathcal{S}(K)\times\mathcal{S}(K)$, where $\mathcal{S}(K)$ is the Schwartz space on K. We establish some properties of this transform, in particular an inversion formula.

For σ in a class of symbols $\mathcal{S}_0^l(K\times\Gamma)$, $l \in \mathbb{R}\cup\{+\infty\}$, where $\Gamma = [0, +\infty[\times\mathbb{N}$, we define the Bessel-Laguerre Weyl transform W_σ, we prove that W_σ is continuous from $\mathcal{S}(K)$ into itself, and can be extended to a linear continuous operator from $L^p(K, m_\alpha)$ (the space of $p-$integrable functions on K with respect to the measure m_α) into $L^q(K, m_\alpha)$, ($p \in [1, +\infty[$, $1/q+1/q = 1$), where m_α is the positive measure defined on K, by

$$dm_\alpha(x,t) = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+1/2)} x^{2\alpha+1}t^{2\alpha}dxdt.$$

Furthermore, we prove that W_σ is a compact operator from $L^2(K, m_\alpha)$ for σ in $\mathcal{S}_0^\infty(K\times\Gamma)$ and in $L^r(K\times\Gamma, m_\alpha\otimes\gamma_\alpha), 1 \leq r \leq 2$, here γ_α is some positive measure defined on Γ. At last, we prove that there exists a symbol σ in $L^r(K\times\Gamma, m_\alpha\otimes\gamma_\alpha), r > 2$, such that the Bessel-Laguerre Weyl transform W_σ is not a bounded linear operator on $L^r(K, m_\alpha)$.

Throughout this paper we use the classic notation.

If (X, Ω) is a measurable space and m a positive measure on X, $L^p(X) = L^p(X, m)$ represent the space of measurable functions $f : X \to \mathbb{C}$, such that

$$||f||_{p,m} = \begin{cases} \left(\displaystyle\int_X |f(x)|^p dm(x)\right)^{\frac{1}{p}} < +\infty, & \text{if } \ 1 \leq p < +\infty, \\ \text{ess sup}_{x\in X}|f(x)| < +\infty, & \text{if } \ p = +\infty. \end{cases}$$

2. HARMONIC ANALYSIS ASSOCIATE WITH THE OPERATORS D_1 AND D_2.

In this section, we recall some results about harmonic analysis associated with the operators D_1 and D_2. (For more details one can see [1]).

Notation . We denote by

- $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.
- $F(\mathbb{R}^*\times\mathbb{N})$ the space of functions defined on $\mathbb{R}^*\times\mathbb{N}$.
- Δ_+ and Δ_- the operators defined on $F(\mathbb{R}^*\times\mathbb{N})$, by

$$\Delta_+ g(\lambda, m) = g(\lambda, m+1) - g(\lambda, m)$$

$$\Delta_- g(\lambda, m) = \begin{cases} g(\lambda, m) - g(\lambda, m-1), & \text{if } \ m \geq 1 \\ g(\lambda, 0), & \text{if } \ m = 0. \end{cases}$$

- Δ_1, Δ_2 and Δ_3 the operators defined on F($\mathbb{R}^* \times \mathbb{N}$), by

$$\begin{aligned}
\Delta_1 g(\lambda, m) &= -\frac{1}{2\lambda}((\alpha+m+1)\Delta_+ g(\lambda, m) + m\Delta_- g(\lambda, m)),\\
\Delta_2 g(\lambda, m) &= \frac{1}{4\lambda^2}\sum_{i=-2}^{i=2} C_i(\lambda, m) g(\lambda, m+i),\\
\Delta_3 g(\lambda, m) &= 2\frac{\partial}{\partial\lambda}(\Delta_1 g(\lambda, m)) + \frac{2\alpha}{\lambda}\Delta_1 g(\lambda, m) + \Delta_2 g(\lambda, m),
\end{aligned}$$

with

$$\begin{aligned}
C_2(\lambda, m) &= (m+\alpha+1)(m+\alpha+2), C_1(\lambda, m) = -2(m+\alpha+1)(\alpha+2),\\
C_0(\lambda, m) &= 2((\alpha+1)+(\alpha+1)^2-(m+1)(m+\alpha+1)-m(m+\alpha)),\\
C_{-1}(\lambda, m) &= 2m(\alpha+2) \text{ and } C_{-2}(\lambda, m) = 2m^2(m-1).
\end{aligned}$$

- Λ_1 , Λ_2 the operators defined on $F(\mathbb{R}^* \times \mathbb{N})$, by

$$\begin{aligned}
\Lambda_1 g(\lambda, m) &= \frac{1}{\lambda}(m\Delta_+\Delta_- g(\lambda, m) + (\alpha+1)\Delta_+ g(\lambda, m))\\
\Lambda_2 &= \Delta_3 + \Delta_{\alpha-1/2},
\end{aligned}$$

where $\Delta_{\alpha-1/2}$ is the Bessel operator defined on $]0, +\infty[$, by

$$\Delta_{\alpha-1/2} = \frac{\partial^2}{\partial\lambda^2} + \frac{2\alpha}{\lambda}\frac{\partial}{\partial\lambda}.$$

The unique solution of the system

$$\begin{cases}
D_1 u = -\lambda^2 u, & \lambda \in [0, +\infty[\\
D_2\, u = -4\lambda(m + \frac{\alpha+1}{2})u, & \\
u(0,0) = 1, \frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial t}(0,0) = 0 &
\end{cases}$$

is the function $\varphi_{\lambda,m}$, $(\lambda, m) \in \Gamma$, given by,

$$\varphi_{\lambda,m}(x,t) = j_{\alpha-1/2}(\lambda t)\mathcal{L}_m^{(\alpha)}(\lambda x^2), \qquad (x,t) \in K,$$

where $j_{\alpha-1/2}$ is the function defined on $[0, +\infty[$, by

$$j_{\alpha-1/2}(\lambda t) = \begin{cases} 2^{\alpha-1/2}\Gamma(\alpha+1/2)\frac{J_{\alpha-1/2}(\lambda t)}{(\lambda t)^{\alpha-1/2}}, & if \;\; \lambda t \neq 0\\ 1, & if \;\; \lambda t = 0 \end{cases}$$

here $J_{\alpha-1/2}$ is the Bessel function of order $\alpha - 1/2$, and $\mathcal{L}_m^{(\alpha)}$, $m \in \mathbb{N}$, is the Laguerre function defined on $[0, +\infty[$, by

$$\mathcal{L}_m^{(\alpha)}(x) = e^{-\frac{x}{2}}\frac{L_m^{(\alpha)}(x)}{L_m^{(\alpha)}(0)},$$

$L_m^{(\alpha)}$ being the Laguerre polynomial of degree m and order α.

For all $(\lambda, m) \in \Gamma$, the functions $\varphi_{\lambda,m}$ satisfy the following product formula

- If $\alpha > 0$. For all $(x,t), (y,s) \in K$, we have

$$\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s) = A_\alpha \int_{[0,\pi]^3} \varphi_{\lambda,m}(\Delta(x,y,\theta+\pi), \Delta(X,Y,\xi)) dZ_\alpha(\theta, \Psi, \xi),$$

with

$$A_\alpha = \frac{(\alpha+1)\Gamma(\alpha+1/2)}{\pi^{3/2}\Gamma(\alpha)}, \quad \Delta(x,y,\theta) = \sqrt{x^2+y^2-2xy\cos\theta}, \tag{2.1}$$

$$X = X(t,s,\psi) = \Delta(t,s,\psi), Y = Y(x,y,\theta) = xy\sin\theta, \tag{2.2}$$

and

$$dZ_\alpha(\theta,\Psi,\xi) = (\sin\xi)^{2\alpha-1}\ (\sin\Psi)^{2\alpha-1}(\sin\theta)^{2\alpha} d\xi d\Psi d\theta.$$

• If $\alpha = 0$, for all $(x,t),(y,s) \in K$, we have

$$\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s) = \frac{1}{4\pi}\sum_{0\le i,j\le 1}\int_0^\pi \varphi_{\lambda,m}(\Delta(x,y,\theta), xy\sin\theta + (-1)^i t + (-1)^j s)d\theta.$$

Properties.
i) For all $(\lambda,m) \in \Gamma$, the functions $\varphi_{\lambda,m}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable and we have

$$\sup_{(x,t)\in K}\left|\varphi_{\lambda,m}(x,t)\right| = 1. \tag{2.3}$$

ii) For all $(\lambda,m) \in \Gamma$, and $(x,t) \in K$, we have

$$\Lambda_1\varphi_{\lambda,m}(x,t) = \ -x^2\varphi_{\lambda,m}(x,t), \tag{2.4}$$

$$\Lambda_2\varphi_{\lambda,m}(x,t) = \ -t^2\varphi_{\lambda,m}(x,t). \tag{2.5}$$

Notation . We denote by
• $\mathcal{S}(K)$ the space of $C^\infty-$functions on $\mathbb{R}^2$, even with respect of each variable and rapidly decreasing together with all their derivatives i.e for all $k,p,q \in \mathbb{N}$, we have

$$N_{k,p,q}(f) = \sup_{(x,t)\in K}\left\{(1+x^2+t^2)^k\left|\frac{\partial^{p+q}}{\partial x^p\partial t^q}f(x,t)\right|\right\} < +\infty.$$

• $\mathcal{S}(\Gamma)$ the space of functions g : $\mathbb{R}\times\mathbb{N} \longrightarrow \mathbb{C}$, even with respect to the first variable and satisfying

i) For all $m,p,q,r,s \in \mathbb{N}$, the function

$$\lambda \ \longrightarrow \ \lambda^p(\lambda\,(\mathrm{m} + \frac{\alpha+1}{2}))^q\Lambda_1^r\Lambda_2^q g(\lambda,m)$$

is bounded and continuous on $[0,+\infty[$, C^∞ on $]0,+\infty[$, and the right derivatives at zero exists.

ii) For all $k,p,q \in \mathbb{N}$, we have

$$\nu_{k,p,q}(g) = \sup_{(\lambda,m)\in\Gamma}(1+\lambda^2(1+m^2))^k\,|\Lambda_1^p\Lambda_2^q g(\lambda,m)| < +\infty.$$

Equipped with the topology defined by the semi-norms $N_{k,p,q}$ (resp $\nu_{k,p,q}$) the space $\mathcal{S}(K)$ (resp S(Γ)) is a Fréchet space.

• γ_α the measure on Γ, given by

$$d\gamma_\alpha(\lambda,m) = \ \frac{\lambda^{3\alpha+1}}{2^{2\alpha-1}\Gamma(\alpha+1/2)}L_m^{(\alpha)}(0)d\lambda\otimes\delta_m,$$

δ_m is the Dirac measure at m and $d\lambda$ is the Lebesgue measure.

We have for f and g be in $\mathcal{S}(\Gamma)$

$$\int_\Gamma f(\lambda,m)\Lambda_1 g(\lambda,m)\mathrm{d}\gamma_\alpha(\lambda,m)=\int_\Gamma \Lambda_1 f(\lambda,m)g(\lambda,m)\mathrm{d}\gamma_\alpha(\lambda,m) \tag{2.6}$$

and

$$\int_\Gamma f(\lambda,m)\Lambda_2 g(\lambda,m)d\gamma_\alpha(\lambda,m)=\int_\Gamma \Lambda_2 f(\lambda,m)g(\lambda,m)\mathrm{d}\gamma_\alpha(\lambda,m). \tag{2.7}$$

Definition 2.1.
i) The translation operators $\tau_{(x,t)}, (x,t)\in K$, are defined for a continuous function f on K, by

- If $\alpha>0$

$$\tau_{(x,t)}f(y,s)=A_\alpha\int_{[0,\pi]^3} f(\Delta(x,y,\theta+\pi),\Delta(X,Y,\xi))dZ_\alpha(\theta,\Psi,\xi).$$

where A_α, $\Delta(x,y,\theta), X$ and Y are given by (2.1) and (2.2).

- If $\alpha=0$.

$$\tau_{(x,t)}f(y,s)=\frac{1}{4\pi}\sum_{0\le i,j\le 1}\int_0^\pi f(\Delta(x,y,\theta+\pi),xy\sin\theta+(-1)^i t+(-1)^j s)d\theta.$$

ii) The convolution product of two continuous functions f,g on K, with compact support is defined by

$$(f\star_\alpha g)(x,t)=\int_K \tau_{(x,t)}f(y,s)g(y,s)dm_\alpha(y,s).$$

We have the following properties :

i) For all $(x,t),(y,s)\in K$ and $f\in\mathcal{S}(K)$

$$\tau_{(0,0)}f(y,s)=f(y,s),\tau_{(x,t)}f(y,s)=\tau_{(y,s)}f(x,t).$$

ii) The functions $\varphi_{\lambda,m},(\lambda,m)\in\Gamma$, satisfy the following product formula:

$$\tau_{(x,t)}\varphi_{\lambda,m}(y,s)=\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s).$$

iii) Let $p\in[1,+\infty]$ and $f\in L^p(K)$. Then for all $(x,t)\in K$, we have

$$\left\|\tau_{(x,t)}f\right\|_{p,m_\alpha}\le\|f\|_{p,m_\alpha}. \tag{2.8}$$

iv) Let g in $L^1(K)$. Then for all $(x,t)\in K$, we have

$$\int_K \tau_{(x,t)}g(y,s)dm_\alpha(y,s)=\int_K g(y,s)dm_\alpha(y,s). \tag{2.9}$$

v) Let f,g be two continuous functions on K with compact support, then

$$f\star_\alpha g=g\star_\alpha f,$$

and furthermore if supp$f\subset[-R_1,R_1]\times[-R_1,R_1]$ and supp$g\subset[-R_2,R_2]\times[-R_2,R_2]$, then

$$\text{supp}(f\star_\alpha g)=[-(R_1+R_2),(R_1+R_2)]\times[-(1+R_2)(R_1+R_2),(1+R_2)(R_1+R_2)]$$

Definition 2.2. The Bessel-Laguerre Fourier transform $\mathcal{F}$, is defined on $L^1(K)$, by

$$\mathcal{F}(f)(\lambda,m)=\int_K\varphi_{\lambda,m}(x,t)f(x,t)dm_\alpha(x,t),\quad(\lambda,m)\in\Gamma.$$

The Bessel-Laguerre Fourier transform $\mathcal{F}$ satisfy the following properties:
i) For all f in $(L^1 \cap L^2)(K)$, we have the following Plancherel formula

$$\|\mathcal{F}(f)\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}. \tag{2.10}$$

ii) The Bessel-Laguerre Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{S}(K)$ onto $\mathcal{S}(\Gamma)$, its inverse is given by

$$\mathcal{F}^{-1}(g)(x,t) = \int_\Gamma g(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \quad (x,t) \in K.$$

iii) Let f be in $L^1(K)$ such that $\mathcal{F}(f) \in L^1(\Gamma)$, then we have the inversion formula

$$f(x,t) = \int_\Gamma \mathcal{F}(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \quad \text{a.e on } K. \tag{2.11}$$

iv) Let $f \in L^1(K)$. Then for all $(x,t) \in K$ and $(\lambda,m) \in \Gamma$, we have

$$\mathcal{F}(\tau_{(x,t)}f)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathcal{F}(\lambda,m). \tag{2.12}$$

v) Let $p \in]1,2]$ and $q = \dfrac{p}{p-1}$, then for all $f \in L^p(K)$, we have

$$||\mathcal{F}(f)||_{q,\gamma_\alpha} \leq ||f||_{p,m_\alpha}. \tag{2.13}$$

3. BESSEL-LAGUERRE WIGNER TRANSFORM

In this section, we define and study the Bessel-Laguerre Wigner transform. We establish an inversion formula for this transform.

Notation . We denote by $\mathcal{S}(K \times \Gamma)$ the space of functions σ defined on $K \times \Gamma$, such that
i) For all $(\lambda,m) \in \Gamma$, the function $(x,t) \mapsto \sigma((x,t),(\lambda,m))$ belongs to $\mathcal{S}(K)$.
ii) For all $(x,t) \in K$, the function $(\lambda,m) \to \sigma((x,t),(\lambda,m))$ belongs to $\mathcal{S}(\Gamma)$.

Definition 3.1. The Bessel-Laguerre Wigner transform V, is defined on $\mathcal{S}(K) \times \mathcal{S}(K)$, by

$$V(f,g)((x,t),(\lambda,m)) = \int_K \varphi_{\lambda,m}(t,s)f(y,s)\tau_{(x,t)}g(y,s)dm_\alpha(y,s),$$

where $(x,t) \in K$ and $(\lambda,m) \in \Gamma$.

Remark 3.1. The transform V can also be written in the forms

$$V(f,g)((x,t),(\lambda,m)) = \mathcal{F}(f\tau_{(x,t)}g)(\lambda,m) = g \star_\alpha (f\varphi_{\lambda,m}(x,t). \tag{3.1}$$

Proposition 3.1.
i) The transform V is a bilinear mapping from $\mathcal{S}(K) \times \mathcal{S}(K)$ into $\mathcal{S}(K \times \Gamma)$.
ii) Let $p,q \in [1,+\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all f,g in $\mathcal{S}(K)$, we have

$$\|V(f,g)\|_{\infty,m_\alpha\otimes\gamma_\alpha} \leq \|f\|_{p,m_\alpha}\|g\|_{q,m_\alpha}.$$

iii) Let f,g be in $\mathcal{S}(K)$, then we have

$$\|V(f,g)\|_{2,m_\alpha\otimes\gamma_\alpha} \leq \|f\|_{2,m_\alpha}\|g\|_{2,m_\alpha}.$$

iv) Let f be in $L^p_\alpha(K), p \in]1,2[$ and g in $L^q(K)$ with q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have

$$\|V(f,g)\|_{q,m_\alpha\otimes\gamma_\alpha} \leq \|f\|_{p,m_\alpha}\|g\|_{q,m_\alpha}.$$

Proof.
i) Let $f \in \mathcal{S}(K)$, then the mapping $(x,t) \to T^{(\alpha)}_{(x,t)} f$ is continuous from K into $\mathcal{S}(K)$. (see [1]).
ii) Using Hölder's inequality, (2.8) and Definition 3.1, we deduce the assertion.
iii) We have

$$\begin{aligned}\|V(f,g)\|_{2,m_\alpha\otimes\gamma_\alpha} &= \int_K\int_\Gamma |V(f,g)((x,t),(\lambda,m))|^2 d\gamma_\alpha(\lambda,m)dm_\alpha(x,t)\\ &= \int_K\int_\Gamma |\mathcal{F}(f\tau_{(x,t)}g)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)dm_\alpha(x,t),\end{aligned}$$

then the result follows from (2.8) and (2.10).
iv) Formulas (2.13), (3.1) and Minkowski's inequality for integral, yields to

$$\begin{aligned}\|V(f,g)\|^2_{q,m_\alpha\otimes\gamma_\alpha} &\leq \int_K [\int_K |f(y,s)|^p |T^{(\alpha)}_{(x,t)} g(y,s)|^p dm_\alpha(y,s)|]^{q/p} dm_\alpha(x,t)\\ &\leq \{\int_K [\int_K |f(y,s)|^p |\tau_{(x,t)} g(y,s)|^q dm_\alpha(x,t)]^{p/q} dm_\alpha(y,s)\}^{q/p}\\ &\leq \|f\|^q_{p,m_\alpha}\|g\|^q_{q,m_\alpha}.\end{aligned}$$

Which finishes the proof of the assertion. ∎

Proposition 3.2. *Let f and g be in $\mathcal{S}(K)$. Then for $(x,t) \in K$ and $(\lambda,m) \in \Gamma$, we have*

$$\mathcal{F}\otimes\mathcal{F}^{-1}[V(f,g)]((\lambda,m),(x,t)) = f(x,t)\varphi_{\lambda,m}(x,t)\mathcal{F}(g)(\lambda,m).$$

Proof. Using the relations (2.11),(2.12), (3.1), (2.10) and Fubini's Theorem, we have $\mathcal{F}\otimes\mathcal{F}^{-1}[V(f,g)]((\lambda,m),(x,t))$

$$= \int_\Gamma\int_K \varphi_{\lambda,m}(y,s)V(f,g)((y,s),(\beta,n))\varphi_{\beta,n}(x,t)dm_\alpha(y,s)d\gamma_\alpha(\beta,n)$$

$$= \int_K f(x,t)\tau_{(y,s)}g(x,t)\varphi_{\lambda,m}(y,s)dm_\alpha(y,s)$$

$$= \int_K f(x,t)\tau_{(y,s)}g)(\beta,n)\varphi_{(\beta,n)}(x,t)d\gamma_\alpha(\beta,n)]\varphi_{\lambda,m}(y,s)dm_\alpha(y,s)$$

$$= f(x,t)\mathcal{F}(\tau_{(x,t)}g)(\lambda,m) = f(x,t)\varphi_{\lambda,m}(x,t)\mathcal{F}(g)(\lambda,m),$$

which proves the result. ∎

Corollary 3.3. *Let f and g be in $S(K)$. For $(\lambda,m) \in \Gamma$ and $(x,t) \in K$, we have the following relations*

$$\begin{aligned}\int_K \mathcal{F}\otimes\mathcal{F}^{-1}[V(f,g)]((\lambda,m),(x,t))dm_\alpha(x,t) &= \mathcal{F}(f)(\lambda,m)\mathcal{F}(g)(\lambda,m),\\ \int_\Gamma \mathcal{F}\otimes\mathcal{F}^{-1}[V(f,g)]((\lambda,m),(x,t))d\gamma_\alpha(\lambda,m) &= f(x,t)g(x,t).\end{aligned}$$

Proof. We deduce these relations from Proposition 3.2, Fubini's Theorem and (2.11). ∎

Theorem 3.4. *Let g be in $L^1(K)\cap L^2(K)$ such that $c = \int_K g(x,t)dm_\alpha(x,t) \neq 0$. Then for all f in $(L^1\cap L^2)(K)$ we have*

$$\mathcal{F}(f)(\lambda,m) = \frac{1}{c}\int_K V(f,g)((x,t),(\lambda,m))dm_\alpha(x,t), \quad (\lambda,m) \in \Gamma.$$

Proof. Using Definition 3.1, Fubini's Theorem and formulas (2.8), (2.9), we obtain the result. ∎

Corollary 3.5. *Under the hypothesis of Theorem 3.4, and if moreover $\mathcal{F}(f)$ belongs to $L^1(\Gamma)$, then we have the following inversion formula for the Bessel-Laguerre Wigner-transform*

$$f(z,r)=\frac{1}{c}\int_\Gamma\int_K \varphi_{\lambda,m}(z,r)V(f,g)((x,t),(\lambda,m))dm_\alpha(x,t)d\gamma_\alpha(\lambda,m),(z,r)\in K.$$

Proof. We obtain the result from (2.10) and Theorem 3.4. ∎

4. BESSEL-LAGUERRE WEYL TRANSFORM

In this section, we consider a class of symbols which allows to define the Bessel-Laguerre Weyl transform. We establish a relation between the Wigner and Weyl transform. At last, we discuss criterion in term of symbols for the boundedness and compactness of the Bessel-Laguerre Weyl transform.

4.1. Bessel-Laguerre Weyl transform with symbols in Schwartz spaces.

Definition 4.1. A function $\sigma((x,t),(\lambda,m))$ belongs to a class of $\mathcal{S}^l(K\times\Gamma)$ (resp $\mathcal{S}^l_0(K\times\Gamma)$) if it satisfy

i) For fixed $(\lambda,m)\in\Gamma$, the function $(x,t)\to\sigma((x,t),(\lambda,m))$ defined on K is C^∞ on $\mathbb{R}^2$ even with respect to each variables.

ii) For fixed $(x,t)\in K$ and for all $m\in\mathbb{N}$, the function $\lambda\to\sigma((x,t),(\lambda,m))$ is C^∞on]0,+∞[, the right derivatives at zero exist and for all $k,r,s,p,q\in N$, there exist a constant $C>0$, such that

$$\left|\frac{\partial^{p+q}}{\partial x^p\partial t^q}\Lambda_1^r\Lambda_2^s\sigma((x,t),(\lambda,m))\right|\leq C\,(1+\lambda^2(1+m^2))^{l-r-s}$$

$(resp.$

$$(1+x^2+t^2)^k\left|\frac{\partial^{p+q}}{\partial x^p\partial t^q}\Lambda_1^r\Lambda_2^s\sigma((x,t),(\lambda,m))\right|\leq C(1+\lambda^2(1+m^2))^{l-r-s})).$$

Notation . We denote by

$$\mathcal{S}^\infty_0(K\times\Gamma)=\bigcap_{l\in\mathbb{R}}\mathcal{S}^l_0(K\times\Gamma).$$

Remark 4.1. Let $q\in[1,+\infty[$ and $l\in\mathbb{R}$ such that $lq<-\frac{3\alpha+2}{2}$. Then $\mathcal{S}^l_0(K\times\Gamma)$ is included in $L^q(K\times\Gamma)$.

Proposition 4.1. *The space $\mathcal{S}^\infty_0(K\times\Gamma)$ is dense in $L^r(K\times\Gamma),\ 1\leq r<+\infty$.*

Proof. The proof is the same as for Proposition II. 13 p. 350 in [3]. ∎

Definition 4.2. Let σ be in $\mathcal{S}^l(K\times\Gamma), l<-(3\alpha+2)/2$. We define the Bessel-Laguerre Weyl transform on $\mathcal{S}(K)$, by

$$W_\sigma(f)(x,t)=\int_\Gamma\int_K\varphi_{\lambda,m}(x,t)\sigma((y,s),(\lambda,m))\tau_{(x,t)}f(y,s)dm_\alpha(y,s)d\gamma_\alpha(\lambda,m). \tag{4.1}$$

Lemma 4.2. *Let σ be in $\mathcal{S}^\infty_0(K\times\Gamma)$. The function k defined on $K\times K$, by*

$$k((x,t),(y,s))=\int_\Gamma\varphi_{\lambda,m}(x,t)\tau_{(x,t)}[\sigma(.,(\lambda,m))](y,s)d\gamma_\alpha(\lambda,m),$$

satisfies

i) For all $q\in[1,+\infty[$ and $p\in\mathbb{N}^*$ such that $pq>\alpha+3/2$, there exists $M_{pq}>0$ such that

$$\int_K|k((x,t),(y,s))|^q\ dm_\alpha(y,s)\leq\frac{M_{pq}}{(1+x^2+t^2)^{pq}}\ ;\quad (x,t)\in K. \tag{4.2}$$

ii) The function k belongs to $L^q(K \times K), q \geq 1$.

Proof.
i) We consider the function

$$\Phi((y,s),(\lambda,m)) = (1+\lambda^2(1+m^2))^{-l}\,[I-\Lambda_1-\Lambda_2]^p\,\sigma((y,s),(\lambda,m)).$$

Let $q \in [1,+\infty[, p, l \in \mathbb{N}$ such that $p > \frac{(2\alpha+3)l}{2q}, l < -(\alpha+2)/2q$. Using the relations (2.4), (2.5), (2.3), (2.6) and (2.7), we get

$$\begin{aligned} &(1+x^2+t^2)^p k((x,t),(y,s)) \\ = &\int_\Gamma [I-\Lambda_1-\Lambda_2]^p\,\varphi_{\lambda,m}(x,t)\tau_{(x,t)}[\sigma(.,(\lambda,m))](y,s)d\gamma_\alpha(\lambda,m) \\ = &\int_\Gamma \tau_{(x,t)}\Phi(.,(\lambda,m))(y,s)\varphi_{\lambda,m}(x,t)(1+\lambda^2(1+m^2))^l d\gamma_\alpha(\lambda,m). \end{aligned}$$

Let q' such that $1/q+1/q'=1$. Using Hölder's inequality and (2.8), we deduce that

$$(1+x^2+t^2)^{qp}\int_K |k((x,t),(y,s))|^q\,dm_\alpha(y,s) \leq \left\|\,(1+\lambda^2(1+m^2))^l\right\|_{q,\gamma_\alpha}^q \|\Phi\|_{q',m_\alpha\otimes\gamma_\alpha}.$$

We obtain (4.2) from this inequality.
ii) We deduce the result from (4.2). ∎

Theorem 4.3. *Let σ be in $\mathcal{S}_0^\infty(K \times \Gamma)$. Then we have*
i) For all f in $\mathcal{S}(K)$

$$W_\sigma(f)(x,t) = \int_K k((x,t),(y,s))\,f(y,s)dm_\alpha(y,s), \quad (x,t) \in K.$$

ii) Let $q \in [1,+\infty[$ and q' be such that $1/q+1/q'=1$. Then for all f in $\mathcal{S}(K)$, we have

$$\|W_\sigma(f)\|_{q,m_\alpha} \leq \|f\|_{q,m_\alpha}\|k\|_{q',m_\alpha\otimes m_\alpha}.$$

iii) The operator W_σ can be extended to a continuous linear operator denoted also W_σ, from $L^q(K)$ into $L^{q'}(K)$. In particular W_σ is a Hilbert-Schmidt operator and a compact operator in $L^2(K)$.

Proof. We deduce these results from Lemma 4.2. ∎

Definition 4.3. For all $\sigma \in \mathcal{S}^l(K\times\Gamma), l \in \mathbb{R}$, we define the operator H_σ on $\mathcal{S}(K)\times\mathcal{S}(K)$, by $H_\sigma(f,g)(z,r) =$

$$\int_\Gamma\int_K \varphi_{\lambda,m}(z,r)\sigma((x,t),(\lambda,m))V(f,g)((x,t),(\lambda,m))dm_\alpha(x,t)d\gamma_\alpha(\lambda,m).$$

We denote by $H_\sigma(f,g) = H_\sigma(f,g)(0,0)$.

Example 4.1. *Let*

$$\sigma_1((x,t),(\lambda,m)) = -\lambda^2; \quad \textit{and} \quad \sigma_2((x,t),(\lambda,m)) = -4\lambda(m+\frac{\alpha+1}{2}).$$

Then for all $f, g \in \mathcal{S}(K)$, we have

$$\begin{cases} H_{\sigma_1}(f,g)(z,r) = c\,(D_1 f)(z,r), \\ H_{\sigma_2}(f,g)(z,r) = c\,(D_2 f)(z,r)\,. \end{cases}$$

with $c = \int_K g(x,t)dm_\alpha(x,t)$.

Corollary 4.4. *Let σ be in $\mathcal{S}_0^l(K\times\Gamma), l<-(3\alpha+2)/2$. Then for $f,g\in\mathcal{S}(K)$, we have*

$$H_\sigma(f,g) =< W_\sigma(g), \overline{f} >,$$

where $< .,. >$ is the inner product in $L^2(K)$.

Proof. Let $l<-(3\alpha+2)/2$, using Definitions (3.1), (4.1), (4.2), the relation (2.8) and Fubini's theorem, we obtain

$$H_\sigma(f,g) = \int_K f(y,s)W_\sigma(g)(y,s)dm_\alpha(y,s) =< W_\sigma(g), \overline{f} >,$$

which finishes the proof. ∎

4.2. **Bessel-Laguerre Weyl transform with symbols in $L^r(K\times\Gamma), 1\leq r\leq 2$.**

Notation . We denote by $B(L^r(K))$ the C^*-algebra of all bounded linear operators Ψ from $L^2(K)$ into itself with the norm

$$\|\Psi\|_* = \sup_{\|f\|_{2,m_\alpha}=1} \|\Psi(f)\|_{2,m_\alpha}.$$

Proposition 4.5. *For $r\in[1,2]$. There exists a unique bounded linear operator*

$$W : L^r(K\times\Gamma) \to B(L^2(K))$$

$$\sigma \to W_\sigma$$

such that for all f,g in $\mathcal{S}(K)$

$$<W_\sigma(g),\overline{f}> = \int_\Gamma\int_K \sigma((x,t),(\lambda,m))V(f,g)((x,t),(\lambda,m))dm_\alpha(x,t)d\gamma_\alpha(\lambda,m).$$

and

$$\|W_\sigma\|_* \leq \|\sigma\|_{r,m_\alpha\otimes\gamma_\alpha}. \tag{4.3}$$

Proof. Let σ be in $\mathcal{S}_0^\infty(K\times\Gamma)$. Then using Proposition 3.1, we have

$$\|W_\sigma\|_* \leq \|\sigma\|_{1,m_\alpha\otimes\gamma_\alpha} \ ; \quad \|W_\sigma\|_* \leq \|\sigma\|_{2,m_\alpha\otimes\gamma_\alpha}.$$

From these relations and the Riesz-Thorin Theorem (see [2]), we deduce that for all $\sigma\in L^r(K\times \Gamma, r\in[1,2]$, we have

$$\|W_\sigma\|_* \leq \|\sigma\|_{r,m_\alpha\otimes\gamma_\alpha},$$

which finishes the proof. ∎

Theorem 4.6. *Let σ be in $L^r(K\times\Gamma), 1\leq r\leq 2$, then W_σ is a compact operator in $L^2(K)$.*

Proof. Let σ be in $L^r(K\times\Gamma)), 1\leq r\leq 2$ and $\{\sigma_k\}_{k\geq1}$ a sequence of functions in $\mathcal{S}_0^\infty(K\times\Gamma)$ such that $\sigma_k\to\sigma$ in $L^r(K\times\Gamma)$ as k $\to+\infty$. Then using Theorem 3.4, W_{σ_k} is compact in $L^2(K)$ for all k $\in\mathbb{N}$. Thus by formula (4.3), W_σ is the limit in $B(L^2(K))$ of the sequence $\{W_{\sigma_k}\}_{k\geq1}$. Then we get the result. ∎

4.3. **Bessel-Laguerre Weyl transform with symbols in** $L^r(K \times \Gamma), 2 < r < +\infty$**.**

Theorem 4.7. *For* $r \in]2, +\infty[$*, there exists a function* $\sigma \in L^r(K \times \Gamma)$*, such that the Bessel-Laguerre Weyl transform is not a bounded linear operator on* $L^2(K)$*.*

Proof. Suppose that for all σ in $L^r(K \times \Gamma), 2 < r < +\infty$, the Weyl transform defined by (4.1) is a bounded linear operator on $L^2(K)$. Then for all σ in $L^r(K \times \Gamma)$, there exists a positive constant C_σ such that

$$\|W_\sigma\|_* \leq C_\sigma.$$

Let f and g be two functions in $\mathcal{S}(K)$ such that $\|f\|_{2,m_\alpha} = \|g\|_{2,m_\alpha} = 1$, consider the bounded linear functional $Q_{f,g} : L^r(K \times \Gamma) \to \mathbb{C}$, defined by

$$Q_{f,g}(\sigma) = \ < W_\sigma(g), \overline{f} > .$$

Then using Banach-Steinhauss Theorem, there exists a positive constant C such that for all f, g in $\mathcal{S}(K)$ with $\|f\|_{2,m_\alpha} = \|g\|_{2,m_\alpha} = 1$, we have

$$\|Q_{f,g}\|_* \leq C.$$

Let $r > 2$ and r' such that $1/r + 1/r' = 1$, then by Corollary 4.4, we have for all f, g in $\mathcal{S}(K)$ with $\|f\|_{2,m_\alpha} = \|g\|_{2,m_\alpha} = 1$,

$$\sup_{\|\sigma\|_{r,m_\alpha\otimes\gamma_\alpha}} \left|< W_\sigma(g), \overline{f} >\right| = \|V(f,g)\|_{r',m_\alpha\otimes\gamma_\alpha} \leq C.$$

Using the density of $\mathcal{S}(K)$ into $L^2(K)$, we deduce that for all f in $L^2(K)$, we have

$$\|V(f,f)\|_{r',m_\alpha\otimes\gamma_\alpha} \leq C \|f\|^2_{2,m_\alpha}. \tag{4.4}$$

Now let f be an even function on K in $L^2(K)$, such that its support is included in $[-1,1] \times [-1,1]$. So for all $(\lambda, m) \in \Gamma$, the support of the function $(x,t) \to V(f,f)((x,t),(\lambda,m))$ is included in $B = [-2,2] \times [-4,4]$.
Using Hölder′s inequality and (4.4), we get

$$\left\| \int_B V(f,f)((x,t),(\lambda,m)) dm_\alpha(x,t) \right\|_{r',\gamma_\alpha} \leq [m_\alpha(B)]^{1/r} \|V(f,f)\|_{r',m_\alpha\otimes\gamma_\alpha}.$$

Thus the function $(\lambda, m) \to \int_B V(f,f)((x,t),(\lambda,m)) dm_\alpha(x,t)$ belongs to $L^r(K \times \Gamma), 1 < r' < 2$. On the other hand using Theorem 3.4, we have

$$\mathcal{F}(f)(\lambda,m) = \frac{1}{c} \int_K V(f,f)((x,t),(\lambda,m)) dm_\alpha(x,t)$$

where $c = \int_K f(x,t) dm_\alpha(x,t) \neq 0$. So we deduce that the function $(\lambda, m) \to \mathcal{F}(f)(\lambda, m)$ belongs to $L^{r'}(\Gamma), 1 < r' < 2$.

Now we consider the function $f(x,t) = h_1(t)h_2(x)$, with

$$h_1(t) = \begin{cases} t^{2(k-\alpha-1/2)}, & \text{, if } 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases} \quad ; \quad h_2(x) = \begin{cases} x^{2(k-\alpha-1)}, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

where $k \in \mathbb{R}$ such that $k > \frac{\alpha+1}{2}$. The function f belongs to $L^2(K)$ and we shall prove that $\mathcal{F}(f)$ does not belongs to $L^{r'}(K \times \Gamma), 1 < r' < 2$, so inequality (4.4) is not valid .

From Definition 2.2, we deduce that

$$\mathcal{F}(f)(\lambda,m)=a_\alpha\frac{1}{\lambda^{3k}}\ \left(\int_0^\lambda u^{2k-1}j_{\alpha-1/2}(u)du\right)\left(\int_0^\lambda e^{-\frac{u}{2}}L_m^{(\alpha)}(u)u^{k-1}du\right),$$

where

$$a_\alpha=\frac{1}{2\Gamma(\alpha+1)\Gamma(\alpha+1/2)L_m^{(\alpha)}(0)}.$$

But we have

$$\begin{aligned}\|\mathcal{F}(f)\|_{r',\gamma_\alpha}^{r'} &= C_\alpha\sum_{m=0}^{\infty}L_m^{(\alpha)}(0)\int_0^{+\infty}|\mathcal{F}(f)(\lambda,m)|^{r'}\lambda^{3\alpha+1}d\lambda\\ &\geq C_\alpha(a_\alpha)^{r'}\int_0^{+\infty}|g(\lambda)|^{r'}\lambda^{3\alpha-3kr'+1}d\lambda,\end{aligned}$$

where

$$C_\alpha=\frac{1}{2^{2\alpha-1}\Gamma(\alpha+1/2)}\ ;\quad g(\lambda)=\left(\int_0^\lambda u^{2k-1}j_{\alpha-1/2}(u)du\right)\left(\int_0^\lambda e^{-\frac{u}{2}}u^{k-1}du\right).$$

But using asymptotic formula for $j_{\alpha-1/2}(u)$, (see [4]), we deduce that there is a positive constant A, such that for $\lambda>R$, we have

$$\left|\int_0^\lambda u^{2k-1}j_{\alpha-1/2}(u)du\right|\geq A.$$

So there exists $M>0$ and $R>0$, such that if $\lambda>R$, then $|g(\lambda)|\geq M$. Thus

$$\|\mathcal{F}(f)\|_{r',\gamma_\alpha}^{r'}\geq C_\alpha a_\alpha M^{r'}\int_R^{+\infty}\lambda^{3\alpha-3kr'+1}d\lambda.$$

Hence

$$\|\mathcal{F}(f)\|_{r',\gamma_\alpha}^{r'}=+\infty\ ,\ \text{if}\quad k<\frac{3\alpha+2}{3r'}.$$

Therefore the inequality (4.4) is impossible if we pick k to be some number in the interval $\left]\frac{3\alpha+2}{6},\frac{3\alpha+2}{3r'}\right[$. ∎

REFERENCES

[1] E. JEBBARI, Harmonic analysis associated with system of partial differential operators D_1 and D_2, *Preprint, 2004.*

[2] G. B. FOLLAND, *Real Analysis. Modern Techniques and Their Applications*, Wiley, New York, 1984.

[3] M. M. NESSIBI and K. TRIMECHE, Inversion of the radon transform on the Laguerre hypergroup by using generalized wavelets, *J. Math. Anal. Appl.* **208** (1997) 337-363.

[4] G. N. WATSON, *A treatise on the theory of Bessel function.* 2^{nd} ed Cambridge London, Univ.Press and New -York, 1996.

[5] H. WEYL, *The Theory of Groups and Quantum Mechanics*, Dover ,1950.

[6] M. W. WONG, *Weyl Transforms*, Springer-Verlag New York (1998).