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A RELATION BETWEEN NUCLEAR CONES AND FULL NUCLEAR CONES

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ABSTRACT. The notion of nuclear cone in locally convex spaces corresponds to the notion of well based cone in normed spaces. Using the bipolar theorem from locally convex spaces it is proved that every closed nuclear cone is a full nuclear cone. Thus every closed nuclear cone can be associated to a mapping from a family of continuous seminorms in the space to the topological dual of the space. The relation with Pareto efficiency is discussed.

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1. INTRODUCTION

The notion of the *nuclear cone* was defined by G. Isac in 1983 in [7] as a mathematical tool for the study of Pareto efficiency, a fundamental notion in the theory of multiobjective optimization. Since its definition the notion of the nuclear cone (known also under the name of *supernormal cone*) has been considered by several authors in relation to several kinds of problems in optimization theory, in the best approximation theory, in the fixed point theory, in the study of nuclearity of topological vector spaces (in Grothendieck's sense), in the study of absolute summability, in the study of C^*-algebras, in the study of some geometrical aspects of Ekeland's principle, etc. About the applications of nuclear cones the reader is referred to the references in the paper [14]. In 1990, C. Pontini introduced the notion of pseudo-nuclear cone and applying it he studied interesting geometrical properties of convex cones in general topological vector spaces [20].

In a recent paper [14] concerning Pareto optimization G. Isac and A. O. Bahya introduced the notion of *full nuclear cone* as a tool of a deeper investigation of the existence of efficient points of some sets in an ordered locally convex spaces. Full nuclearity is for Pareto efficiency the expression of a kind of *multiple scalarization*. In the cited paper the important problem of the relation between nuclearity and full nuclearity is reached. The question is if whether or no a closed nuclear cone is full nuclear? For the particular case of well based closed convex cones in a normed space (which are also nuclear ones) an affirmative answer follows from a paper of M. Petschke [18] and for well based closed convex cones in locally convex Hausdorff spaces from the results in [12] and [13]. The aim of our note is to give an affirmative answer to this question in the case of general closed nuclear cones in locally convex Hausdorff spaces.

From the main result of the note it follows that each closed nuclear cone can be associated to a mapping from a family of continuous seminorms of the locally convex space to its topological dual. This way a new prospective is gained on closed nuclear cones. A relation with Pareto efficiency is also considered.

2. PRELIMINARIES

If E is a real vector space, then a function $p : E \longrightarrow \mathbb{R}$ is a *seminorm* if:

(1) $p(\alpha x) = |\alpha| p(x), \ \forall\, x \in E, \ \forall\, \alpha \in \mathbb{R}$, and
(2) $p(x + y) \leq p(x) + p(y), \ \forall\, x,\, y \in E.$

According to an approach of Treves [27], a locally convex space can be defined as a couple $(E, \mathrm{Spec}(E))$, where E is a real vector space and $\mathrm{Spec}(E)$ is a family of seminorms on E such that:

(1) $\lambda p \in \mathrm{Spec}(E)$, whenever $\lambda \in \mathbb{R}_+ = [0, +\infty)$, and $p \in \mathrm{Spec}(E)$,
(2) if $p \in \mathrm{Spec}(E)$ and q is a seminorm on E such that $q \leq p$, then $q \in \mathrm{Spec}(E)$, and
(3) for every $p_1,\, p_2 \in \mathrm{Spec}(E)$, $\sup(p_1, p_2) \in \mathrm{Spec}(E)$, where $\sup(p_1, p_2)(x) = \sup(p_1(x), p_2(x))$, for any $x \in E$.

It is known (see [27]) that if $\mathrm{Spec}(E)$ is given, then there exists a locally convex topology τ on E such that $E(\tau)$ is a topological vector space, with the property that a seminorm p on E is τ-continuous if and only if $p \in \mathrm{Spec}(E)$. An approach like this, offers some technical facilities in working with locally convex spaces. All our considerations concerning them will be given in the context of this terminology.

A subset $\mathcal{B} \subset \mathrm{Spec}(E)$ is said a *base* of $\mathrm{Spec}(E)$, if for every $p \in \mathrm{Spec}(E)$ there exists $q \in \mathcal{B}$ and a real number $\lambda \in \mathbb{R}^+ = (0, +\infty)$ such that $p \leq \lambda q$. Obviously, $\mathrm{Spec}(E)$ is its own base. If $\mathcal{B}$ is a base of $\mathrm{Spec}(E)$ and $\mathcal{B} \subset \mathcal{D} \subset \mathrm{Spec}(E)$, then $\mathcal{D}$ is also a base of $\mathrm{Spec}(E)$.

The topology τ defined on E by Spec(E) is Hausdorff if

$$\{x \in E : p(x) = 0, \forall p \in \text{Spec}(E)\} = \{0\}.$$

The base $\mathcal{B}$ of Spec(E) is called a *Hausdorff base* if

$$\{x \in E : p(x) = 0, \forall p \in \mathcal{B}\} = \{0\}.$$

If Spec(E) possesses a Hausdorff base then $(E, \text{Spec}(E))$ Hausdorff.

We will denote by E^* the topological dual of the locally convex space E and by K a *pointed convex cone*, which is a subset K of E satisfying the following properties:

(1) (k_1) $K + K \subset K$,
(2) (k_2) $\lambda K \subset K$, for every $\lambda \in \mathbb{R}_+$, and
(3) $(k3)$ $K \cap (-K) = \{0\}$.

If a pointed convex cone K is given, we have an ordering on E defined by $x \leq y$ whenever $y - x \in K$. This ordering is reflexive, transitive and antisymmetric, translation invariant and invariant by multiplication with scalars from $\mathbb{R}_+$.

A nonempty set K satisfying the properties $(k1)$ and $(k2)$ is called a *convex cone*.

The *dual* K^* of the convex cone K is the set

$$K^* = \{f \in E^* : f(x) \geq 0, \forall x \in K\}.$$

K^* is a closed set in E^* satisfying the axioms $(k_1), (k_2)$. K^* is a pointed, convex cone if and only if $(K - K)^- = E$, where the superscript $^-$ denotes closure of the set. Using the notation

$$K^{**} = \{x \in E : f(x) \geq 0, \forall f \in K^*\},$$

we can state the *bipolar theorem* (an abstract version of the lemma of Farkas), which asserts that if K is a closed, convex cone, then

$$K^{**} = K$$

[26, Theorem IV.1.5.].

If τ is the topology defined by Spec(E), then a pointed convex cone K is said *normal* (with respect to τ) if one of the following equivalent assertions are satisfied:

(1) (n_1) there exists a base $\mathcal{B}$ of Spec(E) such that for every $p \in \mathcal{B}$ and $x, y \in K$ such that $x \leq y$ it holds $p(x) \leq p(y)$,
(2) (n_2) if $(x_i)_{i \in I}$, $(y_i)_{i \in I}$ are two arbitrary nets in K such that for every $i \in I$ it holds $x_i \leq y_i$ and $\lim_{i \in I} y_i = 0$, then it holds $\lim_{i \in I} x_i = 0$.

From the second point of the definition it follows that in Hausdorff spaces this condition implies that K is pointed.

If K is a closed normal cone then $E^* = K^* - K^*$ [26, Corollary 3, Theorem V.3.3.].

Let$K$ be a pointed convex cone. It is said that K is *well based* if there exists a convex bounded set $B \subset E$ such that $0 \notin B^-$ and $K = \cup_{\lambda \geq 0} \lambda B$. A well based cone is a normal cone.

We note that the notions and the results listed above are ones of the most important notions in the theory of ordered topological vector spaces. For more details about these notions the reader is referred to [17], [26] and [6].

3. NUCLEAR AND FULL NUCLEAR CONES

Let $(E, \text{Spec}(E))$ be a locally convex space and $K \subset E$ a pointed convex cone therein. The following definition was introduced in [7] (see also [8]):

Definition 1. *The cone K is called nuclear cone with respect to the topology τ induced by Spec(E) if there exists a base $\mathcal{B}$ of Spec(E) such that for every $p \in \mathcal{B}$ there exists $f_p \in E^*$ such that $p(x) \leq f_p(x), \forall x \in K$.*

Several papers were dedicated to the study of this notion (see the papers [1] - [5], [7] - [15], [21] - [24], [28] - [30]).

For examples of nuclear cones the reader is referred to the papers [1] - [5], [7] - [15]. In [12], [13] it was shown that a cone K in a locally convex space $(E, \text{Spec}(E))$ is well based if and only if there exists a base $\mathcal{B}$ of Spec(E) and a functional $f \in K^*$ such that for each $p \in \mathcal{B}$ there exists a $c_p \in \mathbb{R}^+$ with $c_p p(x) \leq f(x),\ \forall\, x \in K$. Thus a well based cone in a locally convex space is always nuclear. In the paper [14] is given an example of a nuclear cone in a locally convex vector space which isn't well based.

Let $(E, \text{Spec}(E))$ be a locally convex space, $\mathcal{B} \subset$ Spec(E) a Hausdorff base of Spec(E), and $K \subset E$ a closed, pointed, convex cone. Let $\varphi : \mathcal{B} \longrightarrow K^*$ an arbitrary mapping. Given the base $\mathcal{B}$ and the mapping φ in [14] is defined the set

$$K_\varphi = \{x \in E :\ p(x) \leq \varphi(p)(x),\ \forall\, p \in \mathcal{B}\}.$$

In the cited paper it was shown that K_φ is a closed, convex, pointed cone (Lemma 2, 3 and 4).

The cone $K \subset E$ is nuclear if and only if there exists a base $\mathcal{B}$ of Spec(E) and a function $\varphi : \mathcal{B} \longrightarrow K^*$ such that $K \subset K_\varphi$.

Definition 2. *Given the cone K, the base $\mathcal{B}$ of Spec(E) and the function $\varphi : \mathcal{B} \to K^*$, we say that K_φ is the full nuclear cone associated to $K,\ \mathcal{B}$ and φ.*

For details concerning full nuclear cones the reader is referred to the papers [14] and [15].

Remark 3.1. Let us remark that K_φ depends more on φ and less on K. Indeed, if $\mathcal{B}$ and φ are given, and K' is a subcone of K, then $K^* \subset K'^*$ and φ can be considered as function from $\mathcal{B}$ to $K'^* \setminus \{0\}$ and obviously $K'_\varphi = K_\varphi$. Further, from the definition of K_φ it follows that $\varphi(p) \in K^*_\varphi \setminus \{0\},\ \forall\, p \in \mathcal{B}$. Hence we can assume that φ has its range in $K^*_\varphi \setminus \{0\}$.

4. NUCLEAR CONES ASSOCIATED TO A NORMAL CONE

Using the fact that for a normal cone K it holds $E^* = K^* - K^*$, in the paper [14] it was proved the following result:

Proposition 4.1. *If K is a closed normal cone in the locally convex space E and $D \subset K$ is a closed, bounded, convex subset with $0 \notin D$, then there exists a base $\mathcal{B}$ of Spec(E) and a mapping $\varphi : \mathcal{B} \longrightarrow K^*$ such that $K_D = \cup_{\lambda \geq 0} \lambda D \subset K_\varphi$. Thus $K_\varphi \neq \{0\}$.*

This proposition justifies the following definition:

Definition 3. *We say that a closed pointed convex cone K_n is a nuclear cone associated to the cone $K \subset E$, if there exists a base $\mathcal{B}$ of Spec(E) such that for each $p \in \mathcal{B}$ there exists an $f \in K^*$ such that $p(x) \leq f(x)$ for each $x \in K_n$.*

The above proposition shows that - at least for the case of normal cones - this notion is consistent: for each normal cone there exist nontrivial associated nuclear cones.

The notion of full nuclear cone K_φ associated to a base $\mathcal{B}$, and to a function $\varphi : \mathcal{B} \longrightarrow K^*$ defined at the end of the preceding section is a particular form of the above introduced notion. A test for Pareto efficiency using these notions was proved in [14] and improved in [15]. In the last section of this note we shall comment these results.

5. CONVEX CONES REPRESENTED AS FULL NUCLEAR CONES

Let us recall first a construction of Bishop and Phelps (see [19] and [18]).

Let $(E, \|.\|)$ be a normed space, $f \in E^*$ with $\|f\| = 1$ and $\rho > 0$. The set

$$K(f, \rho) = \{x \in E :\ \rho\|x\| \leq f(x)\}$$

is a closed, convex, pointed cone. From the definition of the norm of f it follows that $K(f,\rho) = \{0\}$ for $\rho > 1$ and it is a nontrivial cone for $0 < \rho < 1$, (for $\rho = 1$ it is nontrivial if and only if f attains its supremum on the unit ball of E).

Suppose that $0 < \rho < 1$ and consider the set

$$B = \{x \in E : \|x\| \leq 1, \text{ and } f(x) \geq \rho\}.$$

Then $K(f,\rho)$ is a nontrivial closed, convex, pointed cone, B is a bounded, closed convex set with $0 \notin B$ and

$$K(f,\rho) = \cup_{\lambda \geq 0} \lambda B,$$

that is the cone $K(f,\rho)$ is well based. A such cone is called in [18] a *Bishop-Phelps cone*. In the cited paper Petschke introduced the following notion:

Definition 4. *The convex cone K in the normed vector space $(E, \|.\|)$ is representable as a Bishop-Phelps cone if there is a functional $f \in E^*$ and a norm $p : E \longrightarrow \mathbb{R}_+$ equivalent with $\|.\|$ such that*

$$K = \{x \in E : p(x) \leq f(x)\}.$$

The principal result in [18] is the following theorem:

Theorem 5.1. *Let $(E, \|.\|)$ be a real normed space and K a closed, convex cone such that $K \neq \{0\}$. Then the following assertions are equivalent:*

(1) *K is representable as a Bishop-Phelps cone,*
(2) *K possess a closed bounded base.*

Now, it is easy to see that a Bishop-Phelps cone is a full nuclear cone. By the above theorem it follows then that each cone with a closed bounded base is a full nuclear cone.

Remark 5.1. The remark at the end of Section 3 and the above construction of Bishop and Phelps suggests another natural way of constructing full nuclear cones. For an arbitrary base $\mathcal{B}$ of Spec(E) and a function $\gamma : \mathcal{B} \longrightarrow E^*$ we define the set

$$K_\gamma = \{x \in E : p(x) \leq \gamma(p)(x), \ \forall p \in \mathcal{B}\}.$$

If $\mathcal{B}$ is a Hausdorff base, then K_γ is a closed, convex, pointed cone witch is a full nuclear cone. Obviously, every full nuclear cone with Spec(E) a Hausdorff spectrum is of this form.

Let K be a convex cone in the locally convex space $(E, \text{Spec}(E))$. Using the notation introduced in the Section 3 we give the following definition ([14]):

Definition 5. *The cone K is said to be representable as a full nuclear cone if there exists a base $\mathcal{B}$ of Spec(E) and a mapping $\varphi : \mathcal{B} \longrightarrow K^*$ such that $K = K_\varphi$.*

Obviously, the closeness of K is a necessary condition for its representability as a full nuclear cone. In this regard the following natural question occurs (stated in [14] as an open problem): *Is it true, that for an arbitrary locally convex space, any closed nuclear cone is representable as a full nuclear cone?*

The question is answered into affirmative for well based closed cones in a normed space by the above cited result of Petschke [18] and for well based closed cones in locally convex spaces by the results in [12] and [13]. We shall give next the affirmative answer to the above question for locally convex spaces.

6. EVERY CLOSED NUCLEAR CONE IS REPRESENTABLE AS A FULL NUCLEAR CONE

Let $(E, \mathrm{Spec}(E))$ be a locally convex space and let $K \subset E$ be a closed nuclear cone therein. This according to the definition given in Section 3 means that K is closed convex and pointed cone for which there exists a base $\mathcal{B} \subset \mathrm{Spec}(E)$, $\mathcal{B} = \{p_i : i \in I\}$ and a function $\varphi : \mathcal{B} \longrightarrow K^* \setminus \{0\}$ with the property that for $\forall p_i \in \mathcal{B}$ there hold $p_i(x) \leq \varphi(p_i)(x)\ \forall x \in K$.

In the terminology of the last definition of the preceding section we shall see that each closed nuclear cone is representable as a full nuclear cone. That is, we have the following result:

Theorem 6.1. *If K is a closed nuclear cone in the locally convex space $(E, Spec(E))$, then there exists a base $\mathcal{D}$ of Spec(E) and a function $\psi : \mathcal{D} \longrightarrow K^*$ such that $K = K_\psi$, where K_ψ is the full nuclear cone associated to K, $\mathcal{D}$ and ψ.*

Proof. Let $\mathcal{B} = \{p_i : i \in I\}$ be the base of Spec(E) and $\varphi : \mathcal{B} \to K^* \setminus \{0\}$ be the function which occur in the definition of the nuclearity of K. (See the notations at the beginning of the section.)

I. Case $(K - K)^- = E$.

(a) Let us fix $p_j \in \mathcal{B}$. From the bipolar theorem ($K^{**} = K$ since K is closed), for each y in the complementary K^c of K, there exists $f_y \in K^*$ such that $f_y(y) < 0$. Hence for an appropriate $\lambda > 0$ we can realize that

$$(\varphi(p_j) + \lambda f_y)(y) < 0.$$

For the sake of simplicity we shall denote in what follows the element $\lambda f_y \in K^*$ attached in the above way to y simply by f_y.

(b) We shall define another base $\mathcal{D}$ of seminorms indexed by the elements of $I \times K^c$ as follows:

$$q_{iy} = p_i \text{ if } i \neq j \text{ and}$$
$$q_{jy} = p_j + |f_y|,$$

where $y \in K^c$ and f_y is the element of K^* attached to y by the above construction in (a), and $|f_y|$ is defined by $|f_y|(x) = |f_y(x)|$. It can happen that some q_{jy} coincides with some q_{iz} with $i \neq j$; in this case it is considered once in $\mathcal{D}$ represented as q_{jy}.

Obviously, q_{iy} is a continuous seminorm and hence it is in Spec(E). $\mathcal{D}$ is a base of Spec(E) since $\mathcal{B}$ is. If $\mathcal{B}$ is Hausdorff, then so is $\mathcal{D}$.

(c) Let us define now $\psi : \mathcal{D} \longrightarrow K^* \setminus \{0\}$ as follows $\psi(q_{iy}) = \varphi(p_i)$ if q_{iy} is not representable as some q_{jz}, and $\psi(q_{jy}) = \varphi(p_j) + f_y$.

We have to see that ψ is well defined. Problem can occur only in the case when $q_{jy} = q_{jz}$ and $f_y \neq f_z$. From the definition, in this case we must have $|f_y| = |f_z|$. According the condition $(K - K)^- = E$ f_y and f_z are determined by their values on K. But on K f_y and f_z takes nonnegative values and these must coincide since $|f_y| = |f_z|$. Thus we must have $f_y = f_z$.

(d) Let $x \in K$ be arbitrary. If q_{iy} is not representable as q_{jz}, then $\psi(q_{iy}) = \varphi(p_i)$ and we have

$$q_{iy}(x) = p_i(x) \leq \varphi(p_i)(x) = \psi(q_{iy})(x).$$

For q_{jy} it holds

$$q_{jy}(x) = p_j(x) + |f_y(x)| = p_j(x) + f_y(x) \leq \varphi(p_j)(x) + f_y(x) = \psi(q_{jy})(x).$$

The above relations show that

$$K \subset K_\psi = \{x \in E : q_{iy}(x) \leq \psi(q_{iy})(x)\ \forall (i, y) \in I \times K^c\}.$$

(e) Let be $y \in K^c$. Then from the definition of f_y we have $\psi(q_{jy})(y) = \varphi(p_j)(y) + f_y(y) < 0$, relation which shows that $y \notin K_\psi$. Thus $K^c \subset K_\psi^c$ and hence $K_\psi \subset K$.

II. Case $E_0 = (K - K)^- \neq E$.

We handle this case as follows:

(a) We proceed with $K \subset E_0$, $K_0^* = K^* \cap E_0$ after restricting the seminorms p in $\mathcal{B}$ and the linear functionals $\varphi(p)$ to E_0 as we have done it at the case I. Then we obtain a base $\mathcal{D}_0$ in Spec(E_0) and $\psi_0 : \mathcal{D}_0 \longrightarrow K_0^* \setminus \{0\}$ such that

$$K = \{x \in E_0 : q(x) \leq \psi_0(q)(x) \ \forall q \in \mathcal{D}_0\}.$$

We can extend using the geometric form of the Hahn-Banach theorem the functionals $f_y \in K_0^*$ to functionals in E that support $K$and hence are in K^*. We use for extension of f_y the same notation.

(b) The subspace E_0 is the intersection of a family $\{H_j : j \in J\}$ of distinct closed hyperplanes through 0. Let $g_j \in E^* \setminus \{0\}$ be the normal of the hyperplane H_j. Consider the family $\mathcal{C}$ of seminorms defined as follows:

$$\mathcal{C} = \{|g_j|, \ |2g_j| : j \in J\}.$$

Observe that the elements in $\mathcal{C}$ are distinct since the hyperplanes H_j were so. Observe that g_j, $-2g_j \in K^*$ since these functionals vanish on K.

(c) Let be now $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{C}$ where $\mathcal{D}_0$ is the family of seminorms constructed at (a) and extended to E such that the extension of $\varphi|E_0$ is taken φ and the extension of f_y-s are those in K^* with the same notation, extensions whose existence is justified at (a).

(d) Let us define the function $\psi : \mathcal{D} \to K^* \setminus \{0\}$ as follows:

$\psi(q) =$ the extension of $\psi_0(q)$ to K^* if $q \in \mathcal{D}_0 \setminus \mathcal{C}$. (Here in the first formula q denotes the extension of $q \in$ Spec(E_0) to E in the mode outlined at (a)).

$\psi(|g_j|) = g_j$ and $\psi(|2g_j|) = -2g_j$ for the seminorms in $\mathcal{C}$.

(If for some $x \in E$ one has $|g_j(x)| = |g_j|(x) \leq \psi(|g_j|)(x) = g_j(x)$ and $|2g_j(x)| = |2g_j|(x) \leq \psi(|2g_j|(x) = -2g_j(x)$, then we must have $g_j(x) = 0$, that is $x \in H_j$. This is the intuitive reason of the introduction of the family $\mathcal{C}$ of seminorms, which have the role in the definition of K_ψ to restrict x to be in the subspace $\cap H_j = E_0$ where we can follow the construction at I.)

A case analysis similar to those in the points (d) and (e) of the case I shows that we have also in this case $K = K_\psi$.

∎

Remark 6.1. From the above theorem and the remarks at the ends of Section 3 and Section 5 a new prospective is gained on closed nuclear cones. Indeed, a closed nuclear cone is everything associated to a function φ from a base $\mathcal{B}$ of Spec(E) into E^*. The question of whether two such functions give rise to the same nuclear cone is resolved via the bipolar theorem as follows: If φ and ψ are two such functions (may be associated to different bases), then the nuclear cones defined by them are identic if and only if $(\text{cone}\, r(\varphi))^- = (\text{cone}\, r(\psi))^-$, where cone M stands for the conical hull of the set M, (i. e., the minimal convex cone containing M), $r(\gamma)$ denotes the range of the function γ and the closure is taken with respect to the weak* topology in E^*.

7. RELATION WITH PARETO EFFICIENCY

Let $(E, \text{Spec}(E))$ be a locally convex space ordered by a closed convex normal cone K. Let X be a nonempty set and $F : X \longrightarrow E$ a function. In some practical problems we are interested to consider the following optimization problem:

$$\begin{cases} \text{minimize } F(x) \\ \qquad\qquad x \in X \end{cases}$$

In this problem minimization means to find all the solutions that are *Pareto (minimal) efficient points* in X, i.e., the points $x_0 \in X$ such that

$$F(X) \cap (F(x_0) - K) = \{F(x_0)\}.$$

This approach give rise to the following abstract minimization problem:

Given a nonempty subset A of E we ask for conditions of existence of a *Pareto (minimal) efficient points with respect to* K of A, i.e., points $x_* \in A$ with the property

$$A \cap (x_* - K) = \{x_*\}.$$

The geometrical background of a result stated in the paper [14, Theorem 7] is the following.

Proposition 7.1. *If $M \subset E$ is a set having the property that 0 is a (Pareto minimal) efficient point of M and $D \subset E$ contains a point x_* such that $D - x_* \subset M$, then x_* is an efficient point of D.*

Proof. Indeed, in this case $0 \in D - x_*$ and since $D - x_* \subset M$, and 0 is efficient point of M by hypothesis, it will follow that 0 is efficient point of $D - x_*$. That is, $(-K) \cap (D - x_*) = \{0\}$ and hence $(x_* - K) \cap D = \{x_*\}$. ∎

Proposition 7.2. *Let K_n be a nuclear cone associated to the cone K in the Hausdorff locally convex space E. Then 0 is an efficient point of K_n with respect to K, that is, we have $(-K) \cap K_n = \{0\}$.*

Proof. Indeed, if it would exist $x \in (-K) \cap K_n$ with $x \neq 0$, then there would exist p in $\mathrm{Spec}(E)$ such that $p(x) > 0$. Thus there must be a $q \in \mathcal{B}$ where $\mathcal{B}$ is the base which occurs in the definition of K_n, such that $q(x) > 0$. Let be $f \in K^*$ the functional for which $q(x) \leq f(x)$. Then it must be $f(x) > 0$. On the other hand $x \in -K$ and hence we must have $f(x) \leq 0$ and we arrive to a contradiction. ∎

From the above two propositions we conclude:

Proposition 7.3. *If K is a cone in the Hausdorff locally convex space E and K_n is a nuclear cone associated to it, if D is a subset in E which contains a point x_* such that $D - x_* \subset K_n$, then x_* is an efficient point of D with respect to K.*

The result given by Proposition 7.3 is the essential part of main theorem proved in [14] and it was stated and proved for full nuclear cones K_n associated to a closed normal cone K. For the first sight it seems to be rather restrictive in comparison with the geometrical idea deduced from Proposition 7.1. But the minimization with respect to a full nuclear cone offers the possibility of using a sort of multiple scalarization, the basic reason for introducing nuclear cones.

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