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TWO MAPPINGS RELATED TO STEFFENSEN'S INEQUALITIES

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ABSTRACT. In this paper, we define two mappings closely connected with Steffensen's inequalities, investigate their main properties, give some refinements for Steffensen's inequalities and obtain new inequalities.

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1. INTRODUCTION

Let $f : [a, b] \to \mathbf{R}$ be a decreasing function and $g : [a, b] \to [0, 1]$ be a integrable function. We write

$$r(x, y) = \int_x^y g(t)dt, \qquad a \le x, y \le b.$$

In [1], Steffensen showed the following inequalities

$$\int_{b-r(a,b)}^b f(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^{a+r(a,b)} f(x)dx. \tag{1.1}$$

We define two mappings S and s by

$$S : [a, b] \times [a, b] \to \mathbf{R}, \quad S(x, y) = \int_x^{x+r(x,y)} f(t)dt - \int_x^y f(t)g(t)dt$$

and

$$s : [a, b] \times [a, b] \to \mathbf{R}, \quad s(x, y) = \int_x^y f(t)g(t)dt - \int_{y-r(x,y)}^y f(t)dt,$$

they are differences generated by the inequalities (1.1).

The aim of this paper is to study the main properties of $S(x, y)$ and $s(x, y)$, and then obtain some refinements of (1.1) and new inequalities. Similar inequalities connected with (1.1) can be seen in [2, 2.16]; [3, P. 570-572].

2. MAIN RESULTS

The main properties of $S(x, y)$ are given in the following theorem.

Theorem 2.1. *Let f and g be defined as in the first section. Then we have the following.*

(1) $S(a, y)$ is nonnegative and monotonically increasing with y on $[a, b]$, $S(x, b)$ is nonnegative and monotonically decreasing with x on $[a, b]$;

(2) For any $x \in (a, b)$, we have the following three refinements of the right side in (1.1):

$$\int_a^b f(t)g(t)dt \le \left(\int_a^{a+r(a,x)} + \int_x^b g(t) \right) f(t)dt \le \int_a^{a+r(a,b)} f(t)dt, \tag{2.1}$$

$$\int_a^b f(t)g(t)dt \le \left(\int_x^{x+r(x,b)} + \int_a^x g(t) \right) f(t)dt \le \int_a^{a+r(a,b)} f(t)dt \tag{2.2}$$

and

$$\int_a^b f(t)g(t)dt \le \frac{1}{2} \left(\int_a^{a+r(a,x)} + \int_x^{x+r(x,b)} + \int_a^b g(t) \right) f(t)dt \le \int_a^{a+r(a,b)} f(t)dt. \tag{2.3}$$

The main properties of $s(x, y)$ are embodied in the following theorem.

Theorem 2.2. *Let f and g be defined as in Theorem 2.1. Then we have the following.*

(1) $s(a, y)$ is nonnegative and monotonically increasing with y on $[a, b]$, $s(x, b)$ is nonnegative and monotonically decreasing with x on $[a, b]$;

(2) Fox any $x \in (a, b)$, we have the following three refinements of the left side in (1.1):

$$\int_{b-r(a,b)}^b f(t)dt \le \left(\int_{b-r(a,b)}^b - \int_{x-r(a,x)}^x + \int_a^x g(t) \right) f(t)dt \le \int_a^b f(t)g(t)dt, \tag{2.4}$$

$$\int_{b-r(a,b)}^{b} f(t)dt \le \left(\int_{b-r(a,b)}^{b-r(x,b)} + \int_{x}^{b} g(t) \right) f(t)dt \le \int_{a}^{b} f(t)g(t)dt \tag{2.5}$$

and

$$\begin{aligned} \int_{b-r(a,b)}^{b} f(t)dt &\le \frac{1}{2} \left(\int_{b-r(a,b)}^{b} - \int_{x-r(a,x)}^{x} + \int_{b-r(a,b)}^{b-r(x,b)} + \int_{a}^{b} g(t) \right) f(t)dt \\ &\le \int_{a}^{b} f(t)g(t)dt. \end{aligned} \tag{2.6}$$

Theorem 2.3. *Let f and g be defined as in Theorem 2.1. Then we have the following six inequalities.*

$$\begin{aligned} &\int_{b-r(a,b)}^{b} \left(\int_{a}^{x} f(t)g(t)dt \right) dx + \left(\int_{a}^{a+r(a,b)} + \int_{a}^{b} g(x) \right) \left(\int_{a}^{a+r(a,x)} f(t)dt \right) dx \\ &\le \int_{a}^{b} \left(\int_{a}^{x} f(t)g(t)dt \right) g(x)dx + \left(\int_{a}^{a+r(a,b)} + \int_{b-r(a,b)}^{b} \right) \left(\int_{a}^{a+r(a,x)} f(t)dt \right) dx \\ &\le \int_{a}^{a+r(a,b)} \left(\int_{a}^{x} f(t)g(t)dt \right) dx + \left(\int_{b-r(a,b)}^{b} + \int_{a}^{b} g(x) \right) \left(\int_{a}^{a+r(a,x)} f(t)dt \right) dx, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\int_{b-r(a,b)}^{b} \left(\int_{x}^{x+r(x,b)} f(t)dt \right) dx + \left(\int_{a}^{a+r(a,b)} + \int_{a}^{b} g(x) \right) \left(\int_{x}^{b} f(t)g(t)dt \right) dx \\ &\le \int_{a}^{b} \left(\int_{x}^{x+r(x,b)} f(t)dt \right) g(x)dx + \left(\int_{a}^{a+r(a,b)} + \int_{b-r(a,b)}^{b} \right) \left(\int_{x}^{b} f(t)g(t)dt \right) dx \\ &\le \int_{a}^{a+r(a,b)} \left(\int_{x}^{x+r(x,b)} f(t)dt \right) dx + \left(\int_{b-r(a,b)}^{b} + \int_{a}^{b} g(x) \right) \left(\int_{x}^{b} f(t)g(t)dt \right) dx, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\int_{b-r(a,b)}^{b} \left(\int_{x-r(a,x)}^{x} f(t)dt \right) dx + \left(\int_{a}^{a+r(a,b)} + \int_{a}^{b} g(x) \right) \left(\int_{a}^{x} f(t)g(t)dt \right) dx \\ &\le \int_{a}^{b} \left(\int_{x-r(a,x)}^{x} f(t)dt \right) g(x)dx + \left(\int_{a}^{a+r(a,b)} + \int_{b-r(a,b)}^{b} \right) \left(\int_{a}^{x} f(t)g(t)dt \right) dx \\ &\le \int_{a}^{a+r(a,b)} \left(\int_{x-r(a,x)}^{x} f(t)dt \right) dx + \left(\int_{b-r(a,b)}^{b} + \int_{a}^{b} g(x) \right) \left(\int_{a}^{x} f(t)g(t)dt \right) dx, \end{aligned} \tag{2.9}$$

$$\begin{aligned}(2.10)\quad &\int_{b-r(a,b)}^{b}\left(\int_{x}^{b} f(t)g(t)dt\right)dx+\left(\int_{a}^{a+r(a,b)}+\int_{a}^{b} g(x)\right)\left(\int_{b-r(x,b)}^{b} f(t)dt\right)dx\\ &\le \int_{a}^{b}\left(\int_{x}^{b} f(t)g(t)dt\right)g(x)dx+\left(\int_{a}^{a+r(a,b)}+\int_{b-r(a,b)}^{b}\right)\left(\int_{b-r(x,b)}^{b} f(t)dt\right)dx\\ &\le \int_{a}^{a+r(a,b)}\left(\int_{x}^{b} f(t)g(t)dt\right)dx+\left(\int_{b-r(a,b)}^{b}+\int_{a}^{b} g(x)\right)\left(\int_{b-r(x,b)}^{b} f(t)dt\right)dx,\end{aligned}$$

$$\begin{aligned}(2.11)\quad &\int_{b-r(a,b)}^{b}\left(\int_{x-r(a,x)}^{x} f(t)dt\right)dx+\left(\int_{a}^{a+r(a,b)}+\int_{a}^{b} g(x)\right)\left(\int_{a}^{a+r(a,x)} f(t)dt\right)dx\\ &\le \int_{a}^{b}\left(\int_{x-r(a,x)}^{x} f(t)dt\right)g(x)dx+\left(\int_{a}^{a+r(a,b)}+\int_{b-r(a,b)}^{b}\right)\left(\int_{a}^{a+r(a,x)} f(t)dt\right)dx\\ &\le \int_{a}^{a+r(a,b)}\left(\int_{x-r(a,x)}^{x} f(t)dt\right)dx+\left(\int_{b-r(a,b)}^{b}+\int_{a}^{b} g(x)\right)\left(\int_{a}^{a+r(a,x)} f(t)dt\right)dx\end{aligned}$$

and
(2.12)

$$\begin{aligned}&\int_{b-r(a,b)}^{b}\left(\int_{x}^{x+r(x,b)} f(t)dt\right)dx+\left(\int_{a}^{a+r(a,b)}+\int_{a}^{b} g(x)\right)\left(\int_{b-r(x,b)}^{b} f(t)dt\right)dx\\ &\le \int_{a}^{b}\left(\int_{x}^{x+r(x,b)} f(t)dt\right)g(x)dx+\left(\int_{a}^{a+r(a,b)}+\int_{b-r(a,b)}^{b}\right)\left(\int_{b-r(x,b)}^{b} f(t)dt\right)dx\\ &\le \int_{a}^{a+r(a,b)}\left(\int_{x}^{x+r(x,b)} f(t)dt\right)dx+\left(\int_{b-r(a,b)}^{b}+\int_{a}^{b} g(x)\right)\left(\int_{b-r(x,b)}^{b} f(t)dt\right)dx.\end{aligned}$$

3. THE PROOF OF THEOREMS

Proof of Theorem 2.1.
(1) The fact that $S(a,y)$ and $S(x,b)$ are nonnegative follows from the right side of (1.1).

Let any $y_1, y_2 \in [a,b]$ and $y_1 < y_2$. From the property of integrable function (see [4]), we have

$$(3.1)\qquad d\left(a+\int_{a}^{u} g(t)dt\right)=g(u)du \quad (\text{almost} \quad \text{everywhere}).$$

From (3.1) and the property of definite integral (see [4]), we obtain
(3.2)

$$\int_{a+r(a,y_1)}^{a+r(a,y_2)} f(t)dt=\int_{y_1}^{y_2} f\Big(a+r(a,u)\Big)d\left(a+\int_{a}^{u} g(t)dt\right)=\int_{y_1}^{y_2} f\Big(a+r(a,u)\Big)g(u)du.$$

From $0\le g(u)\le 1$ $(u\in[a,b])$, we have

$$(3.3)\qquad a+r(a,t)=a+\int_{a}^{t} g(u)du\le a+\int_{a}^{t} du=t.$$

Using (3.2) and (3.3), we obtain

$$\begin{aligned} S(a,y_2)-S(a,y_1) &= \int_{a+r(a,y_1)}^{a+r(a,y_2)} f(t)dt - \int_{y_1}^{y_2} f(t)g(t)dt \\ &= \int_{y_1}^{y_2} f\Big(a+r(a,u)\Big)g(u)du - \int_{y_1}^{y_2} f(t)g(t)du \\ &= \int_{y_1}^{y_2} g(t)\Big(f\Big(a+r(a,t)\Big)-f(t)\Big)dt \ge 0, \end{aligned}$$

which implies that $S(a,y)$ is monotonically increasing with y on $[a,b]$.

Let any $x_1,x_2\in[a,b]$ and $x_1<x_2$. By the same arguments of proof for (3.2), we obtain

$$\begin{aligned} \int_{x_1+r(x_1,b)}^{x_2+r(x_2,b)} f(t)dt &= \int_{x_1}^{x_2} f\Big(u+r(u,b)\Big)d\left(u+\int_u^b g(t)dt\right) \\ &= \int_{x_1}^{x_2} f\Big(u+r(u,b)\Big)\Big(1-g(u)\Big)du. \end{aligned} \tag{3.4}$$

From (3.4), we obtain

$$\begin{aligned} S(x_2,b)-S(x_1,b) &= \int_{x_2}^{x_2+r(x_2,b)} f(t)dt - \left(\int_{x_1}^{x_2} + \int_{x_2}^{x_1+r(x_1,b)}\right) f(t)dt + \int_{x_1}^{x_2} f(t)g(t)dt \\ &= \int_{x_1+r(x_1,b)}^{x_2+r(x_2,b)} f(t)dt - \int_{x_1}^{x_2} f(t)\Big(1-g(t)\Big)dt \\ &= \int_{x_1}^{x_2} \Big(1-g(t)\Big)\Big(f\Big(t+r(t,b)\Big)-f(t)\Big)dt \le 0, \end{aligned}$$

which implies that $S(x,b)$ is monotonically decreasing with x on $[a,b]$.
(2) Let any $x\in(a,b)$. By the monotonically increasing property of $S(a,y)$ on $[a,b]$ with y, we have

$$0=S(a,a)\le S(a,x)\le S(a,b), \tag{3.5}$$

the expression (3.5) plus $\int_a^b f(t)g(t)dt$ yields (2.1). By the monotonically decreasing property of $S(t,b)$ on $[a,b]$ with t, we have

$$0=S(b,b)\le S(x,b)\le S(a,b), \tag{3.6}$$

the expression (3.6) plus $\int_a^b f(t)g(t)dt$ yields (2.2). The expression (2.1) plus (2.2) yields (2.3).

This completes the proof of Theorem 2.1. ∎

Proof of Theorem 2.2.
(1) The fact that $s(a,y)$ and $s(x,b)$ are nonnegative follows from the left side of (1.1).

Let any $y_1,y_2\in[a,b]$ and $y_1<y_2$. By the same arguments of proof for (3.2), we obtain

$$\begin{aligned} \int_{y_1-r(a,y_1)}^{y_2-r(a,y_2)} f(t)dt &= \int_{y_1}^{y_2} f\Big(u-r(a,u)\Big)d\left(u-\int_a^u g(t)dt\right) \\ &= \int_{y_1}^{y_2} f\Big(u-r(a,u)\Big)\Big(1-g(u)\Big)du. \end{aligned} \tag{3.7}$$

From (3.7), we obtain

$$\begin{aligned} s(a,y_2)-s(a,y_1) &= \int_{y_1}^{y_2} f(t)g(t)dt - \left(\int_{y_2-r(a,y_2)}^{y_1} + \int_{y_1}^{y_2}\right) f(t)dt + \int_{y_1-r(a,y_1)}^{y_1} f(t)dt \\ &= \int_{y_1-r(a,y_1)}^{y_2-r(a,y_2)} f(t)dt - \int_{y_1}^{y_2} f(t)\Big(1-g(t)\Big)dt \\ &= \int_{y_1}^{y_2} \Big(1-g(t)\Big)\Big(f\Big(t-r(a,t)\Big)-f(t)\Big)dt \geq 0, \end{aligned}$$

which implies that $s(a,y)$ is monotonically increasing with y on $[a,b]$.

By $0 \leq g(u) \leq 1$ $(u \in [a,b])$, we have

$$b - r(t,b) = b - \int_t^b g(u)du \geq b - \int_t^b du = t. \tag{3.8}$$

Let any $x_1, x_2 \in [a,b]$ and $x_1 < x_2$. By the same arguments of proof for (3.2), we obtain

$$\begin{aligned} \int_{b-r(x_1,b)}^{b-r(x_2,b)} f(t)dt &= \int_{x_1}^{x_2} f\Big(b-r(u,b)\Big)d\left(b-\int_u^b g(t)dt\right) \\ &= \int_{x_1}^{x_2} f\Big(b-r(u,b)\Big)g(u)du. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\begin{aligned} s(x_2,b)-s(x_1,b) &= \int_{b-r(x_1,b)}^{b-r(x_2,b)} f(t)dt - \int_{x_1}^{x_2} f(t)g(t)dt \\ &= \int_{x_1}^{x_2} g(t)\Big(f\Big(b-r(t,b)\Big)-f(t)\Big)dt \leq 0, \end{aligned}$$

which implies that $s(x,b)$ is monotonically decreasing with x on $[a,b]$.

(2) By the same arguments of proof for case (2) in the Theorem 2.1, on using the monotonically increasing property of $s(a,y)$ on $[a,b]$ with y and the monotonically decreasing property of $s(x,b)$ on $[a,b]$ with x, we obtain (2.4) and (2.5), respectively. The expression (2.4) plus (2.5) yields (2.6).

This completes the proof of Theorem 2.2. ∎

Proof of Theorem 2.3.

Replacing $f(x)$ in the (1.1) with $-S(a,x)$, $S(x,b)$, $-s(a,x)$ and $s(x,b)$, with some simple manipulations we obtain (2.7), (2.8), (2.9) and (2.10), respectively. The expression (2.7) plus (2.9) yields (2.11). The expression (2.8) plus (2.10) yields (2.12).

This completes the proof of Theorem 2.3. ∎

4. APPLICATIONS

Let $h : [a,b] \to \mathbf{R}$ be a continuous convex function, g be defined as in Theorem 2.3. Then we have the following two inequalities.

(4.1)

$$\begin{aligned}
&\int_{b-r(a,b)}^{b} \Big(h\Big(x-r(a,x)\Big)+h\Big(a+r(a,x)\Big)\Big)\,dx+\left(\int_a^{a+r(a,b)}+\int_a^b g(x)\right)h(x)dx\\
&\le \int_a^b \Big(h\Big(x-r(a,x)\Big)+h\Big(a+r(a,x)\Big)\Big)\,g(x)dx+\left(\int_a^{a+r(a,b)}+\int_{b-r(a,b)}^b\right)h(x)dx\\
&\le \int_a^{a+r(a,b)} \Big(h\Big(x-r(a,x)\Big)+h\Big(a+r(a,x)\Big)\Big)\,dx+\left(\int_{b-r(a,b)}^{b}+\int_a^b g(x)\right)h(x)dx
\end{aligned}$$

and

(4.2)

$$\begin{aligned}
&\int_{b-r(a,b)}^{b} h(x)dx+\left(\int_a^{a+r(a,b)}+\int_a^b g(x)\right)\Big(h\Big(b-r(x,b)\Big)+h\Big(x+r(x,b)\Big)\Big)\,dx\\
&\le \int_a^b h(x)g(x)dx+\left(\int_a^{a+r(a,b)}+\int_{b-r(a,b)}^b\right)\Big(h\Big(b-r(x,b)\Big)+h\Big(x+r(x,b)\Big)\Big)\,dx\\
&\le \int_a^{a+r(a,b)} h(x)dx+\left(\int_{b-r(a,b)}^{b}+\int_a^b g(x)\right)\Big(h\Big(b-r(x,b)\Big)+h\Big(x+r(x,b)\Big)\Big)\,dx.
\end{aligned}$$

Indeed, Replacing $f(t)$ in the (2.11) and (2.12) with $-h_-^{'}(t)$, and using

$$\int_x^y h_-^{'}(t)dt = h(y)-h(x) \quad a\le x,y\le b \qquad (see \quad [5,6]),$$

with some simple manipulations we obtain (4.1) and (4.2), respectively.

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