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MEROMORPHIC P-VALENT FUNCTIONS WITH POSITIVE AND FIXED SECOND COEFFICIENTS

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ABSTRACT. We introduce the classes $\Omega_p^*(\alpha, q)$ and $\Lambda_p^*(\alpha, c)$ of meromorphic univalent functions with positive and fixed second coefficients. The aim of the present paper is to obtain coefficient inequalities and closure theorems for these classes. Furthermore, the radii of convexity and starlikeness for functions the classes $\Omega_p^*(\alpha, q)$ and $\Lambda_p^*(\alpha, c)$ are determined.

Key words and phrases: Meromorphic p-valent functions, Coefficient inequalities, Convex linear combinations, Radii of meromorphically convexity and starlikeness.

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1. INTRODUCTION AND PRELIMINARIES

Let Σ_p denotes the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \qquad (a_{p+n-1} \geq 0,\ p \in \mathbb{N}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. Let $\Omega_p(\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ which satisfy the inequality

$$\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{D}; 0 \leq \alpha < p;\ p \in N). \tag{1.2}$$

And let $\Lambda_p(\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ which satisfy the inequality

$$\text{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{D}; 0 \leq \alpha < p;\ p \in N). \tag{1.3}$$

The classes $\Omega_p(\alpha)$ and $\Lambda_p(\alpha)$ were introduced and studied by the authors [2].

For the above classes, the authors [2] obtained the following sufficient conditions for a function of the form (1.1) to be in the classes $\Omega_p(\alpha)$ and $\Lambda_p(\alpha)$.

Lemma 1. *If $f(z) \in \Sigma_p$ satisfies*

$$\sum_{n=1}^{\infty} (p+n+\delta-1+|p+n+2\alpha-\delta-1|)a_{p+n-1} < 2(p-\alpha) \tag{1.4}$$

for some $\alpha(0 \leq \alpha < p)$ and some $\delta(\alpha < \delta \leq p)$, then $f(z) \in \Omega_p(\alpha)$.

Lemma 2. *If $f(z) \in \Sigma_p$ satisfies*

$$\sum_{n=1}^{\infty} (p+n-1)(p+n+\delta-1+|p+n+2\alpha-\delta-1|)a_{p+n-1} < 2(p-\alpha) \tag{1.5}$$

for some $\alpha(0 \leq \alpha < p)$ and some $\delta(\alpha < \delta \leq p)$, then $f(z) \in \Lambda_p(\alpha)$.

In view of Lemma 1 and Lemma 2, we now define the subclasses $\Omega_p^*(\alpha) \subset \Omega_p(\alpha)$ and $\Lambda_p^*(\alpha) \subset \Lambda_p(\alpha)$, which consist of functions $f(z) \in \Sigma_p$ satisfying the conditions (1.4) and (1.5), respectively. (see [2]).

From Lemma 1, we can see that any function $f \in \Omega_p^*(\alpha)$ satisfy the coefficient inequality

$$a_p \leq \frac{p-\alpha}{p+\alpha}. \tag{1.6}$$

Hence we may take

$$a_p = \frac{q(p-\alpha)}{p+\alpha}, \tag{1.7}$$

where $0 \leq q \leq 1$. Let $\Omega_p^*(\alpha, q)$ be the subclass of $\Omega_p^*(\alpha)$ consisting of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p + \sum_{n=2}^{\infty} a_{p+n-1} z^{p+n-1} \qquad (a_{p+n-1} \geq 0). \tag{1.8}$$

Similarly from Lemma 2, we can see that any function $f \in \Lambda_p^*(\alpha)$ satisfy the coefficient inequality

$$a_p \leq \frac{(p-\alpha)}{p(p+\alpha)}. \tag{1.9}$$

Hence we may take

$$a_p = \frac{c(p-\alpha)}{p(p+\alpha)}, \tag{1.10}$$

where $0 \le c \le 1$. Let $\Lambda_p^*(\alpha, c)$ be the subclass of $\Lambda_p^*(\alpha)$ consisting of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{c(p-\alpha)}{p(p+\alpha)} z^p + \sum_{n=2}^{\infty} a_{p+n-1} z^{p+n-1} \qquad (a_{p+n-1} \ge 0). \tag{1.11}$$

In this paper, we obtain coefficient inequalities and closure theorems for the classes $\Omega_p^*(\alpha, q)$ and $\Lambda_p^*(\alpha, c)$. Also we obtain the radii of convexity and starlikeness for functions belonging to these. Techniques used are similar to those of Silverman and Silvia [3] and Uralegaddi [4] (see also [1]).

2. COEFFICIENT INEQUALITIES

The following theorem gives a necessary and sufficient condition for a function to be in the class $\Omega_p^*(\alpha, q)$.

Theorem 2.1. *A function $f(z)$ defined by (1.8) is in the class $\Omega_p^*(\alpha, q)$ if and only if*

$$\sum_{n=2}^{\infty} (p+n+\alpha-1) a_{p+n-1} < (p-\alpha)(1-q). \tag{2.1}$$

The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p + \frac{(p-\alpha)(1-q)}{p+n+\alpha-1} z^{p+n-1}. \tag{2.2}$$

Proof. The result follows by putting

$$a_p = \frac{q(p-\alpha)}{p+\alpha}, \quad 0 \le q \le 1,$$

in the inequality (1.4). ∎

Corollary 1. *Let $f(z)$ defined by (1.8) be in the class $\Omega_p^*(\alpha, q)$, then*

$$a_{p+n-1} \le \frac{(p-\alpha)(1-q)}{p+n+\alpha-1}, \qquad (n \ge 2,\ p \in \mathbb{N}). \tag{2.3}$$

The result is sharp for the function given by (2.2).

Corollary 2. *If $0 \le q_1 \le q_2 \le 1$, then*

$$\Omega_p^*(\alpha, q_2) \subset \Omega_p^*(\alpha, q_1). \tag{2.4}$$

In the same way, we can prove the following theorem for the class $\Lambda_p^*(\alpha, c)$ by using Lemma 2 instead of Lemma 1.

Theorem 2.2. *A function $f(z)$ defined by (1.11) is in the class $\Lambda_p^*(\alpha, c)$ if and only if*

$$\sum_{n=2}^{\infty} (p+n-1)(p+n+\alpha-1) a_{p+n-1} < (p-\alpha)(1-c). \tag{2.5}$$

The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{c(p-\alpha)}{p(p+\alpha)} z^p + \frac{(p-\alpha)(1-c)}{(p+n-1)(p+n+\alpha-1)} z^{p+n-1}. \tag{2.6}$$

Corollary 3. *Let $f(z)$ defined by (1.11) be in the class $\Lambda_p^*(\alpha, c)$, then*

$$a_{p+n-1} \leq \frac{(p-\alpha)(1-c)}{(p+n-1)(p+n+\alpha-1)}, \qquad (n \geq 2,\ p \in \mathbb{N}). \tag{2.7}$$

The result is sharp for the function given by (2.6).

Corollary 4. *If $0 \leq c_1 \leq c_2 \leq 1$, then*

$$\Omega_p^*(\alpha, c_2) \subset \Omega_p^*(\alpha, c_1). \tag{2.8}$$

3. CLOSURE PROPERTIES

In this section, we shall show that the classes $\Omega_p^*(\alpha, q)$ and $\Lambda_p^*(\alpha, c)$ are closed under arithmetic means and convex linear combinations.

Theorem 3.1. *Let*

$$f_i(z) = \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p + \sum_{n=2}^{\infty} a_{p+n-1,i} z^{p+n-1} \qquad (a_{p+n-1,i} \geq 0) \tag{3.1}$$

for $i = 1, 2, 3, \ldots, m$. If $f_i(z) \in \Omega_p^*(\alpha, q)$ for each $i = 1, 2, 3, \ldots, m$, then the function

$$g(z) = \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p + \sum_{n=2}^{\infty} b_{p+n-1} z^{p+n-1}, \qquad (b_{p+n-1} \geq 0), \tag{3.2}$$

also is a member of $\Omega_p^*(\alpha, q)$, where

$$b_{p+n-1} = \frac{1}{m} \sum_{i=1}^{m} a_{p+n-1,i}. \tag{3.3}$$

Proof. Since $f_i(z) \in \Omega_p^*(\alpha, q)$, it follows from Theorem 2.1 that

$$\sum_{n=2}^{\infty} (p+n+\alpha-1) a_{p+n-1,i} < (p-\alpha)(1-q).$$

Hence

$$\begin{aligned} \sum_{n=2}^{\infty} (p+n+\alpha-1) b_{p+n-1} &= \sum_{n=2}^{\infty} \left(\frac{1}{m} \sum_{i=1}^{m} a_{p+n-1,i} \right) (p+n+\alpha-1) \\ &= \frac{1}{m} \sum_{i=1}^{m} \sum_{n=2}^{\infty} (p+n+\alpha-1) a_{p+n-1,i} \\ &\leq (p-\alpha)(1-q) \end{aligned}$$

and the result follows. ∎

With the aid of Theorem 2.2, we can similarly prove the following theorem.

Theorem 3.2. *Let*

$$f_i(z) = \frac{1}{z^p} + \frac{c(p-\alpha)}{p(p+\alpha)} z^p + \sum_{n=2}^{\infty} a_{p+n-1,i} z^{p+n-1} \qquad (a_{p+n-1,i} \geq 0) \tag{3.4}$$

for $i = 1, 2, 3, \ldots, m$. If $f_i(z) \in \Lambda_p^*(\alpha, c)$ for each $i = 1, 2, 3, \ldots, m$, then the function

$$g(z) = \frac{1}{z^p} + \frac{c(p-\alpha)}{p(p+\alpha)} z^p + \sum_{n=2}^{\infty} b_{p+n-1} z^{p+n-1}, \qquad (b_{p+n-1} \geq 0), \tag{3.5}$$

with b_{p+n-1} *defined by (3.3), is also in the class* $\Lambda_p^*(\alpha, c)$.

Theorem 3.3. *Let*

$$f_p(z) = \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p, \qquad (z \in \mathcal{D}) \tag{3.6}$$

and

$$f_{n+p-1}(z) = \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p + \frac{(p-\alpha)(1-q)}{p+n+\alpha-1} z^{p+n-1}, \qquad (z \in \mathcal{D}), \tag{3.7}$$

where $n \geq 2,\ p \in \mathbb{N}$. *Then* $f(z)$ *is in the class* $\Omega_p^*(\alpha, q)$ *if and only if* $f(z)$ *can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z) \tag{3.8}$$

where $\lambda_{p+n-1} \geq 0$, $(n \in \mathbb{N})$ *and* $\sum\limits_{n=1}^{\infty} \lambda_{p+n-1} = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z) \\ &= \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p + \sum_{n=2}^{\infty} \lambda_{p+n-1} \left(\frac{(p-\alpha)(1-q)}{p+n+\alpha-1} \right) z^{p+n-1}. \end{aligned}$$

Since

$$\sum_{n=2}^{\infty} \lambda_{p+n-1} \left(\frac{(p-\alpha)(1-q)}{p+n+\alpha-1} \right) \left(\frac{p+n+\alpha-1}{(p-\alpha)(1-q)} \right) = \sum_{n=2}^{\infty} \lambda_{p+n-1} = 1 - \lambda_{p-1} \leq 1,$$

it follows from Theorem 2.1 that $f(z) \in \Omega_p^*(\alpha, q)$.

Conversely, suppose that

$$f(z) = \frac{1}{z^p} + \frac{q(p-\alpha)}{p+\alpha} z^p + \frac{(p-\alpha)(1-q)}{p+n+\alpha-1} z^{p+n-1}$$

is in the class $\Omega_p^*(\alpha, q)$. From Corollary 1, we have

$$a_{p+n-1} \leq \frac{(p-\alpha)(1-q)}{p+n+\alpha-1}, \qquad (n \geq 2,\ p \in \mathbb{N}).$$

Taking

$$\lambda_{p+n-1} = \frac{p+n+\alpha-1}{(p-\alpha)(1-q)} a_{p+n-1}, \qquad (n \geq 2,\ p \in \mathbb{N}),$$

and

$$\lambda_p = 1 - \sum_{n=2}^{\infty} \lambda_{p+n-1},$$

we get $f(z) = \sum\limits_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z)$. ∎

In a similar manner, we can prove the following.

Theorem 3.4. *Let*

$$f_p(z) = \frac{1}{z^p} + \frac{c(p-\alpha)}{p(p+\alpha)} z^p, \qquad (z \in \mathcal{D}) \tag{3.9}$$

and

$$f_{n+p-1}(z) = \frac{1}{z^p} + \frac{c(p-\alpha)}{p(p+\alpha)} z^p + \frac{(p-\alpha)(1-c)}{(p+n-1)(p+n+\alpha-1)} z^{p+n-1}, \qquad (z \in \mathcal{D}). \tag{3.10}$$

where $n \geq 2,\ p \in \mathbb{N}$. Then $f(z)$ is in the class $\Lambda_p^*(\alpha, c)$ if and only if $f(z)$can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z) \tag{3.11}$$

where $\lambda_{p+n-1} \geq 0$, $(n \in \mathbb{N})$ and $\sum\limits_{n=1}^{\infty} \lambda_{p+n-1} = 1$.

4. RADII OF CONVEXITY AND STARLIKENESS

We next obtain the radii of meromorphically convexity and meromorphically starlikeness for functions in $\Omega_p^*(\alpha, q)$.

Theorem 4.1. *Let the function $f(z)$ defined by (1.8) be in the class $\Omega_p^*(\alpha, q)$, then it is meromorphically convex in $0 \leq |z| \leq r_0 = r_0(p, q, \alpha)$, where $r_0(p, q, \alpha)$ is the largest value of r for which*

$$\frac{3qp^2(p-\alpha)}{p+\alpha} r^{2p} + \frac{(p+n-1)(3p+n-1)(p-\alpha)(1-q)}{p+n+\alpha-1} r^{2p+n-1} \leq p^2, \quad (n \geq 2). \tag{4.1}$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It suffices to show that $|p + 1 + zf''(z)/f'(z)| \leq p$ for $0 \leq |z| \leq r_0 = r_0(p, q, \delta, \alpha)$. Observe that, if $f \in \Omega_p^*(\alpha, q)$ is given by (1.8), we have

$$\left| p + 1 + \frac{zf''(z)}{f'(z)} \right| \leq \frac{\frac{2p^2 q(p-\alpha)}{p+\alpha} r^{2p} + \sum\limits_{n=2}^{\infty} (p+n-1)(2p+n-1) a_{p+n-1} r^{2p+n-1}}{p - \frac{pq(p-\alpha)}{p+\alpha} r^{2p} - \sum\limits_{n=2}^{\infty} (p+n-1) a_{p+n-1} r^{2p+n-1}} \leq p,$$

whenever

$$\frac{2qp^2(p-\alpha)}{p+\alpha} r^{2p} + \sum_{n=2}^{\infty} (p+n-1)(3p+n-1) a_{p+n-1} r^{2p+n-1} \leq p^2.$$

Since $f(z) \in \Omega_p^*(\alpha, q)$, in view of Theorem 2.1, we may take

$$a_{p+n-1} = \frac{(p-\alpha)(1-q)}{p+n+\alpha-1} \lambda_{p+n-1},$$

where $\sum\limits_{n=2}^{\infty} \lambda_{p+n-1} \leq 1$. For each fixed r, choose the integer $n_0 = n_0(r)$ for which

$$\frac{(p+n-1)(3p+n-1) r^{2p+n-1}}{p+n+\alpha-1}$$

is maximal. Then

$$\sum_{n=2}^{\infty}(p+n-1)(3p+n-1)r^{2p+n-1}a_{p+n-1} \leq \frac{(p+n_0-1)(3p+n_0-1)(p-\alpha)(1-q)}{p+n_0+\alpha-1}r^{2p+n_0-1}.$$

We now find the value $r_0 = r_0(p,q,\delta,\alpha)$ and the corresponding $n_0 = n_0(r_0)$ so that

$$\frac{3qp^2(p-\alpha)}{p+\alpha}r_0^{2p} + \frac{(p+n_0-1)(3p+n_0-1)(p-\alpha)(1-q)}{p+n_0+\alpha-1}r_0^{2p+n_0-1} = p^2.$$

This gives the radius of convexity for $\Omega_p^*(\alpha,q)$. ∎

Remark 4.1. By taking $p=q=1$ and $\alpha=0$ in Theorem 4.1, we have:
If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ is meromorphically starlike, then $f(z)$ is meromorphically convex in $0 \leq |z| \leq 1/\sqrt{3}$.

Theorem 4.2. *Let the function $f(z)$ defined by (1.8) be in class $\Omega_p^*(\alpha,q)$, then it is meromorphically starlike in $0 \leq |z| \leq r_1 = r_1(p,q,\alpha)$, where $r_1(p,q,\alpha)$ is the largest value of r for which*

$$\frac{3pq(p-\alpha)}{p+\alpha}r^{2p} + \frac{(3p+n-1)(p-\alpha)(1-q)}{p+n+\alpha-1}r^{2p+n-1} \leq p, \quad (n \geq 2). \tag{4.2}$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. It suffices to show that $|p + zf'(z)/f(z)| \leq p$ for $0 \leq |z| \leq r_1 = r_1(k,q,\alpha)$. But

$$\left|p + \frac{zf'(z)}{f(z)}\right| \leq \frac{\frac{2pq(p-\alpha)}{p+\alpha}r^{2p} + \sum_{n=2}^{\infty}(2p+n-1)a_{p+n-1}r^{2p+n-1}}{1 - \frac{q(p-\alpha)}{p+\alpha}r^{2p} - \sum_{n=2}^{\infty}a_{p+n-1}r^{2p+n-1}} \leq p,$$

if

$$\frac{3pq(p-\alpha)}{p+\alpha}r^{2p} + \sum_{n=2}^{\infty}(3p+n-1)a_{p+n-1}r^{2p+n-1} \leq p. \tag{4.3}$$

The radius of meromorphically starlikeness now follows from (2.1). ∎

In a similar manner, we can obtain the radii of meromorphically convexity and meromorphically starlikeness for functions in the class $\Lambda_p^*(\alpha,c)$.

Theorem 4.3. *Let the function $f(z)$ defined by (1.11) be in the class $\Lambda_p^*(\alpha,c)$, then it is meromorphically convex in $0 \leq |z| \leq r_2 = r_2(p,c,\alpha)$, where $r_2(p,c,\alpha)$ is the largest value of r for which*

$$\frac{3cp(p-\alpha)}{p+\alpha}r^{2p} + \frac{(3p+n-1)(p-\alpha)(1-c)}{p+n+\alpha-1}r^{2p+n-1} \leq p^2, \quad (n \geq 2). \tag{4.4}$$

The result is sharp for the function $f(z)$ given by (2.6).

Theorem 4.4. *Let the function $f(z)$ defined by (1.11) be in the class $\Lambda_p^*(\alpha, c)$, then it is meromorphically starlike in $0 \leq |z| \leq r_3 = r_3(p, c, \alpha)$, where $r_3(p, c, \alpha)$ is the largest value of r for which*

$$\frac{3c(p-\alpha)}{p+\alpha} r^{2p} + \frac{(3p+n-1)(p-\alpha)(1-c)}{(p+n-1)(p+n+\alpha-1)} r^{2p+n-1} \leq p, \qquad (n \geq 2). \tag{4.5}$$

The result is sharp for the function $f(z)$ given by (2.6).

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