WEAK SOLUTION FOR HYPERBOLIC EQUATIONS WITH A NON-LOCAL CONDITION

LAZHAR BOUGOFFA

Received 17 July, 2004; revised 13 February, 2005; accepted 14 January, 2005; published 7 March, 2005.

KING KHALID UNIVERSITY, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, P.O. BOX 9004, ABHA, SAUDI ARABIA

ABSTRACT. In this paper, we study hyperbolic equations with a non-local condition. We prove the existence and uniqueness of weak solutions, using energy inequality and the density of the range of the operator generated by the problem.

Key words and phrases: Non-local condition, A priori estimate, Weak solution.

1991 Mathematics Subject Classification 35L20, 35D05, 35B45, 35B30.

ISSN (electronic): 1449-5910
© 2005 Austral Internet Publishing. All rights reserved.
1. Introduction

Various problems arising in heat conduction [5], [6], [8], chemical engineering [7], thermoelasticity [14], and plasma physics [12] can be reduced to the non-local problems with integral boundary conditions. This type of boundary value problems has been investigated in [1], [3], [5], [6], [7], [8], [14], [16] for parabolic equations and in [2], [11], [15] for hyperbolic equations. Boundary value problems with integral conditions constitute a very interesting and important class of problems. For comments on their importance, we refer the reader to the above papers. This paper is a continuation of the mentioned papers, our goal is to prove the existence and uniqueness of weak solutions for one-dimensional wave equations with a non-local boundary condition.

Consider the equation

\[ \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U = F(t, x), \]

in the rectangular domain \( \Omega = (0, T) \times (0, 1) \).

To equation (1.1) we attach the initial conditions

\[ U(0, x) = \Phi(x), \]
\[ U_t(0, x) = \Psi(x), \]

Dirichlet boundary condition

\[ U(t, 1) - U(t, 0) = 0, \]

and the non-local boundary condition

\[ \int_0^1 U(t, x)dx = 0. \]

We assume that \( \Phi(x), \Psi(x) \in L_2(0, 1) \) are known functions and satisfy the compatibility conditions

\[ \Phi(1) - \Phi(0) = 0, \Psi(1) - \Psi(0) = 0 \text{ and } \int_0^1 \Phi(x)dx = \int_0^1 \Psi(x)dx = 0. \]

Such equations become more complicated when studied with a non-local boundary condition. For that, we reduce (1.1)-(1.5) to an equivalent problem.

Lemma 1.1. Problem (1.1)-(1.5) is equivalent to the following problem

\[
\begin{cases}
\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U = F(t, x), \\
U(0, x) = \Phi(x), \\
U_t(0, x) = \Psi(x), \\
U(t, 1) - U(t, 0) = 0, \\
U_x(t, 1) - U_x(t, 0) = -\int_0^1 F(t, x)dx.
\end{cases}
\]

Proof. Let \( U(t, x) \) be a solution of (1.1)-(1.5). Integrating (1.1) with respect to \( x \) over \((0, 1)\), and taking in account (1.5)-(1.5), we obtain

\[ U_x(t, 1) - U_x(t, 0) = -\int_0^1 F(t, x)dx. \]
Let now \( U(t, x) \) be a solution of \((PR)\), we are required to show that
\[
\int_0^1 U(t, x) dx = 0, \forall t \in (0, T).
\]
For this end we integrate again \((1.1)\) with respect to \(x\) and obtain
\[
\frac{d^2}{dt^2} \int_0^1 U(t, x) dx + \frac{d}{dt} \int_0^1 U(t, x) dx + \int_0^1 U(t, x) dx = 0, \forall t \in (0, T),
\]
by virtue of the compatibility conditions
\[
\int_0^1 U(0, x) dx = 0 \text{ and } \int_0^1 U_t(0, x) dx = 0,
\]
we get
\[
\int_0^1 U(t, x) dx = 0.
\]

Introduce now the new unknown function \( u(t, x) = U(t, x) - w(t, x) \), where
\[
w(t, x) = \frac{x(1 - x)}{2} \int_0^1 F(t, x) dx.
\]
Then \((PR)\) is transformed into
\[
\begin{align*}
\ell u & \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u = f(t, x), \\
\ell_0 u & \equiv u(0, x) = \varphi(x), \\
\ell_1 u & \equiv u_t(0, x) = \psi(x),
\end{align*}
\]
where
\[
f(t, x) = F(t, x) - \frac{x(1 - x)}{2} \int_0^1 F_{tt}(t, x) dx - \frac{x(1 - x)}{2} \int_0^1 F_t(t, x) dx + \frac{(x^2 + x - 3)}{2} \int_0^1 F(t, x) dx,
\]
\[
\varphi(x) = \Phi(x) - \frac{x(1 - x)}{2} \int_0^1 F(0, x) dx,
\]
and
\[
\psi(x) = \Psi(x) - \frac{x(1 - x)}{2} \int_0^1 F_t(0, x) dx.
\]

### 2. Abstract formulation of the boundary problem

We consider the problem \((Pr)\) as a solution of the operational equation
\[
Lu = F,
\]
where \( L = (\ell, \ell_0, \ell_1) \) with domain of definition \( D(L) \) consisting of functions \( u \) belonging to the Sobolev space \( H^2(\Omega) \) and satisfying the boundary conditions of \((Pr)\). The operator \( L \) is
considered from $E$ to $W$, where $E$ is the Banach space consisting of functions $u \in L_2(\Omega)$ having the finite norm
\[ ||u||^2_E = \sup_{0 \leq \tau \leq T} \left[ \int_0^\tau (u^2 + u_t^2 + u_x^2)(\tau, x) dx + \int_\Omega u_t^2(t, x) dtdx < \infty \right] \]
and satisfying the boundary conditions of $(\Pr)$ and $W$ is the Hilbert space obtained by completion of $L_2(\Omega) \times H^1(0, 1) \times L_2(0, 1)$ with respect to the norm
\[ ||F||^2_W = \int_\Omega f^2(t, x) dtdx + \int_0^1 \left[ \varphi^2(x) + \varphi'^2(x) \right] dx + \int_0^1 \psi^2(x) dx. \]
The inner product in $W$ is defined by:
\[ (F, Z)_W = (f, w, \varphi)_{0, \Omega} + (\varphi, w_0)_{1, (0, 1)} + (\psi, w_1)_{0, (0, 1)} , \]
where $F = (f, \varphi, \psi)$, $Z = (w, w_0, w_1)$ belongs to $W$ and $(\cdot, \cdot)_{0, \Omega}$, $(\cdot, \cdot)_{0, (0, 1)}$ and $(\cdot, \cdot)_{1, (0, 1)}$ denote the inner product in $L_2(\Omega)$, $L_2(0, 1)$ and $H^1(0, 1)$ respectively.

3. A PRIORI ESTIMATES

Here we establish an energy inequality which ensures the uniqueness of the weak solution.

**Theorem 3.1.** For the problem $(\Pr)$, we have
\[ ||u||_E \leq c_0 ||Lu||_W, \forall u \in D(L), \]
where $c_0 > 0$ is independent on $u$.

**Proof.** Define the operator
\[ Mu = 2u_t, \]
and consider the scalar product $(\ell u, Mu)_{0, \Omega^\tau}$, where $0 \leq \tau \leq T$, and $\Omega^\tau = (0, \tau) \times (0, 1)$. Employing integration by parts, we obtain
\[ 2(\ell u, u_t)_{0, \Omega^\tau} = \int_0^1 \left[ u^2 + u_t^2 + u_x^2 \right] (\tau, x) dx + 2 \int_{\Omega^\tau} u_t^2(t, x) dtdx \]
\[ + 2 \int_{\Omega^\tau} [u_x(t, x) \times u_t(t, x)] dtdx - \int_0^1 u^2(0, x) dx \]
\[ - \int_0^1 u_t^2(0, x) dx - \int_0^1 u_x^2(0, x) dx \]
\[ - 2 \int_\Omega [u_x(t, 1) \times u_t(t, 1) - u_x(t, 0) \times u_t(t, 0)] dt. \]
Taking into account the initial and boundary conditions of $(\Pr)$, we see that
\[ \int_0^1 [u^2 + u_t^2 + u_x^2](\tau, x) dx + 2 \int_{\Omega^\tau} u_t^2(t, x) dtdx = 2(\ell u, u_t)_{0, \Omega^\tau} \]
\[ - 2 \int_{\Omega^\tau} [u_x(t, x) \times u_t(t, x)] dtdx \]
\[ + \int_0^1 \psi^2(x) dx + \int_0^1 \left[ \varphi^2(x) + \varphi'^2(x) \right] dx. \]
We now apply the $\varepsilon$-inequality $2|ab| \leq \varepsilon a^2 + \frac{1}{2} b^2$, $\varepsilon > 0$ to the first and second terms on the right-hand side of (4.2) and employ Gronwall’s Lemma (see e.g. Lemma 3.4 [13]), we get the inequality
\[
\int_0^1 (u^2 + u_x^2) (\tau, x) dx + \int_{\Omega\tau} u^2(t, x) dtdx \leq c_0^2 ||Lu||_W^2
\]
where
\[
||Lu||_W^2 = \int_{\Omega} f^2(t, x) dtdx + \int_0^1 \left[ \varphi^2(x) + \varphi_x^2(x) \right] dx + \int_0^1 \psi^2(x) dx.
\]
Now, as the right-hand side of (3.3) is independent of $\tau$, replacing the left-hand side by its upper bound with respect to $\tau$ in the interval $(0, T)$, we obtain the desired inequality. This completes the proof.}

4. Existence and uniqueness

For existence of the weak solution for (Pr), we shall prove that the range $\mathcal{R}(L)$ is dense in $W'$, where $W' = E^* \times H^{-1}(0, 1) \times L_2(0, 1)$, $W \subset W'$ and $E^*$ is the dual space of $E$ with respect to the canonical bilinear form $\langle u, v \rangle$, $u \in E$ and $v \in E^*$, which is the extension by continuity of the bilinear form $\langle u, v \rangle$, where $u \in L_2(\Omega)$ and $v \in E$. First consider $u \in D_0(L)$ where $D_0(L) = \{ u \in D(L) / \ell_0 u = \ell_1 u = 0 \}$, then (Pr) becomes
\[
(Pr)_0 \left\{ \begin{array}{ll}
\ell u & \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u = f(t, x), \\
\ell_0 u & \equiv u(0, x) = 0, \\
\ell_1 u & \equiv u_t(0, x) = 0, \\
& u(t, 1) - u(t, 0) = 0, \\
& u_x(t, 1) - u_x(t, 0) = 0.
\end{array} \right. 
\]

Our aim here is to prove existence of weak solutions of $(Pr)_0$. The proof is based on an energy inequality and the density of the range of the operators generated by the studied problem.

Analogous to the problem $(Pr)_0$, we consider its dual problem. We denote by $\ell^*$ the formal dual of the operator $\ell$, which is defined with respect to the inner product in the space $L_2(\Omega)$ using
\[
\langle \ell u, v \rangle = \langle u, \ell^* v \rangle \quad \text{for all } u, v \in C^2_0(\Omega),
\]
where $\langle \cdot, \cdot \rangle$ stands for the inner product in $L_2(\Omega)$.

Let
\[
(Pr)^*_0 \left\{ \begin{array}{ll}
\ell^* v & \equiv \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + v = g(t, x), \\
\ell^*_0 v & \equiv v(T, x) = 0, \\
\ell^*_1 v & \equiv v_t(T, x) = 0, \\
& v(t, 1) - v(t, 0) = 0, \\
& v_x(t, 1) - v_x(t, 0) = 0.
\end{array} \right. 
\]

The solution of $(Pr)_0$ will be considered as a solution of the operational equation:
\[
\ell u = f, \quad u \in D(\ell),
\]
and the solution of $(Pr)^*_0$ as a solution of the operational equation:
\[
\ell^* v = g, \quad v \in D(\ell^*).
\]

To solve the equation (4.2) for every $f \in E^*$, we establish the following existence and uniqueness theorems of weak solutions for problems $(Pr)_0$ and $(Pr)^*_0$. 

AJMAA, Vol. 2, No. 1, Art. 7, pp. 1-7, 2005
Theorem 4.1. For the problem (Pr)$_0$(resp. (Pr)$_0'$) we have
\begin{align}
(4.4) & \quad ||u||_E \leq c_1||\ell u||_{E^*}, \forall u \in E \\
(4.5) & \quad ||v||_E \leq c_1^*||\ell^* v||_{E^*}, \forall v \in E^*.
\end{align}
where the constants $c_1 > 0$ and $c_1^* > 0$ are independent on $u$ and $v$.

Proof. An application of (3.1) gives (4.4) for $u \in D(\ell)$. For $u \in E$, we use the regularization operators of Friedreichs [9], [10] to conclude that (4.4) holds true. ■

Theorem 4.2. For all functions $f \in E^*$ (resp. $g \in E^*$) there exists one and only one weak solution of the problem (Pr)$_0$ (resp. (Pr)$_0'$).

Proof. We mention that from the inequality (4.4) follows immediately the uniqueness of the solutions. It also ensures the closure of the range set $\mathcal{R}(\ell)$ of the operator $\ell$.

An application of the Theorem II.19 in [4] with the inequality (4.5) give $\mathcal{R}(\ell) = E^*$. ■

The second part of Theorems 4.1-4.2 can be proved in a similar way by using the operator $M^*v = 2v_1$.

Now we need the following Lemma.

Lemma 4.3. If $w \in E$ and for all $u \in D_0(L)$, we have
\[ \langle \ell u, w \rangle_{E^*, E} = 0, \]
then $w = 0$.

Proof. It sufficient to show that $\mathcal{R}(\ell)$ is dense in $E^*$. The fact that $\mathcal{R}(\ell) = E^*$; results directly from Theorem 4.1. It remains to prove that the inclusion $\overline{\mathcal{R}(\ell)} \subseteq \mathcal{R}(\ell)$. Indeed, let $\{f_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence in the space $E^*$, which consists of elements of set $\mathcal{R}(\ell)$. Then it corresponds to a sequence $\{u_k\}_{k \in \mathbb{N}} \subseteq D(\ell)$ such that: $\ell u_k = f_k, k \in \mathbb{N}$.

From the inequality (3.1), we conclude that the sequence $\{u_k\}$ is also a Cauchy sequence in the space $E$ and converges to an element $u$ in $E$, we define the element
\[ \ell u = f (f = \lim_{k \to \infty} f_k). \]
This establishes the inclusion $\overline{\mathcal{R}(\ell)} \subseteq \mathcal{R}(\ell)$. ■

To this end, we show that the following existence theorem:

Theorem 4.4. The range $\mathcal{R}(L)$ of the operator $L$ is dense in $W'$.

Proof. Since $W'$ is a Hilbert space, the density of $\mathcal{R}(L)$ in $W'$ is equivalent to the orthogonality of the vector $Z = \langle w, w_0, w_1 \rangle \in W'$ to the set $\mathcal{R}(L)$, i.e. the equality
\begin{equation}
(4.6) \quad \langle \ell u, w \rangle_{E^*, E} + (\varphi, w_0)_{1,(0,1)} + (\psi, w_1)_{0,(0,1)} = 0,
\end{equation}
implying that $Z = 0$. In particular, put $u \in D_0(L)$ in (4.6). Then, $\langle \ell u, w \rangle_{E^*, E} = 0$ and we conclude by Lemma 4.3 that $w = 0$, so it follows from (4.6) that $(\varphi, w_0)_{1,(0,1)} + (\psi, w_1)_{0,(0,1)} = 0$. But since the range of the trace operators $\ell_0$ and $\ell_1$ are dense in $H^1(0,1)$ and $L^2(0,1)$ respectively, thus $w_0 = w_1 = 0$. Consequently, $Z = 0$. Hence $\overline{\mathcal{R}(L)} = W'$. ■
REFERENCES


