The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 2, Issue 1, Article 5, pp. 1-5, 2005

A SIMPLE NEW PROOF OF FAN-TAUSSKY-TODD INEQUALITIES

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Received 8 July, 2004; accepted 27 January, 2005; published 15 February, 2005.
Communicated by: P. Cerone

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ABSTRACT. In this paper we present simple new proofs of the inequalities:

$$2\left(1-\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n}a_k^2 \le \sum_{k=1}^{n+1}(a_k-a_{k-1})^2 \le 2\left(1+\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n}a_k^2,$$

which holds for all real numbers $a_0=0, a_1, \cdots, a_n, a_{n+1}=0$ and the coefficients $2(1-\cos(\pi/(n+1)))$ and $2(1+\cos(\pi/(n+1)))$ are the best possible; and

$$2\left(1-\cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n}a_k^2 \le \sum_{k=1}^{n}(a_k-a_{k-1})^2 \le 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n}a_k^2,$$

which holds for all real numbers $a_0=0, a_1, \cdots, a_n$ and the coefficients $2(1-\cos(\pi/(2n+1)))$ and $2(1+\cos(\pi/(2n+1)))$ are the best possible.

Key words and phrases: Fan-Taussky-Todd inequalities, Reverse inequalities.

2000 *Mathematics Subject Classification.* Secondary 26D15, 26D10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In 1955, K. Fan, O. Taussky and J. Todd published a remarkable paper proving discrete analogues of several well-known integral inequalities. Among their results is the following theorem in [2]:

Theorem 1.1. *Assume a_i are real numbers for $1 \le i \le n$, we have*
(a) *if $a_0 = a_{n+1} = 0$, then*

$$2\left(1-\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n+1}(a_k - a_{k-1})^2, \tag{1.1}$$

with equality holding if and only if $a_k = c\sin\dfrac{k\pi}{n+1}$ $(k = 1, 2, \cdots, n, c$ is a real constant$)$.
(b) *if $a_0 = 0$, then*

$$2\left(1-\cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n}(a_k - a_{k-1})^2, \tag{1.2}$$

with the equality holding if and only if $a_k = c\sin\dfrac{k\pi}{2n+1}$ $(k = 1, 2, \cdots, n$, c is a real constant$)$.

The two constants $2\left(1-\cos\dfrac{\pi}{n+1}\right)$ and $2\left(1-\cos\dfrac{\pi}{2n+1}\right)$ given in inequalities (1.1) and (1.2), respectively, are the best possible.

Redheffer [4] presented an "ingenious proof" for these results based on an analysis of the characteristic values and vectors of Hermitean matrices. The main tool is an intriguing inequality of D. E. Rutherford who investigated the structure of Hermitean matrices "because of their great importance in a number of mathematical models of chemical and physical processes" [1]. E. F. Beckenbach and R. Bellman mention the Theorem1.1 as well as similar results are important for the numerical integration of differential equations. Motivated to find "easy proofs" of the inequalities (1.1) and (1.2), R. M. Redheffer [4] presented in 1983 a very elegant elementary method for the proof of Theorem1.1.

In 1982, G. V. Milovanović and I. Z. Milovanović [3] obtained reversed of inequalities (1.1) and (1.2). By using techniques similar to those of Fan-Taussky-Todd they proved:

Theorem 1.2. *Assume a_i are real numbers for $1 \le i \le n$, we have*
(a) *if $a_0 = a_{n+1} = 0$, then*

$$\sum_{k=1}^{n+1}(a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2, \tag{1.3}$$

with the equality holding if and only if $a_k = (-1)^{k-1}c\sin\dfrac{k\pi}{n+1}(k = 1, 2, \cdots, n$, c is a real constant$)$.
(b) *if $a_0 = 0$, then*

$$\sum_{k=1}^{n}(a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2, \tag{1.4}$$

with the equality holding if and only if $a_k = (-1)^{k-1}c\sin\dfrac{k\pi}{2n+1}(k = 1, 2, \cdots, n$, c is a real constant$)$.

By using a modification of Redheffer's technique [4], H. Alzer [5] gave a simple proof of inequalities (1.3) and (1.4) in 1991.

Combining Theorem1.1 and Theorem1.2, we obtain

Theorem 1.3. *Assume a_i are real numbers for $1 \le i \le n$, we have*

(a) *if $a_0 = a_{n+1} = 0$, then*

$$2\left(1-\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n+1}(a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2, \tag{1.5}$$

and the coefficients $2(1-\cos\frac{\pi}{n+1})$ and $2(1+\cos\frac{\pi}{n+1})$ are the best possible.

(b) *if $a_0 = 0$, then*

$$2\left(1-\cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n}(a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2, \tag{1.6}$$

and the coefficients $2(1-\cos\frac{\pi}{2n+1})$ and $2(1+\cos\frac{\pi}{2n+1})$ are the best possible.

Therefore (1.1) and (1.2) are called Fan-Taussky-Todd inequalities, and (1.3) and (1.4) are called the reverse Fan-Taussky-Todd inequalities.

In this paper, we give a simple elementary proof of inequalities (1.5) and (1.6).

2. Main Result

Theorem 2.1. *Let n be an integer with $n > 1$, $a_i (1 \le i \le n)$ be n real numbers, and $t \in \left(0, \frac{\pi}{n}\right)$, then we have*

$$2\cos t \sum_{k=1}^{n} a_k^2 \ge \frac{\sin(n+1)t}{\sin nt} a_n^2 \pm 2\sum_{k=1}^{n-1} a_k a_{k+1}, \tag{2.1}$$

with the equality holding if and only if $a_k = c_1 \sin kt$ for "$+$", and $a_k = (-1)^{k-1} c_2 \sin kt$ for "$-$", where $k = 1, 2, \cdots, n$, and c_1, c_2 are two real constants.

Proof. From $t \in \left(0, \frac{\pi}{n}\right)$, we have $\sin kt > 0 (k = 1, 2, \cdots, n-1)$, and

$$\frac{\sin(k+1)t}{\sin kt} a_k^2 + \frac{\sin kt}{\sin(k+1)t} a_{k+1}^2 \ge \pm 2 a_k a_{k+1}, \tag{2.2}$$

with equality holding if and only if

$$a_{k+1}\sin kt \pm a_k \sin(k+1)t = 0.$$

From (2.2), summing from 1 to $n-1$, we get

$$\sum_{k=1}^{n-1}\left[\frac{\sin(k+1)t}{\sin kt} a_k^2 + \frac{\sin kt}{\sin(k+1)t} a_{k+1}^2\right] \ge \pm 2\sum_{k=1}^{n-1} a_k a_{k+1},$$

that is

$$\sum_{k=1}^{n} \frac{\sin(k-1)t + \sin(k+1)t}{\sin kt} a_k^2 \ge \frac{\sin(n+1)t}{\sin nt} a_n^2 \pm 2\sum_{k=1}^{n-1} a_k a_{k+1}. \tag{2.3}$$

Utilizing the fact that

$$\sin(k-1)t + \sin(k+1)t = 2\sin kt\cos t,$$

inequalities (2.3) become (2.1), with the equality holding if and only if

$$a_{k+1}\sin kt=a_k\sin\left(k+1\right)t,$$

or $a_k=c\sin kt$ $(k=1,2,\cdots,n,c$ is a constant) for "+"; and

$$a_{k+1}\sin kt+a_k\sin\left(k+1\right)t=0,$$

or $a_k=(-1)^{k-1}c\sin kt$ $(k=1,2,\cdots,n$, c is a constant) for "$-$".

This proves Theorem2.1. ∎

3. REMARKS

Remark 3.1. Let (2.1) for "$+$", let $t=\dfrac{\pi}{n+1}$, or $t=\dfrac{\pi}{2n+1}$, since

$$\sin\pi=0,\ \sin\frac{(n+1)\pi}{2n+1}=\sin\frac{\pi}{2n+1},$$

then we have

$$\cos\frac{\pi}{n+1}\sum_{k=1}^{n}a_k^2\geq\sum_{k=1}^{n}a_ka_{k+1}, \tag{3.1}$$

with the equality holding if and only if $a_k=c\sin\dfrac{k\pi}{n+1}$ $(k=1,2,\cdots,n$, c is a constant$)$, and

$$2\cos\frac{\pi}{2n+1}\sum_{k=1}^{n}a_k^2\geq a_n^2+2\sum_{k=1}^{n}a_ka_{k+1}, \tag{3.2}$$

with the equality holding if and only if $a_k=c\sin\dfrac{k\pi}{2n+1}$ $(k=1,2,\cdots,n$, c is a constant$)$, respectively.

Remark 3.2. Let (2.1) for "$-$", and $t=\dfrac{\pi}{n+1},\dfrac{2\pi}{n+1}$, because

$$\sin\frac{2(n+1)\pi}{2n+1}=\sin\frac{2n\pi}{2n+1},$$

therefore

$$\cos\frac{\pi}{n+1}\sum_{k=1}^{n}a_k^2\geq-\sum_{k=1}^{n-1}a_ka_{k+1}, \tag{3.3}$$

with the equality holding if and only if $a_k=(-1)^{k-1}c\sin\dfrac{k\pi}{n+1}$ $(k=1,2,\cdots,n$, c is a constant$)$, and

$$2\cos\frac{2\pi}{2n+1}\sum_{k=1}^{n}a_k^2\geq-a_n^2-2\sum_{k=1}^{n-1}a_ka_{k+1}, \tag{3.4}$$

with the equality holding if and only if $a_k=(-1)^{k-1}c\sin\dfrac{k\pi}{2n+1}$ $(k=1,2,\cdots,n$, c is a constant$)$, respectively.

Remark 3.3. It is easy to see that inequalities (3.1) and (3.3) are equivalent to (1.5), and inequalities (3.2) and (3.4) are equivalent to (1.6), respectively.

Remark 3.4. Combining inequalities (3.1) and (3.3), that is to say: assume $a_i (1 \leq i \leq n)$ be n real numbers for $\sum_{i=1}^{n} a_i^2 \neq 0$, and

$$f(a_1, a_2, \cdots, a_n) = \frac{\sum_{k=1}^{n-1} a_k a_{k+1}}{\sum_{k=1}^{n} a_k^2}, \tag{3.5}$$

then

$$f_{max} = \cos \frac{\pi}{n+1}, \tag{3.6}$$

and

$$f_{min} = -\cos \frac{\pi}{n+1}. \tag{3.7}$$

4. AN OPEN PROBLEM

The paper of Fan-Taussky-Todd contains also noteworthy inequalities involving the second differences

$$\triangle^2 \, a_k = a_k - 2a_{k+1} + a_{k+2}. \tag{4.1}$$

One of their results states that if $a_1, \cdots, a_n$ are real numbers and if $a_0 = a_{n+1} = 0$, then

$$\left(2 \sin \frac{\pi}{2(n+1)}\right)^2 \sum_{k=1}^{n} a_k^2 \leq \sum_{k=0}^{n-1} (a_k - 2a_{k+1} + a_{k+2})^2 , \tag{4.2}$$

with the equality holding if and only if $a_k = c \sin((k\pi)/(n+1))$ ($k = 1, \cdots, n, c$ be a real constant).

We conclude the paper by asking: Does there exist converse inequalities of (4.2) and of related inequalities given in [2], and if the answer is "yes" which are the best possible constants?

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