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ON CERTAIN CLASSES OF HARMONIC UNIVALENT FUNCTIONS BASED ON SALAGEAN OPERATOR

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ABSTRACT. We define and investigate a class of complex-valued harmonic univalent functions of the form $f = h + \overline{g}$ using Salagean operator where h and g are analytic in the unit disc $U = \{z : |z| < 1\}$. A necessary and sufficient coefficient conditions are given for functions in these classes. Furthermore, distortion theorems, inclusion relations, extreme points, convolution conditions and convex combinations for this family of harmonic functions are obtained.

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1. INTRODUCTION

A continuous functions $f = u + iv$ is a complex-valued harmonic function in a complex domain Ω if both u and v are real harmonic in Ω. In any simply connected domain $C \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in C. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in C is that $|g'(z)| < |h'(z)|$ (see Clunie and Sheil-Small [3]).

Denote by H the family of functions $f = h + \overline{g}$ that are harmonic, univalent and orientation preserving in the open disc $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Therefore, for $f = h + \overline{g} \in H$ we can express the analytic functions h and g by the following power series expansion

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}, \quad |b_1| < 1. \tag{1.1}$$

Note that the family H consisting of orientation preserving, normalized, harmonic univalent functions reduces to the class S of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero, ie., $g(z) \equiv 0$. Further denote by $\overline{H}$ the subfamily of H consisting of harmonic functions $f_n = h + \overline{g_n}$ of the form

$$f_n(z) = z - \sum_{m=2}^{\infty} a_m z^m + (-1)^n \overline{\sum_{m=1}^{\infty} b_m z^m}, \quad a_m, b_m \geq 0 \text{ and } |b_1| < 1. \tag{1.2}$$

For $f = h + \overline{g}$, given by (1.1), recently Jahangiri et al. [5], defined the Salagean derivative of harmonic functions $f = h + \overline{g}$ in H by

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \ \ n \in N \cup \{0\} \tag{1.3}$$

where the Salagean derivative [7] of power series $\phi(z) = \sum\limits_{m=1}^{\infty} \phi_m z^m$ is given by $D^0\phi(z) = \phi(z)$, $D^1\phi(z) = z\phi'(z)$ and $D^n\phi(z) = D(D^{n-1}\phi(z)) = \sum\limits_{m=1}^{\infty} m^n \phi_m z^m$.

For fixed values of n, we let $F_H(n, \lambda, \alpha)$ to consist of harmonic functions $f = h + \overline{g}$ in H so that

$$Re\left\{(1-\lambda)[D^n f(z)/z] + \lambda \left[\frac{\frac{\partial D^n f(z)}{\partial \theta}}{\frac{\partial z}{\partial \theta}}\right]\right\} > \alpha, \tag{1.4}$$

where $0 \leq \alpha < 1$,$0 \leq \lambda \leq 1$ and $z = re^{i\theta} \in U$. We also let $F_{\overline{H}}(n, \lambda, \alpha) = F_H(n, \lambda, \alpha) \cap \overline{H}$.

As λ changes from 0 to 1, the family $F_H(n, \lambda, \alpha)$ provides a passage from the class of harmonic functions $P_H(n, \alpha) \equiv F_H(n, 0, \alpha)$ consisting of functions f where $Re\left\{\frac{D^n f(z)}{z}\right\} \geq \alpha$ to the class of harmonic functions $Q_H(n, \alpha) \equiv F_H(n, 1, \alpha)$ consisting of functions f where $\left\{\frac{\partial}{\partial\theta} D^n f(z) / \frac{\partial}{\partial\theta} z\right\} > \alpha$. Note that if $n = 0$ and the co-analytic part $g \equiv 0$, the class $F_{\overline{H}}(n, \lambda, \alpha) = F_\lambda(\alpha)$ [2]. Further if $n = 0$, $Q_H(n, \alpha) = N_H(\alpha)[1]$ and if $n = 0$, $Q_{\overline{H}}(n, \alpha) = N_{\overline{H}}(\alpha)$.

Recently there has been triggering interest to study harmonic complex functions (details see [1], [3], [4], [5], [6], [8]), motivated by Jahangiri et al., [5] and using the techniques of Silverman [8], in this paper, we have obtained the coefficient conditions for the classes $F_H(n, \lambda, \alpha)$ and $F_{\overline{H}}(n, \lambda, \alpha)$. Further a representation theorem, inclusion properties and distortion bound for the class $F_{\overline{H}}(n, \lambda, \alpha)$ are established.

2. MAIN RESULTS

First we give the sufficient coefficient bound for functions in the class $F_H(n,\lambda,\alpha)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{m=1}^{\infty} m^n \left\{|(m-1)\lambda+1||a_m| + |(m+1)\lambda-1||b_m|\right\} \le 2(1-\alpha) \tag{2.1}$$

where $a_1 = 1$ and $0 \le \alpha < 1$ then f is orientation preserving, harmonic univalent in U and $f \in F_H(n,\lambda,\alpha)$.

Proof. If the inequality (2.1) holds for the coefficients of $f = h + \overline{g}$, then by (1.3), f is orientation preserving and harmonic univalent in U. It remains to show that

$$\text{Re}\left\{(1-\lambda)\frac{D^n f}{z} + \lambda\left(\frac{\frac{\partial}{\partial\theta}D^n f}{\frac{\partial}{\partial\theta}z}\right)\right\} \ge \alpha.$$

According to (1.2) and (1.3) we have

$$\begin{aligned} &\text{Re}\left\{(1-\lambda)\frac{D^n f}{z} + \lambda\left(\frac{\frac{\partial}{\partial\theta}D^n f}{\frac{\partial}{\partial\theta}z}\right)\right\} \\ = \ &\text{Re}\left\{\frac{(1-\lambda)[D^n h(z) + (-1)^n\overline{D^n g(z)}] + \lambda[z(D^n h(z))' - \overline{(-1)^n z(D^n g(z))'}]}{z}\right\} \ge \alpha. \end{aligned}$$

Using the fact that Re $w \ge \alpha$ it suffices to show that $|1-\alpha+w| \ge |1+\alpha-w|$. This is equivalent to showing that if the condition (2.1) holds then

$$\begin{aligned} &\left|(1-\alpha)z + (1-\lambda)\left[D^n h(z) + (-1)^n\overline{D^n g(z)}\right] + \lambda\left[z(D^n h(z))' - (-1)^n\overline{z(D^n g(z))'}\right]\right| \\ - &\left|(1-\lambda)\left[D^n h(z) + (-1)^n\overline{D^n g(z)}\right] + \lambda\left[z(D^n h(z))' - (-1)^n\overline{z(D^n g(z))'}\right] - (1+\alpha)z\right| \\ \ge\ & 0 \end{aligned}$$

substituting for $D^n h(z)$, $D^n g(z)$, $(D^n h(z))'$ and $(D^n g(z))'$, simple computation leads yield

$$\begin{aligned} \ge\ & z - \alpha z - \sum_{m=2}^{\infty} m^n|1-\lambda+m\lambda|\ |a_m||z|^m - \sum_{m=1}^{\infty} m^n|-1+\lambda+m\lambda|\ |b_m||z|^m \\ & -\alpha z + z - \sum_{m=2}^{\infty} m^n|1-\lambda+m\lambda|\ |a_m||z|^m - \sum_{m=1}^{\infty} m^n|-1+\lambda+m\lambda|\ |b_m||z|^m \\ =\ & 2(1-\alpha)|z| \\ & \left\{1 - \sum_{m=2}^{\infty}\frac{m^n|(m-1)\lambda+1|}{(1-\alpha)}|a_m||z|^{m-1} - \sum_{m=1}^{\infty}\frac{m^n|(m+1)\lambda-1|}{(1-\alpha)}|b_m||z|^{m-1}\right\} \\ \ge\ & 2(1-\alpha)|z|\left\{1 - \sum_{m=2}^{\infty}\frac{m^n|(m-1)\lambda+1|}{(1-\alpha)}|a_m| - \sum_{m=1}^{\infty}\frac{m^n|(m+1)\lambda-1|}{(1-\alpha)}|b_m|\right\}. \end{aligned}$$

The last expression is nonnegative by (2.1) and so the proof is complete.

∎

The starlike harmonic function

$$f(z) = z + \sum_{m=2}^{\infty} \frac{(1-\alpha)}{m^n|(m-1)\lambda+1|} x_m z^m + \sum_{m=1}^{\infty} \frac{(1-\alpha)}{m^n|(m+1)\lambda-1|} \overline{y_m}\ \overline{z}^m \tag{2.2}$$

where $\sum_{m=2}^{\infty} |x_m| + \sum_{m=1}^{\infty} |y_m| = 1$ shows that the coefficient bound given by (2.1) is sharp.

The function of the form (2.2) are in $F_H(n, \lambda, \alpha)$ because

$$\begin{aligned} &\sum_{m=1}^{\infty} \left(\frac{m^n|(m-1)\lambda+1|}{(1-\alpha)} |a_m| + \frac{m^n|(m+1)\lambda-1|}{(1-\alpha)} |b_m| \right) \\ &= \ 1 + \sum_{m=2}^{\infty} |x_m| + \sum_{m=1}^{\infty} |y_m| = 2. \end{aligned}$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n = h_n + \overline{g}_n$ where f_n are of the form (1.4).

Theorem 2.2. *Let $f_n = h + \overline{g}_n$ be given by (1.2). Then $f_n \in F_{\overline{H}}(n, \lambda, \alpha)$ if and only if*

$$\sum_{m=1}^{\infty} m^n \left\{ |(m-1)\lambda+1| a_m + |(m+1)\lambda-1| b_m \right\} \leq 2(1-\alpha). \tag{2.3}$$

Proof. Since $F_{\overline{H}}(n, \lambda, \alpha) \subset F_H(n, \lambda, \alpha)$, we need to prove the "only if " part of the theorem. To this end, for functions f_n of the form (1.2), we notice that the condition

$$\text{Re} \left\{ (1-\lambda) \left[\frac{D^n f(z)}{z} \right] + \lambda \left[\frac{\frac{\partial}{\partial\theta} D^n f(z)}{\frac{\partial}{\partial\theta} z} \right] \right\} \geq \alpha$$

is equivalent to

$$\text{Re} \left\{ \frac{(1-\alpha)z - \sum_{m=2}^{\infty} m^n|(m-1)\lambda+1| a_m z^m - (-1)^{2n} \sum_{m=1}^{\infty} m^n |(m+1)\lambda-1| b_m \overline{z}^m}{z} \right\} \geq 0. \tag{2.4}$$

The above required condition (2.4) must hold for all values of z in U and from choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we have

$$1 - \alpha - \sum_{m=2}^{\infty} m^n|(m-1)\lambda+1| a_m r^{m-1} - \sum_{m=1}^{\infty} m^n|(m+1)\lambda-1| b_m r^{m-1} \geq 0. \tag{2.5}$$

If the condition (2.3) does not hold then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exists a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.5) is negative. This contradicts the required results for $f_n \in F_{\overline{H}}(n, \lambda, \alpha)$ and so the proof is complete. ∎

Next we determine the extreme points of the closed convex hulls of $F_{\overline{H}}(n, \lambda, \alpha)$ denoted by $clc_0 F_{\overline{H}}(n, \lambda, \alpha)$.

Theorem 2.3. *Let f_n be given by (1.2). Then $f_n \in F_{\overline{H}}(n, \lambda, \alpha)$ if and only if*

$$f_n(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_{n_m}(z)) \tag{2.6}$$

where $h_1(z) = z$, $h_m(z) = z - \frac{1-\alpha}{m^n|(m-1)\lambda+1|}z^m$, $(m = 2, 3, \dots)$,

$$\begin{aligned} g_{n_m}(z) &= z + (-1)^n \frac{(1-\alpha)}{m^n|(m+1)\lambda - 1|}\overline{z}^m, \ (m = 1, 2, \dots), \\ \sum_{m=1}^{\infty}(X_m + Y_m) &= 1, \ X_m \geq 0, \ Y_m \geq 0. \end{aligned}$$

In Particular, the extreme points of $F_{\overline{H}}(n, \lambda, \alpha)$ *are* $\{h_m\}$ *and* $\{g_{n_m}\}$.

Proof. For functions f_n of the form (2.6) we have

$$\begin{aligned} f_n(z) &= \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_{n_m}(z)) \\ &= \sum_{m=1}^{\infty}(X_m + Y_m)z - \sum_{m=2}^{\infty}\frac{1-\alpha}{m^n|(m-1)\lambda+1|}X_m z^m \\ &+ (-1)^n \sum_{m=1}^{\infty}\frac{1-\alpha}{m^n|(m+1)\lambda-1|}Y_m \overline{z}^m. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{m=2}^{\infty}\frac{m^n|(m-1)\lambda+1|}{1-\alpha}a_m + \sum_{m=1}^{\infty}\frac{m^n|(m+1)\lambda-1|}{1-\alpha}b_m \\ = &\sum_{m=2}^{\infty}X_m + \sum_{m=1}^{\infty}Y_m = 1 - X_1 \leq 1 \end{aligned}$$

and so $f_n \in clc_0 F_{\overline{H}}(n, \lambda, \alpha)$.
Conversely, suppose that $f_n \in clc_0 F_{\overline{H}}(n, \lambda, \alpha)$. Setting

$$\begin{aligned} X_m &= \frac{m^n|(m-1)\lambda+1|}{1-\alpha}a_m; (m = 2, 3, 4, \dots) \\ Y_m &= \frac{m^n|(m+1)\lambda-1|}{1-\alpha}b_m; (m = 1, 2, 3, \dots) \end{aligned}$$

where $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$ we obtain $f_n(z) = \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_{n_m}(z))$ as required. ∎

The following theorem gives the distortion bounds for functions in $F_{\overline{H}}(n, \lambda, \alpha)$ which yields a covering result for this class.

Theorem 2.4. *Let* $f_n \in F_{\overline{H}}(n, \lambda, \alpha)$. *Then for* $|z| = r < 1$ *we have*

$$\begin{aligned} &(1 - b_1)r - \frac{1}{2^n}\left(\frac{1-\alpha}{1+\lambda} - \frac{|2\lambda-1|}{1+\lambda}b_1\right)r^2 \leq |f_n(z)| \\ \leq &(1 + b_1)r + \frac{1}{2^n}\left(\frac{1-\alpha}{1+\lambda} - \frac{|2\lambda-1|}{1+\lambda}b_1\right)r^2. \end{aligned}$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted.

Let $f_n \in F_{\overline{H}}(n,\lambda,\alpha)$. Taking the absolute value of f_n we obtain

$$\begin{aligned} |f_n(z)| &\leq (1+b_1)r + \sum_{m=2}^{\infty}(a_m+b_m)r^m \\ &\leq (1+b_1)r + \sum_{m=2}^{\infty}(a_m+b_m)r^2 \\ &\leq (1+b_1)r + \frac{1-\alpha}{2^n(1+\lambda)}\sum_{m=2}^{\infty}\left(\frac{2^n(1+\lambda)}{1-\alpha}a_m + \frac{2^n(|2\lambda-1|)}{1-\alpha}b_m\right)r^2 \\ &\leq (1+b_1)r + \frac{1}{2^n}\left[1-\frac{|2\lambda-1|}{1-\alpha}b_1\right]r^2 \\ &= (1+b_1)r + \frac{1}{2^n}\left[\frac{1-\alpha}{1+\lambda}-\frac{|2\lambda-1|}{1+\lambda}b_1\right]r^2. \end{aligned}$$

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Corollary 2.5. *Let f_n be of the form (1.2) so that $f_n \in F_{\overline{H}}(n,\lambda,\alpha)$. Then*

$$\left\{w : |w| < \frac{2^n(1+\lambda)-1+\alpha}{2^n(1+\lambda)} - \frac{2^n(1+\lambda)-|2\lambda-1|}{2^n(1+\lambda)}b_1\right\} \subset f_n(U).$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form $f_n(z) = z - \sum\limits_{m=2}^{\infty} a_m z^m + (-1)^n \sum\limits_{m=1}^{\infty} b_m \overline{z}^m$ and $F_n(z) = z - \sum\limits_{m=2}^{\infty} A_m z^m + (-1)^n \sum\limits_{m=1}^{\infty} B_m \overline{z}^m$ we define the convolution of f_n and F_n as

$$\begin{aligned} (f_n * F_n)(z) &= f_n(z) * F_n(z) \\ &= z - \sum_{m=2}^{\infty} a_m A_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m B_m \overline{z}^m. \end{aligned} \tag{2.7}$$

Theorem 2.6. *For $0 \leq \beta \leq \alpha < 1$, let $f_n \in F_{\overline{H}}(n,\lambda,\alpha)$ and $F_n \in F_{\overline{H}}(n,\lambda,\beta)$. Then the convolution $f_n * F_n \in F_{\overline{H}}(n,\lambda,\alpha) \subset F_{\overline{H}}(n,\lambda,\beta)$.*

Proof. For f_n and F_n as in Theorem 2.3, write $f_n(z) = z - \sum\limits_{m=2}^{\infty} a_m z^m + (-1)^n \sum\limits_{m=1}^{\infty} b_m \overline{z}^m$ and $F_n(z) = z - \sum\limits_{m=2}^{\infty} A_m z^m + (-1)^n \sum\limits_{m=1}^{\infty} B_m \overline{z}^m$. Then the convolution $f_n * F_n$ is given by (2.7). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in F_{\overline{H}}(n,\lambda,\alpha)$ we note that $|A_m| \leq 1$ and $|B_m| \leq 1$.
Now for the convolution function $f_n * F_n$ we obtain

$$\begin{aligned} &\sum_{m=2}^{\infty} \frac{m^n|(m-1)\lambda+1|}{1-\beta}a_m A_m + \sum_{m=1}^{\infty} \frac{m^n|(m+1)\lambda-1|}{1-\beta}b_m B_m \\ &\leq \sum_{m=2}^{\infty} \frac{m^n|(m-1)\lambda+1|}{1-\beta}a_m + \sum_{m=1}^{\infty} \frac{m^n|(m+1)\lambda-1|}{1-\beta}b_m \\ &\leq \sum_{m=2}^{\infty} \frac{m^n|(m-1)\lambda+1|}{1-\alpha}a_m + \sum_{m=1}^{\infty} \frac{m^n|(m+1)\lambda-1|}{1-\alpha}b_m \\ &\leq 1. \end{aligned}$$

Since $0 \leq \beta < \alpha < 1$ and $f_n \in F_{\overline{H}}(n,\lambda,\alpha)$. Therefore $f_n * F_n \in F_{\overline{H}}(n,\lambda,\alpha) \subset F_{\overline{H}}(n,\lambda,\beta)$. ∎

Finally we show that the class $F_{\overline{H}}(n, \lambda, \alpha)$ is closed under convex combination of its members.

Theorem 2.7. *The family $F_{\overline{H}}(n, \lambda, \alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2 \ldots$ suppose $f_{n_i} \in F_{\overline{H}}(n, \lambda, \alpha)$ where

$$f_{n_i}(z) = z - \sum_{m=2}^{\infty} a_{i_m} z^m + (-1)^n \sum_{m=1}^{\infty} b_{i_m} z^m.$$

Then by Theorem 2.2,

$$\sum_{m=2}^{\infty} \frac{m^n|(m-1)\lambda+1|}{1-\alpha} a_{i_m} + \sum_{m=1}^{\infty} \frac{m^n|(m+1)\lambda-1|}{1-\alpha} b_{i_m} \leq 1. \tag{2.8}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{m=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i_m} \right) z^m + (-1)^n \sum_{m=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i_m} \right) \overline{z}^m.$$

Then, by (2.8),

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{m^n|(m-1)\lambda+1|}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{i_m} \right) + \sum_{m=1}^{\infty} \frac{m^n|(m+1)\lambda-1|}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i b_{i_m} \right) \\
&= \sum_{i=1}^{\infty} t_i \left(\sum_{m=2}^{\infty} \frac{m^n|(m-1)\lambda+1|}{1-\alpha} a_{i_m} + \sum_{m=1}^{\infty} \frac{m^n|(m+1)\lambda-1|}{1-\alpha} b_{i_m} \right) \\
&\leq \sum_{i=1}^{\infty} t_i = 1 \quad \text{and therefore} \quad \sum_{i=1}^{\infty} t_i f_{n_i} \in F_{\overline{H}}(n, \lambda, \alpha).
\end{aligned}$$

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