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# QUANTITATIVE ESTIMATES FOR POSITIVE LINEAR OPERATORS OBTAINED BY MEANS OF PIECEWISE LINEAR FUNCTIONS

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ABSTRACT. In this paper we obtain estimates for the remainder in approximating continuous functions by positive linear operators, using piecewise linear functions.

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## 1. INTRODUCTION

The most natural way to approximate a continuous function of which data are know at certain discrete points only is to use piecewise linear interpolation. This technique has some most advantages: it is very easy, it requires data at certain discrete points, it interpolates at the given data, it reproduces linear functions. The major disadvantage of this technique is that it does not generate approximating functions which are smooth.

Our approach in order to eliminate this disadvantage, will be to compose certain linear operators with the piecewise linear interpolator.

We obtain estimates for the remainder in approximating of continuous functions by means of positive linear operators, using the second order modulus of smoothness.

## 2. PIECEWISE LINEAR FUNCTIONS ON $[a,b]$

Let $f:[a,b]\to\mathbb{R}$ and $\Delta_n : a=x_0<x_1<\cdots<x_n=b$ be a partition of the interval $[a,b]$. There exists exactly one continuous functions $S_{\Delta_n}f$ on the interval $[a,b]$, whose restriction to each of the intervals $[x_k,x_{k+1}]$, $k=0,1,\dots,n-1$ is a polynomial of degree $\le 1$ and which interpolates the function $f$ at the points $x_k$, i.e. $(S_{\Delta_n}f)(x_k)=f(x_k),\ k=0,1,\dots,n.$

So, on each interval $[x_k,x_{k+1}]$, $k=0,1,\dots,n-1$, $S_{\Delta_n}f$ is the Lagrange polynomial of degree 1

$$(S_{\Delta_n}f)(x)=\frac{x_{k+1}-x}{x_{k+1}-x_k}f(x_k)+\frac{x-x_k}{x_{k+1}-x_k}f(x_{k+1}).$$

Several representations for the operator $S_{\Delta_n}f$ are known. We will use in the sequel the following representation, given by T. Popoviciu [8].

**Theorem 2.1.** *For every function $f$ defined on the points $x_k$, $k=0,1,\dots,n$ there holds:*

$$(S_{\Delta_n}f)(x)=f(x_0)+(x-x_0)[x_0,x_1,f] \tag{2.1}$$

$$+\sum_{k=1}^{n-1}(x_{k+1}-x_{k-1})[x_{k-1},x_k,x_{k+1};f]\left(\frac{x-x_k+|x-x_k|}{2}\right),$$

$$(S_{\Delta_n}f)(x)=\sum_{k=1}^{n}\frac{x_{k+1}-x_{k-1}}{2}[x_{k-1},x_k,x_{k+1};|t-x|]_t f(x_k) \tag{2.2}$$

*where, for mutually distinct $a,b,c$ we denote by $[a,b,c;f(t,x)]_t$ the fact that the divided difference is applied in the variable $t$.*

The operator $S_{\Delta_n}:C[a,b]\to C[a,b]$ is a positive linear operator, which has the following properties:

**Lemma 2.2.** *For $x\in[x_k,x_{k+1}]$, $k=0,1,\dots,n-1$ the following equality holds:*

*i)* $(S_{\Delta_n}f)(x)-f(x)=(x-x_k)(x_{k+1}-x)[x_k,x,x_{k+1};f]$;

*ii)* $S_{\Delta_n}e_i=e_i$, $i=0,1$,

$$(S_{\Delta_n}t^2)(x)=x^2+(x-x_k)(x_{k+1}-x),$$

$$S_{\Delta_n}((t-x)^2;x)=(x-x_k)(x_{k+1}-x);$$

*iii)* $S_{\Delta_n}(|t-x|;x)=2\dfrac{(x-x_k)(x_{k+1}-x)}{x_{k+1}-x_k}$;

*iv)* $S_{\Delta_n}(|t-\lambda|;x)=\dfrac{x_{k+1}-x}{x_{k+1}-x_k}|x_k-\lambda|+\dfrac{x-x_k}{x_{k+1}-x_k}|x_{k+1}-\lambda|.$

*Proof.* i) $(S_{\Delta_n} f)(x) - f(x)$

$$\begin{aligned}
&= \frac{x_{k+1}-x}{x_{k+1}-x_k} f(x_k) + \frac{x-x_k}{x_{k+1}-x_k} f(x_{k+1}) - f(x) \\
&= (x-x_k)(x_{k+1}-x)\left(\frac{f(x_k)}{(x_k-x)(x_k-x_{k+1})}+\right. \\
&\left. + \frac{f(x_{k+1})}{(x_{k+1}-x)(x_{k+1}-x_k)} + \frac{f(x)}{(x-x_k)(x-x_{k+1})}\right) \\
&= (x-x_k)(x_{k+1}-x)[x_k, x, x_{k+1}; f].
\end{aligned}$$

ii) By i) we obtain for $f = e_0, e_1, e_2$:

$$S_{\Delta_n} e_0 = e_0, \quad S_{\Delta_n} e_1 = e_1,$$
$$(S_{\Delta_n} e_2)(x) = x^2 + (x-x_k)(x_{k+1}-x)$$

and evidently

$$S_{\Delta_n}((t-x)^2; x) = (x-x_k)(x_{k+1}-x).$$

iii) Replacing in i) $f(t) = |t-x|$ we obtain

$$\begin{aligned}
S_{\Delta_n}(|t-x|; x) &= (x-x_k)(x_{k+1}-x)[x_k, x, x_{k+1}; |t-x|]_t \\
&= 2\frac{(x-x_k)(x_{k+1}-x)}{x_{k+1}-x_k}.
\end{aligned}$$

iv) It is obtained by i) for $f(t) = |t-\lambda|$. ∎

We will need in the sequel the following result given by H. Burkhill [1] and H. Whitney [9].

**Lemma 2.3.** *Let $f : [a,b] \to \mathbb{R}$ and denote by $L_1(f; a, b)$ the Lagrange polynomial of degree 1 interpolating the function $f$ at a and $b$.*

*Then for all $x \in [a,b]$, one has*

$$|f(x) - L_1(f; a, b)| \le \omega_2\left(f; \frac{b-a}{2}\right).$$

By Lemma 2.3 we obtain for $x \in [x_k, x_{k+1}]$ the following estimate

$$|(S_{\Delta_n} f)(x) - f(x)| \le \omega_2\left(f; \frac{x_{k+1}-x_k}{2}\right). \tag{2.3}$$

Using Lemma 2.2.i) and formula (2.3), we can give an upper bound for the absolute value of the divided difference of function $f$ on three distinct points in terms of the modulus of smoothness

$$|[x_k, x, x_{k+1}; f]| \le \frac{1}{(x-x_k)(x_{k+1}-x)}\omega_2\left(f; \frac{x_{k+1}-x_k}{2}\right). \tag{2.4}$$

We will use this result under the following form.

For $f \in C[a,b]$ and $a \le x_{k-1} < x_k < x_{k+1} \le b$ we have

$$\begin{aligned}
&|[x_{k-1}, x_k, x_{k+1}; f]| \\
&\le \frac{1}{(x_k-x_{k-1})(x_{k+1}-x_k)}\omega_2\left(f; \frac{x_{k+1}-x_{k-1}}{2}\right).
\end{aligned} \tag{2.5}$$

Let $L_n : C[a,b] \to C[a,b]$ be a positive linear operator of the form:

$$(L_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(x_k), \text{ with } L_n e_0 = e_0,\ L_n e_1 = e_1, \tag{2.6}$$

where $p_{n,k}$ are continuous functions defined on $[a,b]$, $p_{n,k}(x) \geq 0$, $\forall\ x \in [a,b]$, $x_k$, $k = 0,1,\ldots,n$ are the nodes of $\Delta_n$ and $e_i(x) := x^i$, $i \in \mathbb{N}$.

We denote by $\mathcal{L}_{\Delta_n}$ the set of discretely defined operators $L_n : C[a,b] \to C[a,b]$ satisfying (2.6).

**Lemma 2.4.** *The operators $L_n \in \mathcal{L}_{\Delta_n}$ have the following discretely representation:*

$$(L_n f)(x) = f(x_0) + (x - x_0)[x_0, x_1; f] \tag{2.7}$$

$$+\frac{1}{2}\sum_{k=1}^{n-1}(x_{k+1} - x_{k-1})[x_{k-1}, x_k, x_{k+1}; f](x - x_k + L_n(|t - x_k|; x))$$

$$(L_n f)(x) = \sum_{k=0}^{n} \frac{x_{k+1} - x_{k-1}}{2}[x_{k-1}, x_k, x_{k+1}; L_n(|u - t|; x)]_u f(x_k). \tag{2.8}$$

*Proof.* For every $f : [a,b] \to \mathbb{R}$ we have

$$(S_{\Delta_n} f)(x_k) = f(x_k)$$

and

$$L_n(S_{\Delta_n} f)(x) = \sum_{k=0}^{n}(S_{\Delta_n} f)(x_k) p_{n,k}(x)$$

$$= \sum_{k=0}^{n} f(x_k) p_{n,k}(x) = (L_n f)(x).$$

We apply the operator $L_n$ to the identities (2.1) and (2.2) and we obtain the results. ∎

## 3. QUANTITATIVE ESTIMATES ON $[a,b]$

Making use of Theorem 2.1 and formula (2.4), we can give estimates for the remainder in approximating a continuous functions $f$ by operator $L_n \in \mathcal{L}_{\Delta_n}$.

**Theorem 3.1.** *Let $L_n \in \mathcal{L}_{\Delta_n}$ and $f \in C[a,b]$. Then for every $x \in (x_k, x_{k+1})$ there holds:*

$$|(L_n f)(x) - f(x)|$$

$$\leq \left(1 + \frac{L_n(t-x)^2 - (x - x_k)(x_{k+1} - x)}{\rho^2}\right) \omega_2(f; \|\Delta_n\|),$$

*where $\rho = \min(x_{k+1} - x_k)$ and $\Delta_n = \max(x_{k+1} - x_k)$, $k = 0, 1, \ldots, n-1$.*

*Proof.* From Theorem 2.1(i) we have

$$(S_{\Delta_n} f)(t) - (S_{\Delta_n} f)(x) = [x_0, x_1; f](t - x) +$$

$$+\sum_{k=1}^{n-1} \frac{x_{k+1} - x_{k-1}}{2}[x_{k-1}, x_k, x_{k+1}; f](|t - x_k| - |x - x_k| + (t - x)).$$

Applying the operator $L_n$ on the variable $t$ we obtain

$$(L_n f)(x) - (S_{\Delta_n} f)(x) \tag{3.1}$$

$$= \sum_{k=1}^{n-1} \frac{x_{k+1} - x_{k-1}}{2}[x_{k-1}, x_k, x_{k+1}; f](L_n(|t - x_k|; x) - |x - x_k|).$$

Replacing in (3.1) $f(t) = (t - x)^2$ we obtain

$$L_n((t-x)^2; x) - S_{\Delta_n}((t-x)^2; x) = \tag{3.2}$$

$$= \sum_{k=1}^{n-1} \frac{x_{k+1} - x_k}{2} (L_n(|t - x_k|; x) - |x - x_k|) \geq 0.$$

By (2.5) we obtain

$$|[x_{k-1}, x_k, x_{k+1}; f]| \leq \frac{\omega_2(f; \|\Delta_n\|)}{(x_{k+1} - x_k)(x_k - x_{k-1})} \leq \frac{1}{\rho^2} \omega_2(f; \|\Delta_n\|). \tag{3.3}$$

By (3.1), (3.2) and (3.3) it result

$$|(L_n f)(x) - (S_{\Delta_n} f)(x)| \leq \tag{3.4}$$

$$\leq \frac{1}{\rho^2} \omega_2(f; \|\Delta_n\|) \sum_{k=1}^{n-1} \frac{x_{k+1} - x_{k-1}}{2} (|L_n(|t - x_k|; x) - |x - x_k|) =$$

$$= \frac{1}{\rho^2} \omega_2(f; \|\Delta_n\|)(L_n((t - x)^2; x) - S_{\Delta_n}((t - x)^2; x)).$$

By (2.5) we have

$$|(S_{\Delta_n} f)(x) - f(x)| \leq \omega_2 \left( f; \frac{\|\Delta_n\|}{2} \right) \leq \omega_2(f; \|\Delta_n\|). \tag{3.5}$$

By (3.4), (3.5) and Lemma 2.2 (ii) we obtain

$$|(L_n f)(x) - f(x)| \leq |(L_n f)(x) - (S_{\Delta_n} f)(x)| + |(S_{\Delta_n} f)(x) - f(x)| \leq$$

$$\leq \left( 1 + \frac{L_n(t - x)^2 - (x - x_k)(x_{k+1} - x)}{\rho^2} \right) \omega_2(f; \|\Delta_n\|).$$

∎

We can observe that during the proof it appears the estimate of the remainder between $S_{\Delta_n}$ and $f$. This estimate can be improved for function $f \in C^{(1)}[a, b]$.

**Theorem 3.2.** *If* $f \in C^{(1)}[a, b]$ *and* $x \in [x_k, x_{k+1}]$ *then*

$$|(S_{\Delta_n} f)(x) - f(x)| \leq \frac{(x - x_k)(x_{k+1} - x)}{x_{k+1} - x_k} \omega(f', x_{k+1} - x_k).$$

*Proof.* For $x \in [x_k, x_{k+1}]$ we have

$$(S_{\Delta_n} f)(x) - f(x) = \frac{x_{k+1} - x}{x_{k+1} - x_k} f(x_k)$$

$$+ \frac{x - x_k}{x_{k+1} - x_k} f(x_{k+1}) - f(x)$$

$$= \frac{x_{k+1} - x}{x_{k+1} - x_k} (f(x_k) - f(x)) + \frac{x - x_k}{x_{k+1} - x_k} (f(x_{k+1}) - f(x))$$

$$= \frac{x_{k+1} - x}{x_{k+1} - x_k} (x_k - x) f'(\xi_1) + \frac{x - x_k}{x_{k+1} - x_k} (x_{k+1} - x) f'(\xi_2)$$

$$= \frac{(x - x_k)(x_{k+1} - x)}{x_{k+1} - x_k} (f'(\xi_2) - f'(\xi_1))$$

with $\xi_1 \in (x_k, x)$ and $\xi_2 \in (x, x_{k+1})$, hence $\xi_2 - \xi_1 \leq x_{k+1} - x_k$. Therefore

$$|(S_{\Delta_n} f)(x) - f(x)| \leq \frac{(x - x_k)(x_{k+1} - x)}{x_{k+1} - x_k} \omega(f', x_{k+1} - x_k).$$

∎

**Corollary 3.3.** *For $f \in C^{(1)}[a,b]$ the following inequalities hold:*

$$|(S_{\Delta_n} f)(x) - f(x)| \leq \frac{\|\Delta_n\|}{4}\omega(f', \|\Delta_n\|),\ x \in [a,b]; \tag{3.6}$$

$$|[x_1, x_2, x_3; f]| \leq \frac{\omega(f', x_3 - x_1)}{x_3 - x_1}, \tag{3.7}$$

*where $a \leq x_1 < x_2 < x_3 \leq b$.*

**Theorem 3.4.** *Let $L_n \in \mathcal{L}_{\Delta_n}$ and $f \in C^{(1)}[a,b]$. Then for every $x \in [x_k, x_{k+1}]$ there holds*

$$|(L_n f)(x) - f(x)|$$
$$\leq \left(\frac{1}{4} + \frac{L_n((t-x)^2;x) - (x_{k+1}-x)(x-x_k)}{\rho^2}\right)\|\Delta_n\|\omega(f', \|\Delta_n\|).$$

*Proof.* It is obtained by Theorem 3.2, Corollary 3.3 and using the inequality

$$\omega_2(f,h) \leq h\omega(f',h).$$

∎

## 4. QUANTITATIVE ESTIMATES FOR EQUIDISTANT NODES ON $[0,1]$

The operator $S_{\Delta_n} f$ relative to the interval $[0,1]$ and the nodes of $\Delta_n : x_k = k/n$, $k = 0,1,\dots,n$ will be denoted in the sequel by $S_n$.

The operator $S_n : C[0,1] \to C[0,1]$ can be written as:

$$(S_n f)(x) = \frac{1}{n}\sum_{k=0}^{n}\left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |t-x|\right]_t f\left(\frac{k}{n}\right). \tag{4.1}$$

This operator has the following properties [3], which can be deduced by Lemma 2.2.

**Corollary 4.1.** *i) $S_n : C[0,1] \to C[0,1]$ is positive and linear.*
*ii) $(S_n f)(k/n) = f(k/n),\ k = 0,1,\dots,n$.*
*iii) $S_n e_i = e_i,\ i = 0,1$.*
*iv) $S_n(|t-x|;x) = 2\dfrac{\{nx\}(1-\{nx\})}{n}$*

$$S_n((t-x)^2;x) = \frac{\{nx\}(1-\{nx\})}{n^2},$$

*where $\{nx\} = nx - [nx]$.*

We denote by $\mathcal{L}_n$ the set of discretely defined operators $L_n : C[0,1] \to C[0,1]$ satisfying (2.6) and the nodes of $\Delta_n$ are $x_k = k/n$, $k = 0,1,\dots,n$.

For representation of the operators $L_n \in \mathcal{L}_n$ see [4], [5].

As an application of Theorem 3.1 we can given estimates for the remainder in approximation of function $f \in C[0,1]$ by means of positive linear operators $L_n \in \mathcal{L}_n$ (see also [2]).

**Corollary 4.2.** *Let $L_n \in \mathcal{L}_n$ and $f \in C[0,1]$. Then for every $x \in [0,1]$ there holds*

$$|(L_n f)(x) - f(x)|$$
$$\leq \left(1 + n^2\frac{L_n((t-x)^2;x)}{2} - \frac{\{nx\}(1-\{nx\})}{2}\right)\omega_2\left(f, \frac{1}{n}\right).$$

*Proof.* It is obtained by Theorem 3.1, for $\rho = 1/n$ and
$$\left|\left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; f\right]\right| \leq \frac{n^2}{2}\omega_2\left(f; \frac{1}{n}\right).$$
∎

For $f \in C^{(1)}[0,1]$, using Theorem 3.4 we can obtain the following results.

**Corollary 4.3.** *Let $L_n \in \mathcal{L}_n$ and $f \in C^{(1)}[0,1]$. Then for every $x \in [0,1]$ there holds*
$$|(L_n f)(x) - f(x)| \leq \frac{1}{n}\left(\frac{1}{8} + n^2\frac{L_n((t-x)^2;x)}{2}\right)\omega\left(f', \frac{1}{n}\right).$$

*Proof.*
$$|(L_n f)(x) - f(x)| \leq \frac{\{nx\}(1-\{nx\})}{n}\omega\left(f', \frac{1}{n}\right)$$
$$+\left(n^2\frac{L_n((t-x)^2;x)}{2} - \frac{\{nx\}(1-\{nx\})}{2}\right)\omega_2(f,h).$$
Using inequality $\omega_2(f,h) \leq h\omega(f',h)$ we obtain
$$|(L_n f)(x) - f(x)| \leq \frac{\{nx\}(1-\{nx\})}{n}\omega\left(f', \frac{1}{n}\right)$$
$$+\frac{1}{n}\left(n^2\frac{L_n((t-x)^2;x)}{2} - \frac{\{nx\}(1-\{nx\})}{2}\right)\omega\left(f', \frac{1}{n}\right)$$
$$\leq \frac{1}{n}\left(\frac{1}{8} + n^2\frac{L_n((t-x)^2;x)}{2}\right)\omega\left(f', \frac{1}{n}\right).$$
∎

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