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ON THE ULAM STABILITY FOR EULER-LAGRANGE TYPE QUADRATIC FUNCTIONAL EQUATIONS

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ABSTRACT. In 1940 (and 1968) S. M. Ulam proposed the well-known Ulam stability problem. In 1941 D.H. Hyers solved the Hyers-Ulam problem for linear mappings. In 1951 D. G. Bourgin has been the second author treating the Ulam problem for additive mappings. In 1978 according to P.M. Gruber this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982-2004 we established the Hyers-Ulam stability for the Ulam problem for different mappings. In 1992-2000 J.M. Rassias investigated the Ulam stability for Euler-Lagrange mappings. In this article we solve the Ulam problem for Euler-Lagrange type quadratic functional equations. These stability results can be applied in mathematical statistics, stochastic analysis, algebra, geometry, as well as in psychology and sociology.

Key words and phrases: Ulam stability, Euler-Lagrange type mapping, Quadratic equation.

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1. INTRODUCTION

In 1940 (and 1968) S. M. Ulam [28] proposed *the Ulam stability problem*:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?"

In particular he stated *the stability question*:
"Let G_1 be a group and G_2 a metric group with the metric$\rho(\cdot,\cdot)$ Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x) f(y)) < c$ for all $x, y \in G_1$, then a unique homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941 D.H. Hyers [13] solved this problem for linear mappings. In 1951, D.G. Bourgin [3] was the second author to treat the Ulam problem for additive mappings. In 1978, according to P.M. Gruber [12], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1980 and in 1987, I. Fenyö [7], [8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Z. Gajda and R. Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors: J. Aczél [1], C. Borelli and G.L. Forti [2], [9], P.W. Cholewa [4], St. Czerwik [5], and H. Drljevic [6], and Pl. Kannappan [15]. In 1982-2004, J.M. Rassias [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] solved the above Ulam problem for different mappings. In 1999, P. Găvruţă [11] answered a question of J.M. Rassias [18] concerning the stability of the Cauchy equation. In 1998, S.-M. Jung [14] and in 2002-2003 we [25, 26] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In 1992-2000 the second author ([19, 21, 22, 23, 24]) investigated the Ulam stability for Euler-Lagrange mappings.

In this article we solve the Ulam stability problem for Euler-Lagrange type quadratic functional equations.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations.

Let us introduce the Euler-Lagrange type quadratic functional equation

$$\begin{aligned} &Q(m_1a_1x_1 + m_2a_2x_2) + m_1m_2Q(a_2x_1 - a_1x_2) \\ = &\left(m_1a_1^2 + m_2a_2^2\right)[m_1Q(x_1) + m_2Q(x_2)] \end{aligned} \tag{1.1}$$

with mappings $Q : X \to Y$, for all $x_1, x_2 \in X$, and any fixed pair (a_1, a_2) of reals $a_i \neq 0$ and any fixed pair (m_1, m_2) of positive reals $m_i (i = 1, 2) : 0 < m = \frac{m_1+m_2}{m_1m_2+1}\left(m_1a_1^2 + m_2a_2^2\right) \neq 1$.

Definition 1.1. A mapping $Q : X \to Y$ is called *Euler-Lagrange type quadratic*, if the above-mentioned functional equation (1.1) holds for every $(x_1, x_2, \ldots, x_p) \in X^p$ with an arbitrary but fixed $p = 2, 3, 4, \ldots$

In this paper, we establish an approximation of approximately Euler-Lagrange type quadratic mappings $f : X \to Y$ by Euler-Lagrange type quadratic mappings $Q : X \to Y$, such that the corresponding functional inequality

$$\begin{aligned} &\|f(m_1a_1x_1 + m_2a_2x_2) + m_1m_2f(a_2x_1 - a_1x_2) \\ - &\left(m_1a_1^2 + m_2a_2^2\right)[m_1f(x_1) + m_2f(x_2)]\| \\ \leq &\ c \end{aligned} \tag{1.2}$$

holds with a constant $c \geq 0$ (independent of $x_1, x_2 \in X$).

It is useful for the following, to observe that, from (1.1) with $x_1 = x_2 = 0$, and $0 < m \neq 1$, we get $(m_1 m_2 + 1)\,|1-m|\,Q(0) = 0$, or

$$Q(0) = 0. \tag{1.3}$$

Similarly, from (1.2), one finds $(m_1 m_2 + 1)\,|1-m|\,\|f(0)\| \leq c$, or

$$\|f(0)\| \leq \frac{c}{(m_1 m_2 + 1)\,|1-m|} = \frac{c}{m_1 m_2 + 1} \begin{cases} \frac{1}{m-1}, \text{ if } m > 1 \\ \frac{1}{1-m}, \text{ if } 0 < m < 1. \end{cases} \tag{1.4}$$

Let us denote

$$\overline{Q}(x) = m_0 \begin{cases} \frac{m_1 Q\left(\frac{a_1}{m_0}x\right) + m_2 Q\left(\frac{a_2}{m_0}x\right)}{m}, \text{ if } m > 1 \\ m\left[m_1 Q\left(\frac{b_1}{m_0}x\right) + m_2 Q\left(\frac{b_2}{m_0}x\right)\right], \text{ if } 0 < m < 1, \end{cases} \tag{1.5}$$

and $b_i = \frac{a_i}{m}(i = 1, 2)$ and $m_0 = \frac{(m_1+m_2)}{(m_1 m_2+1)}$, as well as

$$\overline{\overline{Q}}(x) = \frac{1}{m_0 m_1} \begin{cases} \frac{Q(m_1 a_1 x) + m_1 m_2 Q(a_2 x)}{m}, \text{ if } m > 1 \\ m\left[Q(m_1 b_1 x) + m_1 m_2 Q(b_2 x)\right], \text{ if } 0 < m < 1, \end{cases} \tag{1.6}$$

for all $x \in X$.

Definition 1.2. Let X and Y be real linear spaces and $m > 1$. Then the following equation

$$F^a(Q) = Q(m_1 a_1 x) + m_1 m_2 Q(a_2 x) - m_0^2 m_1 \left[m_1 Q\left(\frac{a_1}{m_0}x\right) + m_2 Q\left(\frac{a_2}{m_0}x\right)\right] = 0, \tag{1.7}$$

is called *fundamental functional equation of first type*. This (1.7) is equivalent to

$$\left(M^a(Q) =\right) \frac{F^a(Q)}{m_0 m_1 m} = \overline{\overline{Q}}(x) - \overline{Q}(x) = 0,\ m > 1. \tag{1.8}$$

Note that if X and Y are real normed linear spaces and $m > 1$, then

$$\|F^a(f)\| \leq \varepsilon_1, \tag{1.9}$$

with a constant $\varepsilon_1 \geq 0$ (independent of $x_1, x_2 \in X$). This inequality (1.9) is equivalent to

$$\left(\|M_a(f)\| =\right) \frac{1}{m_0 m_1 m}\|F_a(f)\| = \left\|\overline{\overline{f}}(x) - \overline{f}(x)\right\| \leq \frac{\varepsilon_1}{m_0 m_1 m},\ m > 1. \tag{1.10}$$

Definition 1.3. Let X and Y be real linear spaces, $b_i = \frac{a_i}{m}$ $(i = 1, 2)$ and $0 < m < 1$. Then

$$F^b(Q) = Q(m_1 b_1 x) + m_1 m_2 Q(b_2 x) - m_0^2 m_1 \left[m_1 Q\left(\frac{b_1}{m_0}x\right) + m_2 Q\left(\frac{b_2}{m_0}x\right)\right] = 0 \tag{1.11}$$

is called *fundamental functional equation of second type*. This (1.11) is equivalent to

$$\left(M^b(Q) =\right) \frac{m}{m_0 m_1} F^b(Q) = \overline{\overline{Q}}(x) - \overline{Q}(x) = 0,\ 0 < m < 1. \tag{1.12}$$

Note that if X and Y are real normed linear spaces and $0 < m < 1$, then

$$\left\| F^b(f) \right\| \leq \varepsilon_2, \tag{1.13}$$

with a constant $\varepsilon_2 \geq 0$ (independent of $x_1, x_2 \in X$). This inequality (1.13) is equivalent to

$$\left(\left\| M^b(f) \right\| = \right) \frac{m}{m_0 m_1} \left\| F^b(f) \right\| = \left\| \overline{\overline{f}}(x) - \overline{f}(x) \right\| \leq \frac{m}{m_0 m_1} \varepsilon_2, \, 0 < m < 1. \tag{1.14}$$

Now, claim that for $n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\} = \{0, 1, 2, ...\}$

$$Q(x) = \begin{cases} m^{-2n} Q(m^n x), \text{ if } m > 1 \\ \\ m^{2n} Q(m^{-n} x), \text{ if } 0 < m < 1 \end{cases} \tag{1.15}$$

holds for all $x \in X$.

Let us consider first <u>*the case* $m > 1$.</u> For $n = 0$, it is trivial. From (1.3), (1.5) and (1.1), with $x_i = \frac{a_i}{m_0} x \ \ (i = 1, 2)$, we obtain

$$Q(mx) = m m_0 \left[m_1 Q\left(\frac{a_1}{m_0} x \right) + m_2 Q\left(\frac{a_2}{m_0} x \right) \right],$$

or

$$\overline{Q}(x) = m^{-2} Q(mx). \tag{1.16}$$

Besides from (1.3), (1.6) and (1.1), with $x_1 = x, x_2 = 0$, one gets

$$Q(m_1 a_1 x) + m_1 m_2 Q(a_2 x) = m m_0 m_1 Q(x),$$

or

$$\overline{\overline{Q}}(x) = Q(x). \tag{1.17}$$

Therefore from (1.8), (1.16) and (1.17) we have

$$Q(x) = m^{-2} Q(mx), \tag{1.18}$$

which is (1.15) for $n = 1$, and $m > 1$. Assume (1.15), $m > 1$, is true and from (1.18), with $m^n x$ on place of x, we get :

$$Q\left(m^{n+1} x \right) = m^2 Q(m^n x) = m^2 (m^n)^2 Q(x) = \left(m^{n+1} \right)^2 Q(x). \tag{1.19}$$

This formula (1.19), by induction, proves formula (1.15) for $m > 1$.

Let us consider now <u>*the case* $0 < m < 1$.</u> Similarly from (1.3), (1.5) and (1.1), with $x_i = \frac{a_i}{m_0} \frac{x}{m} \ (i = 1, 2)$, we obtain

$$Q(x) = m m_0 \left[m_1 Q\left(\frac{a_1}{m_0} \frac{x}{m} \right) + m_2 Q\left(\frac{a_2}{m_0} \frac{x}{m} \right) \right],$$

or

$$Q(x) = \overline{Q}(x). \tag{1.20}$$

Besides from (1.3), (1.6) and (1.1), with $x_1 = \frac{x}{m}, x_2 = 0$, one gets

$$Q\left(\frac{m_1 a_1}{m} x \right) + m_1 m_2 Q\left(\frac{a_2}{m} x \right) = m m_0 m_1 Q\left(\frac{x}{m} \right),$$

or

$$\overline{\overline{Q}}(x) = m^2 Q\left(m^{-1} x \right). \tag{1.21}$$

Therefore from (1.12), (1.20) and (1.21) we have

$$Q(x) = m^2 Q\left(m^{-1}x\right), \tag{1.22}$$

which is (1.15) for $n = 1$, and $0 < m < 1$.

Assume (1.15), $0 < m < 1$, is true and from (1.22), with $m^{-n}x$ on place of x, we get :

$$Q\left(m^{-(n+1)}x\right) = m^{-2}Q\left(m^{-n}x\right) = m^{-2}\left(m^{-n}\right)^2 Q(x) = \left(m^{-(n+1)}\right)^2 Q(x). \tag{1.23}$$

This formula (1.23), by induction, proves formula (1.15) for $0 < m < 1$.

2. ULAM STABILITY FOR EULER-LAGRANGE TYPE QUADRATIC FUNCTIONAL EQUATIONS

Theorem 2.1. *Let X and Y be real normed linear spaces. Assume that Y is complete. Take $0 < m = \frac{m_1+m_2}{m_1m_2+1}\left(m_1a_1^2 + m_2a_2^2\right) \neq 1$ for any fixed non-zero reals a_i and positive reals m_i $(i = 1, 2)$. Assume in addition that mappings $Q : X \to Y$ and $f : X \to Y$ satisfy the Euler-Lagrange type functional equation (1.1) and inequality (1.2), respectively, and conditions*

$$\overline{\overline{Q}}(x) = \overline{Q}(x), \tag{2.1}$$

and

$$\left\|\overline{\overline{f}}(x) - \overline{f}(x)\right\| \leq \frac{1}{m_0m_1}\begin{cases} \frac{1}{m}\varepsilon_1, \text{ if } m > 1 \\ \\ m\varepsilon_2, \text{ if } 0 < m < 1, \end{cases} \tag{2.2}$$

with constants $\varepsilon_1, \varepsilon_2 \geq 0$ (independent of $x_1, x_2 \in X$), and a positive constant $m_0 = \frac{m_1m_2+1}{m_1+m_2}$. Define

$$f_n(x) = \begin{cases} m^{-2n}f\left(m^n x\right), \text{ if } m > 1 \\ \\ m^{2n}f\left(m^{-n}x\right), \text{ if } 0 < m < 1, \end{cases} \tag{2.3}$$

for all $x \in X$ and $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. Then the limit

$$Q(x) = \lim_{n\to\infty} f_n(x) \tag{2.4}$$

exists for all $x \in X$ and $Q : X \to Y$ is the unique Euler-Lagrange type quadratic mapping, such that

$$\begin{aligned} &\|f(x) - Q(x)\| \\ \leq\; &\frac{\left\{(m + m_0m_1)(m_1m_2 + 1)|m - 1| + m_0m_2|m^2 - m_1^2|\right\}c + m(m_1m_2 + 1)|m - 1|\varepsilon}{m_0m_1(m_1m_2 + 1)(m - 1)^2(m + 1)} \\ =\; &\delta \end{aligned} \tag{2.5}$$

holds for all $x \in X$, with non-negative constants c and

$$\varepsilon = \begin{cases} \varepsilon_1, \text{ if } m > 1 \\ \\ \varepsilon_2, \text{ if } 0 < m < 1 \end{cases} \quad \text{independent of } x \in X. \tag{2.6}$$

Proof. Now claim for $n \in \mathbb{N}_0$ that inequality

$$\|f(x) - f_n(x)\| \leq \begin{cases} \delta_1\left(1 - m^{-2n}\right), \text{ if } m > 1 \\ \\ \delta_2\left(1 - m^{2n}\right), \text{ if } 0 < m < 1, \end{cases} \tag{2.7}$$

holds for all $x \in X$, where $\delta_1 = \delta$ for $m > 1$ and $\delta_2 = \delta$ for $0 < m < 1$:

$$\delta_1 = \frac{\left\{(m + m_0 m_1)(m_1 m_2 + 1)(m-1) + m_0 m_2 |m^2 - m_1^2|\right\} c + m (m_1 m_2 + 1)(m-1) \varepsilon_1}{m_0 m_1 (m_1 m_2 + 1)(m-1)^2 (m+1)}$$

(2.8)

$$\delta_2 = \frac{\left\{(m + m_0 m_1)(m_1 m_2 + 1)(1-m) + m_0 m_2 |m_1^2 - m^2|\right\} c + m (m_1 m_2 + 1)(1-m) \varepsilon_2}{m_0 m_1 (m_1 m_2 + 1)(1-m)^2 (m+1)}$$

are two non-negative constants independent of $x \in X$. For $n = 0$, it is trivial.
Assume $m > 1$. From (1.2), with $x_i = \frac{a_i}{m_0} x$, $(i = 1, 2)$, we obtain

$$\left\| f(mx) + m_1 m_2 f(0) - m m_0 \left[m_1 f(\frac{a_1}{m_0} x) + m_2 f(\frac{a_2}{m_0} x) \right] \right\| \leq c,$$

or

$$\left\| \overline{f}(x) - m^{-2} f(mx) - \frac{m_1 m_2}{m^2} f(0) \right\| \leq \frac{c}{m^2}, \tag{2.9}$$

where $\overline{f}(x) = \frac{m_0}{m} \left[m_1 f(\frac{a_1}{m_0} x) + m_2 f(\frac{a_2}{m_0} x) \right]$. Besides, from (1.2), with $x_1 = x, x_2 = 0$, we get $\| f(m_1 a_1 x) + m_1 m_2 f(a_2 x) - m m_0 [m_1 f(x) + m_2 f(0)] \| \leq c$, or

$$\left\| f(x) - \overline{\overline{f}}(x) + \frac{m_2}{m_1} f(0) \right\| \leq \frac{c}{m_0 m_1 m}, \tag{2.10}$$

where $\overline{\overline{f}}(x) = \frac{1}{m_0 m_1 m} [f(m_1 a_1 x) + m_1 m_2 f(a_2 x)]$. Therefore from (1.4), (2.2), (2.9), (2.10) and triangle inequality we have

$$\begin{aligned} &\left\| f(x) - m^{-2} f(mx) + \frac{m_2 (m^2 - m_1^2)}{m_1 m^2} f(0) \right\| \\ \leq\ & \left\| f(x) - \overline{\overline{f}}(x) + \frac{m_2}{m_1} f(0) \right\| + \left\| \overline{\overline{f}}(x) - \overline{f}(x) \right\| + \left\| \overline{f}(x) - m^{-2} f(mx) - \frac{m_1 m_2}{m^2} f(0) \right\| \\ \leq\ & \frac{c}{m_0 m_1 m} + \frac{\varepsilon_1}{m_0 m_1 m} + \frac{c}{m^2} = \frac{(m + m_0 m_1) c + m \varepsilon_1}{m_0 m_1 m^2}, \end{aligned}$$

or

$$\left\| f(x) - m^{-2} f(mx) \right\| \leq \delta_1 \left(1 - m^{-2}\right). \tag{2.11}$$

Assume $0 < m < 1$. Similarly, from (1.2), with $x_i = \frac{a_i}{m_0} \frac{x}{m} = \frac{b_i}{m_0} x$, $b_i = \frac{a_i}{m}$ $(i = 1, 2)$, we obtain $\left\| f(x) + m_1 m_2 f(0) - m m_0 \left[m_1 f(\frac{b_1}{m_0} x) + m_2 f(\frac{b_2}{m_0} x) \right] \right\| \leq c$, or

$$\left\| f(x) - \overline{f}(x) + m_1 m_2 f(0) \right\| \leq c, \tag{2.12}$$

where $\overline{f}(x) = m_0 m \left[m_1 f(\frac{b_1}{m_0} x) + m_2 f(\frac{b_2}{m_0} x) \right]$. Besides, from (1.2), with $x_1 = \frac{x}{m}$, $x_2 = 0$, we get $\left\| f(m_1 b_1 x) + m_1 m_2 f(b_2 x) - m m_0 \left[m_1 f(\frac{x}{m}) + m_2 f(0) \right] \right\| \leq c$, or

$$\left\| \overline{\overline{f}}(x) - m^2 f\left(m^{-1} x\right) - \frac{m_2}{m_1} m^2 f(0) \right\| \leq \frac{m}{m_0 m_1} c, \tag{2.13}$$

where $\overline{\overline{f}}(x) = \frac{m}{m_0 m_1}\left[f(m_1 b_1 x) + m_1 m_2 f(b_2 x)\right]$. Therefore from (1.4), (2.2), (2.12), (2.13) and triangle inequality we have

$$\begin{aligned} & \left\| f(x) - m^2 f\left(m^{-1}x\right) + \frac{m_2\left(m_1^2 - m^2\right)}{m_1} f(0) \right\| \\ \leq & \left\| f(x) - \overline{f}(x) + m_1 m_2 f(0) \right\| + \left\| \overline{f}(x) - \overline{\overline{f}}(x) \right\| + \left\| \overline{\overline{f}}(x) - m^2 f\left(m^{-1}x\right) - \frac{m_2}{m_1} m^2 f(0) \right\| \\ \leq & \frac{(m + m_0 m_1)c + m\varepsilon_2}{m_0 m_1}, \end{aligned}$$

or

$$\left\| f(x) - m^2 f\left(m^{-1}x\right) \right\| \leq \delta_2\left(1 - m^2\right). \tag{2.14}$$

Assume $m > 1$. From (2.11), with $m^i x$ $(i = 1, 2, \ldots, n)$, on place of x, and the triangle inequality, we have, *without induction*

$$\begin{aligned} & \left\| f(x) - f_n(x) \right\| = \left\| f(x) - m^{-2n} f(m^n x) \right\| \\ \leq & \left\| f(x) - m^{-2} f(mx) \right\| + m^{-2} \left\| f(mx) - m^{-2} f\left(m^2 x\right) \right\| \\ + & \ldots + m^{-2(n-1)} \left\| f\left(m^{n-1}x\right) - m^{-2} f(m^n x) \right\| \\ \leq & \delta_1\left(1 + m^{-2} + \ldots + m^{-2(n-1)}\right)\left(1 - m^{-2}\right) = \delta_1\left(1 - m^{-2n}\right). \end{aligned}$$

Similarly if we assume $0 < m < 1$, we have from (2.14) that

$$\begin{aligned} & \left\| f(x) - f_n(x) \right\| = \left\| f(x) - m^{2n} f\left(m^{-n}x\right) \right\| \\ \leq & \delta_2\left(1 + m^2 + \ldots + m^{2(n-1)}\right)\left(1 - m^2\right) = \delta_2\left(1 - m^{2n}\right). \end{aligned}$$

Therefore we prove inequality (2.7).

Claim now that the sequence $\{f_n(x)\}$ *converges*. To do this it suffices to prove that it is a Cauchy sequence. Inequality (2.7) is involved.

In fact, if $m > 1$ and $i > j > 0$, and $h_1 = m^j x$, we have

$$\begin{aligned} & \left\| f_i(x) - f_j(x) \right\| \\ = & m^{-2j} \left\| m^{-2(i-j)} f\left(m^{i-j} h_1\right) - f(h_1) \right\| \\ \leq & \delta_1 m^{-2j}\left(1 - m^{-2(i-j)}\right) \\ = & \delta_1\left(m^{-2j} - m^{-2i}\right) \\ < & \delta_1 m^{-2j} \underset{j\to\infty}{\longrightarrow} 0. \end{aligned}$$

Similarly, if $0 < m < 1$, and $h_2 = m^{-j}x$, we have :

$$\left\| f_i(x) - f_j(x) \right\| = m^{2j} \left\| m^{2(i-j)} f(m^{-(i-j)} h_2) - f(h_2) \right\| < \delta_2 m^{2j} \underset{j\to\infty}{\longrightarrow} 0.$$

Thus we can *define* a mapping $Q : X \to Y$, by (2.4).

Claim that from (1.2) and (2.4) we can get (1.1), or equivalently that the afore-mentioned well-defined mapping $Q : X \to Y$ is *Euler-Lagrange type quadratic*. In fact, it is clear from the functional inequality (1.2) and the limit (2.4) with $m > 1$ that

$$\begin{aligned} & m^{-2n} \| f\left(m_1 a_1 m^n x_1 + m_2 a_2 m^n x_2\right) + m_1 m_2 f\left(a_2 m^n x_1 - a_1 m^n x_2\right) \\ - & \left(m_1 a_1^2 + m_2 a_2^2\right)\left[m_1 f(m^n x_1) + m_2 f(m^n x_2)\right] \| \leq m^{-2n} c, \end{aligned}$$

or

$$\begin{aligned} & \|f_n(m_1a_1x_1+m_2a_2x_2)+m_1m_2f_n(a_2x_1-a_1x_2) \\ - & \left(m_1a_1^2+m_2a_2^2\right)[m_1f_n(x_1)+m_2f_n(x_2)]\,\| \\ \le & \ m^{-2n}c \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

or

$$\|Q(m_1a_1x_1+m_2a_2x_2)+m_1m_2Q(a_2x_1-a_1x_2)-(m_1a_1^2+m_2a_2^2)\,[m_1Q(x_1)+m_2Q(x_2)]\| = 0$$

or the mapping Q satisfies (1.1) if $m>1$. Similarly, from (1.2) and (2.4) we get that Q satisfies (1.1) if $0<m<1$. Therefore Q satisfies (1.1) if $0<m\neq 1$, completing the proof that Q is *Euler-Lagrange type quadratic mapping* in X.

It is now clear from inequality (2.7) with $n\to\infty$, as well as formula (2.4) that the required inequality (2.5) holds in X. This completes ***the existence proof*** of the above-mentioned Theorem 2.1.

We claim that Q is **unique.** Let $Q' : X \to Y$ *be another Euler-Lagrange type quadratic mapping satisfying* (2.5). *Then* $Q' = Q$.

In fact, assume $m>1$. Remember both Q and Q' satisfy (1.15). Then for every $x\in X$ and $n\in\mathbb{N}_0$,

$$\begin{aligned} & \|Q(x)-Q'(x)\| \\ \le & \ m^{-2n}\left\{\|Q(m^nx)-f(m^nx)\|+\|Q'(m^nx)-f(m^nx)\|\right\} \\ \le & \ 2\delta m^{-2n} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

or $Q(x)=Q'(x)$. Similarly we establish uniqueness results if $0<m<1$. This completes the proof of *the uniqueness* and ***the Ulam stability*** for Euler-Lagrange type quadratic functional equations of the form (1.1). ∎

Corollary 2.2. *Let* X *and* Y *be real normed linear spaces. Assume that* Y *is complete. Take* $m_1=m_2=1 : 0<m=a_1^2+a_2^2\neq 1$ *and* $m_0=1$ *for any fixed non-zero reals* a_i $(i=1,2)$. *Define functions* $f_n=f_n(x)$ *as in (2.3). Then the limit in (2.4) exists for all* $x\in X$ *and* $Q: X\to Y$ *is the unique Euler-Lagrange type quadratic mapping such that*

$$\|f(x)-Q(x)\| \le \frac{3}{2}\frac{1}{|m-1|}c\,(=\delta) \tag{2.15}$$

for all $x\in X$ *with constant* $c\ge 0$ *(independent of* $x\in X$*).*

Note that in this case there is **no constant** ε in the right-hand side of (2.15) because $\overline{\overline{f}}(x)=\overline{f}(x)$. Besides δ given by (2.15) is sharper than the corresponding one in [21, 22] which is of the form

$$\begin{aligned} \delta = & \ \frac{1}{2}\frac{c}{(m-1)^2(m+1)}\begin{cases} 3m^2-1, \text{ if } m>1 \\ 3-m^2, \text{ if } 0<m<1 \end{cases} \\ \ge & \ \frac{1}{2}\frac{c}{(m-1)^2(m+1)}\begin{cases} 3m^2-3, \text{ if } m>1 \\ 3-3m^2, \text{ if } 0<m<1 \end{cases} \\ = & \ \frac{3}{2}\frac{1}{|m-1|}c. \end{aligned}$$

If $a_1 = a_2 = 1$, then $m = 2$ and from (2.15) we have $\delta = \frac{3c}{2}$. We note that in this case **a sharper constant** $\delta = \frac{c}{2}$ may be found, if new substitution $x_1 = x_2 = x$ is applied in (1.2), because $\overline{\overline{f}}(x) = \overline{f}(x) = f(x)$ [21, 22].

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