The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 2, Issue 1, Article 1, pp. 1-10, 2005

POSITIVE SOLUTIONS OF EVOLUTION OPERATOR EQUATIONS

RADU PRECUP

Received 17 September, 2004; accepted 3 October, 2004; published 22 January, 2005.
Communicated by: G. Isac

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ABSTRACT. Existence and localization results are derived from Krasnoselskii's compression-expansion fixed point theorem in cones, for operator equations in spaces of continuous functions from a compact real interval to an abstract space. The main idea, first used in [12], is to handle two equivalent operator forms of the equation, one of fixed point type giving the operator to which Krasnoselskii's theorem applies and an other one of coincidence type which is used to localize a positive solution in a shell. An application is presented for a boundary value problem associated to a fourth order partial differential equation on a rectangular domain.

Key words and phrases: Positive solution, Boundary value problem, Fixed point theorem in cones.

2000 *Mathematics Subject Classification.* 47J05, 47H10, 47J35.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND PRELIMINARIES

Krasnoselskii's compression-expansion fixed point theorem in cones [6] is one of the most useful results of nonlinear functional analysis, for the investigation of the existence, localization and multiplicity of nonnegative solutions to two-point boundary value problems. Such applications can be found in Agarwal, Meehan, O'Regan and Precup [1], Erbe, Hu and Wang [2], Erbe and Wang [3], Lan and Webb [7], Lian, Wong and Yeh [8], Meehan and O'Regan [10] and O'Regan and Precup [11]. All these applications are based on upper and lower inequalities for the appropriate Green's functions. Similar inequalities for boundary value problems related to partial differential equations are not known and Krasnoselskii's Theorem has appeared quite unapplicable to this type of problems. Recently, in [12], we have proposed a method of application of Krasnoselskii's Theorem to a class of nonlinear wave equations. The idea has been to use two equivalent operator forms of the equation, one of fixed point type giving the operator to which Krasnoselskii's theorem applies and an other one of coincidence type for the localization of a solution in a shell. The main goal of this paper is to present a general version of this method in order to make it applicable to other problems involving partial differential equations. By this we extend previous results established in [10] and [5] for scalar ordinary differential equations and ordinary differential equations in Banach spaces, respectively.

Let us recall Krasnoselskii's compression-expansion fixed point theorem in the form given in Granas and Dugundji [4, p. 325].

Theorem 1.1 (Krasnoselskii)**.** *Let* $(E, |.|)$ *be a normed linear space,* $C \subset E$ *a proper wedge and* $N : C \to C$ *a completely continuous map. Assume that for some numbers* ρ *and* R *with* $0 < \rho < R$, *one of the following conditions is satisfied:*

(a) $|N(x)| \leq |x|$ *for* $|x| = \rho$ *and* $|N(x)| \geq |x|$ *for* $|x| = R$,

(b) $|N(x)| \geq |x|$ *for* $|x| = \rho$ *and* $|N(x)| \leq |x|$ *for* $|x| = R$.

Then N *has a fixed point* x *with* $\rho \leq |x| \leq R$.

In this paper we work in the space $E = C([0,T];X)$. Here $0 < T < \infty$ and X is a Banach space with norm $|.|_X$. We denote by $|.|_{\infty,X}$ the norm on $C([0,T];X)$ defined by $|u|_{\infty,X} = \max_{t\in[0,T]} |u(t)|_X$.

We also use the notation $|.|_p$, or $|.|_{L^p[a,b]}$, for the norm of $L^p[a,b]$ ($1 \leq p \leq \infty$).

We close this section by stating two known results from the literature whose proofs are reproduced for the reader's convenience. The first one is Lemma 5.1 in Lions [9].

Lemma 1.2. *Let* V, X *and* W *be Banach spaces such that* $V \subset X \subset W$, *the injection of* V *in* X *is compact and the injection of* X *in* W *is continuous. Then for each* $\eta > 0$, *there exists a constant* $c(\eta)$ *such that*

$$|x|_X \leq \eta |x|_V + c(\eta) |x|_W \quad \textit{for all } x \in V.$$

Proof. Assume the contrary. Then there is a $\eta > 0$ and a sequence of elements $x_n \in V$ with $|x_n|_X \geq \eta |x_n|_V + n |x_n|_W$. Let $y_n = |x_n|_V^{-1} x_n$. Then

$$|y_n|_X \geq \eta + n |y_n|_W. \tag{1.1}$$

Since $|y_n|_V = 1$, we have $|y_n|_X \leq$constant and then by (1.1), $|y_n|_W \to 0$. On the other hand, since $|y_n|_V = 1$ and the injection of V in X is compact, there is a subsequence of (y_n) convergent in X, necessarily to 0. This yields a contradiction to (1.1). ∎

Lemma 1.3. *Let* V, X *and* W *be Banach spaces such that* $V \subset X \subset W$, *the injection of* V *in* X *is compact and the injection of* X *in* W *is continuous. Then any bounded subset of* $C([0,T];V)$ *which is relatively compact in* $C([0,T];W)$ *is relatively compact in* $C([0,T];X)$.

Proof. Let D be bounded in $C([0,T];V)$ and relatively compact in $C([0,T];W)$. Then $|u-v|_{\infty,V} \leq d < \infty$ for all $u,v \in D$ and for each $\varepsilon > 0$, there exists a finite subset $\{u_k\}$ of D such that for every $u \in D$ there is a k with $|u-u_k|_{\infty,W} \leq \varepsilon$. Lemma 1.2 guarantees

$$|u-u_k|_{\infty,X} \leq \eta\,|u-u_k|_{\infty,V} + c(\eta)\,|u-u_k|_{\infty,W} \leq \eta\, d + c(\eta)\,\varepsilon.$$

Now for any $\varepsilon' > 0$ we choose $\eta = \varepsilon'/(2d)$ and $\varepsilon = \varepsilon'/(2c(\eta))$ to obtain $|u-u_k|_{\infty,X} \leq \varepsilon'$. This implies that D is relatively compact in $C([0,T];X)$. ∎

2. Main Existence and Localization Results

Let V, W be Banach spaces, $A : D(A) \subset C([0,T];V) \to C([0,T];W)$ be a linear map, $B : V \to W$ a linear continuous map and $F : C([0,T];V) \to C([0,T];W)$ be a nonlinear map. We discuss the operator equation

$$\begin{cases} (Au)(t) - Bu(t) = F(u)(t), & t \in [0,T] \\ u \in D(A) \subset C([0,T];V) \end{cases} \tag{2.1}$$

which is seen as

$$(A-B)\,u = F(u), \quad u \in D(A)$$

for short. Throughout, for a function $u \in C([0,T];V)$, by Bu we shall mean the function $(Bu)(t) = Bu(t)$, $t \in [0,T]$. Obviously, if the operator $A-B$ from $D(A)$ to $C([0,T];W)$ is invertible, then (2.1) is equivalent to the fixed point problem

$$u = (A-B)^{-1} F(u), \quad u \in C([0,T];V). \tag{2.2}$$

Assume in addition that $V \subset W$ with continuous injection, A and B are invertible and $A^{-1}B^{-1} = B^{-1}A^{-1}$ on $C([0,T];W)$. Here again, for $u \in C([0,T];W)$, by $B^{-1}u$ we mean the function in $C([0,T];V)$ defined as $(B^{-1}u)(t) = B^{-1}u(t)$ for $t \in [0,T]$. Notice that since $V \subset W$ with continuous injection, any function in $C([0,T];V)$ also belongs to $C([0,T];W)$ making possible the compositions $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$. Now (2.1) is equivalent to the coincidence equation

$$\left(B^{-1} - A^{-1}\right) u = A^{-1}B^{-1}F(u), \quad u \in C([0,T];V) \tag{2.3}$$

since

$$(A-B)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} A^{-1}B^{-1} \quad \text{on } C([0,T];W). \tag{2.4}$$

Indeed, if $v \in C([0,T];W)$ and we denote $u = (A-B)^{-1} v$, then $Au - Bu = v$. It follows succesively that $B^{-1}Au - u = B^{-1}v$ and $A^{-1}B^{-1}Au - A^{-1}u = A^{-1}B^{-1}v$. Hence $B^{-1}A^{-1}Au - A^{-1}u = A^{-1}B^{-1}v$, that is $(B^{-1} - A^{-1})\,u = A^{-1}B^{-1}v$. Therefore $u = (B^{-1} - A^{-1})^{-1} A^{-1}B^{-1}v$. This proves (2.4).

Equation (2.2) gives us the operator $(A-B)^{-1} F$ to which Krasnoselskii's Theorem applies, while (2.3) is used to localize a positive solution in a shell.

Our assumptions are as follows:

(h1) there exist two Banach spaces $(X, |.|_X)$, $(Y, |.|_Y)$ such that $V \subset X \subset W$, $V \subset Y$, the injection of V in X is compact and the injections of X in W and of V in Y are continuous. Also W is ordered by the closed positive cone W_+, $V \cap W_+ \neq \{0\}$ and $|.|_Y$ is monotone on V with respect to the order relation induced in V by the cone $V \cap W_+$.

(h2) A is invertible and its inverse A^{-1} has the representation

$$\left(A^{-1}v\right)(t) = \int_0^T k(t,s)\,v(s)\,ds, \quad v \in C([0,T];W) \tag{2.5}$$

where $k : [0,T]^2 \to \mathbf{R}_+$ is such that $k(t,.) \in L^1[0,T]$ for each $t \in [0,T]$ and the map $t \mapsto k(t,.)$ is continuous from $[0,T]$ to $L^1[0,T]$.

(h3) B is continuous from V to W, invertible and the linear map $B^{-1} : W \to V$ is positive (i.e. $B^{-1}(W_+) \subset W_+$) or negative (i.e. $-B^{-1}$ is positive). Also $A^{-1}B^{-1} = B^{-1}A^{-1}$ on $C([0,T];W)$.

(h4) the linear map $(A-B)^{-1}$ is continuous from $C([0,T];W)$ in $C([0,T];V)$, and compact from $C([0,T];W)$ in $C([0,T];W)$.

(h5) the map $F : C \to C([0,T];W_+)$ is continuous and sends bounded sets into bounded sets, when C is endowed with the topology of $C([0,T];X)$. Here C is the cone of $C([0,T];V)$

$$C = (A-B)^{-1} C([0,T];W_+).$$

(h6) there exists $\alpha > 0$ such that

$$\left|B^{-1}F(u)\right|_{\infty,Y} \le \frac{\alpha}{\max\limits_{t\in[0,T]} |\kappa(t,.)|_{L^1[0,T]}}$$

for every $u \in C$ with $\|u\| = |(B^{-1} - A^{-1})u|_{\infty,Y} = \alpha$.

(h7) there exists an interval $[a,b] \subseteq [0,T]$, a map $\phi : C \to W_+$, a number $\beta > 0$, $\beta \neq \alpha$ and a point $t^* \in [0,T]$ such that

$$\phi(u) \le F(u)(t), \quad t \in [a,b] \quad \text{and} \tag{2.6}$$

$$\left|B^{-1}\phi(u)\right|_Y \ge \frac{\beta}{|k(t^*,.)|_{L^1[a,b]}} \tag{2.7}$$

for all $u \in C$ with $\|u\| = \beta$.

Theorem 2.1. *If the conditions* (h1)-(h7) *are satisfied, then* (2.1) *has at least one solution* u *with* $u \in C$ *and*

$$\min\{\alpha,\beta\} \le \|u\| \le \max\{\alpha,\beta\}.$$

Proof. First note that $|.|_{\infty,X}$ and $\|.\|$ are equivalent norms on $C([0,T];X)$. Indeed, if for any $u \in C([0,T];X)$ we let $v = (B^{-1} - A^{-1})u$, then

$$\left|\left(B^{-1} - A^{-1}\right)^{-1}\right|^{-1} |u|_{\infty,X} \le \|u\| = |v|_{\infty,Y} \le \left|B^{-1} - A^{-1}\right| |u|_{\infty,X}.$$

Hence

$$c_0 |u|_{\infty,X} \le \|u\| \le c_1 |u|_{\infty,X}, \quad u \in C([0,T];X)$$

where

$$c_0 = \left|\left(B^{-1} - A^{-1}\right)^{-1}\right|^{-1} \text{ and } c_1 = \left|B^{-1} - A^{-1}\right|.$$

Here c_0^{-1} is the norm of operator $(B^{-1} - A^{-1})^{-1}$ acting from $C([0,T];Y)$ to $C([0,T];X)$, while c_1 is the norm of $B^{-1} - A^{-1}$ from $C([0,T];X)$ to $C([0,T];Y)$.

Also note that since W_+ is a cone of W and $(A-B)^{-1}$ is linear, the set $C = (A-B)^{-1} C([0,T];W_+)$ is a cone of $C([0,T];X)$.

We shall apply Theorem 1.1 to the space $E = C([0,T];X)$ endowed with norm $\|.\|$ and cone C and to the operator $N = (A-B)^{-1} F$.

From (h5), we have that $F(u) \in C([0,T];W_+)$ for each $u \in C$. Then, $N(u) = (A-B)^{-1} F(u) \in C$ by the definition of C. Hence N maps C into C. Furthermore, (h1), (h4), (h5) and Lemma 1.2 guarantee that N is completely continuous.

We show that condition (a) or (b) holds (with $\rho = \min\{\alpha, \beta\}$ and $R = \max\{\alpha, \beta\}$) if $\alpha < \beta$ or $\beta < \alpha$, respectively. To this end, let $u \in C$ with $\|u\| = \alpha$. Since $N(u) = (B^{-1} - A^{-1})^{-1} A^{-1}B^{-1}F(u)$, we have

$$\|N(u)\| = \left|A^{-1}B^{-1}F(u)\right|_{\infty,Y}.$$

On the other hand, from $u \in C$ we have $F(u) \in C([0,T]; W_+)$ and since σB^{-1} is positive, $\sigma B^{-1}F(u) \in C([0,T]; W_+)$. Here

$$\sigma = \begin{cases} 1 & \text{if } B^{-1} \text{ is positive} \\ -1 & \text{if } B^{-1} \text{ is negative.} \end{cases}$$

Hence

$$0 \le \sigma A^{-1}B^{-1}F(u)(t) = \sigma \int_0^T k(t,s)\, B^{-1}F(u)(s)\, ds.$$

Then, since the norm $|.|_Y$ is monotone on V, by (h6),

$$\left|A^{-1}B^{-1}F(u)(t)\right|_Y \le \int_0^T k(t,s)\left|B^{-1}F(u)(s)\right|_Y ds \le \alpha.$$

Hence

$$\|N(u)\| \le \alpha = \|u\|. \tag{2.8}$$

Furthermore, assume that $u \in C$ and $\|u\| = \beta$. Using (2.6), we obtain

$$\begin{aligned} \sigma\left(B^{-1} - A^{-1}\right) N(u)(t^*) &= \sigma A^{-1}B^{-1}F(u)(t^*) \\ &= \sigma \int_0^T k(t^*,s)\, B^{-1}F(u)(s)\, ds \\ &\ge \sigma \int_a^b k(t^*,s)\, B^{-1}F(u)(s)\, ds \\ &\ge \sigma\, |k(t^*,.)|_{L^1[a,b]}\, B^{-1}\phi(u). \end{aligned}$$

Then, by (2.7), we deduce that

$$\left|\left(B^{-1} - A^{-1}\right) N(u)(t^*)\right|_Y \ge |k(t^*,.)|_{L^1[a,b]} \left|B^{-1}\phi(u)\right|_Y \ge \beta.$$

It follows that

$$\|N(u)\| \ge \beta = \|u\|.$$

This together with (2.8) shows that (a) or (b) holds if $\alpha < \beta$, respectively $\beta < \alpha$. Thus Krasnoselskii's Theorem applies. ∎

Now we shall apply Theorem 2.1 to discuss the existence, localization and multiplicity of solutions for the problem

$$\begin{cases} (Au)(t) - Bu(t) = f\left(\sigma\left(B^{-1} - A^{-1}\right) u(t)\right), & t \in [0,T] \\ u \in D(A) \subset C([0,T]; V). \end{cases} \tag{2.9}$$

We shall succeed this under an additional assumption on kernel k.

Theorem 2.2. *Assume that conditions* (h1)-(h4) *hold. In addition assume that there exists* $0 < M < 1$, $\kappa \in L^1[0,T]$ *and an interval* $[a,b] \subseteq [0,T]$, $a < b$, *such that*

$$\begin{matrix} k(t,s) \le \kappa(s), & t \in [0,T], & \textit{a.e. } s \in [0,T] \\ M\,\kappa(s) \le k(t,s), & t \in [a,b], & \textit{a.e. } s \in [0,T]. \end{matrix} \tag{2.10}$$

Let $f : V \cap W_+ \to W_+$ be continuous from its domain with the topology of V to $W,$ nondecreasing with respect to the order induced by W_+, and let f send bounded sets into bounded sets. In addition assume that the following conditions are satisfied:

(i) *there exists $\alpha > 0$ such that*

$$\frac{\alpha}{\sup\limits_{w\in B^{-1}(W_+),\ |w|_Y=\alpha} |B^{-1}f(\sigma w)|_Y} \geq \max_{t\in[0,T]} |\kappa(t,.)|_{L^1[0,T]}\,; \tag{2.11}$$

(ii) *there exists $\beta > 0$, $\beta \neq \alpha$ and $t^* \in [0,T]$ such that*

$$\frac{\beta}{\inf\limits_{w\in B^{-1}(W_+),\ |w|_Y=\beta} |B^{-1}f(\sigma M v)|_Y} \leq |k(t^*,.)|_{L^1[a,b]}\,. \tag{2.12}$$

Then (2.9) *has at least one solution u with*

$$\min\{\alpha,\beta\} \leq \|u\| \leq \max\{\alpha,\beta\}$$

and

$$0 \leq \sigma M \left(B^{-1} - A^{-1}\right) u(t) \leq \sigma \left(B^{-1} - A^{-1}\right) u(t') \tag{2.13}$$

for all $t \in [0,T]$ and $t' \in [a,b]$. (*Inequalities* (2.13) *are understand with respect to the order induced by W_+*).

Proof. First note that if $u \in C$, then $u = (A-B)^{-1} v$ for some $v \in C([0,T];W_+)$ and so $(B^{-1}-A^{-1})u(t) = A^{-1}B^{-1}v(t) = B^{-1}[(A^{-1}v)(t)]$. Since $v(s) \in W_+$ for all $s \in [0,T]$, by the representation formula (2.5) and the fact that W_+ is closed, we have that $(A^{-1}v)(t) \in W_+$ for every $t \in [0,T]$. Consequently, $(B^{-1}-A^{-1})u(t) \in B^{-1}(W_+)$ for every $t \in [0,T]$. In particular, since σB^{-1} is positive, one has that $\sigma(B^{-1}-A^{-1})u(t) \in V \cap W_+$. Hence $f(\sigma(B^{-1}-A^{-1})u(t))$ is well defined. Now, the continuity of f and its property of sending bounded sets into bounded sets, imply that the map $F(u)(t) = f(\sigma(B^{-1}-A^{-1})u(t))$ satisfies (h5).

Furthermore, inequality (i) guarantees (h6). Now we prove that (ii) guarantees (h7). First we show that for each $u \in C$ and all $t \in [0,T]$ and $t' \in [a,b]$, inequalities (2.13) hold. Indeed, if $u = (A-B)^{-1} v$ with $v \in C([0,T];W_+)$, then using (2.10) we obtain

$$\begin{aligned}\sigma M\left(B^{-1}-A^{-1}\right)u(t) &= \sigma M\, A^{-1}B^{-1}v(t) = \sigma M \int_0^T k(t,s)\, B^{-1}v(s)\, ds \\ &\leq \sigma M \int_0^T \kappa(s)\, B^{-1}v(s)\, ds \leq \sigma \int_0^T k(t',s)\, B^{-1}v(s)\, ds \\ &= \sigma A^{-1}B^{-1}v(t') = \sigma\left(B^{-1}-A^{-1}\right)u(t').\end{aligned}$$

Also, the positivity of A^{-1} and σB^{-1} guarantees $\sigma A^{-1}B^{-1}v(t) \geq 0$. Hence

$$\sigma\left(B^{-1}-A^{-1}\right)u(t) \geq 0, \quad t \in [0,T].$$

Now, if $u \in C$, then (2.13) and the monotonicity of f imply

$$F(u)(t) \geq f\left(\sigma M\left(B^{-1}-A^{-1}\right)u(t_u)\right), \quad t \in [a,b],$$

where $t_u \in [0,T]$ and $|(B^{-1}-A^{-1})u(t_u)|_Y = \|u\|$. Hence (2.6) holds with $\phi(u) = f(\sigma M(B^{-1}-A^{-1})u(t_u))$. Now (2.12) guarantees (2.7). ∎

Remark 2.1. If in Theorem 2.2, f is nonincreasing instead of nondecreasing, then (2.6) holds for $\phi(u) = f(\sigma M^{-1}(B^{-1}-A^{-1})u(t_u))$ and the conclusion of the theorem is still valid with the only modification in (ii), where we must take M^{-1}in the place of M.

Remark 2.2. From Theorem 2.2 one can immediately obtain multiplicity results for problem (2.9).

The assumptions of Theorem 2.2 can be compared to those from Theorem 2.1 in [5] and Theorem 2.1 in [10].

Remark 2.3. The use of different spaces V, W, X and Y is convenient when treating partial differential equations in terms of weak solutions. See [12] for an example, where V is the Sobolev space $H_0^1(\Omega)$, W is its dual $H^{-1}(\Omega)$, $X = L^p(\Omega)$ and $Y = L^q(\Omega)$ with suitable exponents p and q.

3. Application

We shall apply Theorem 2.2 to discuss the existence and the localization of solutions for the nonlinear problem associated to a fourth order partial differential equation:

$$\begin{cases} v_{ttxx} + g(v) = 0, & (t,x) \in [0,T] \times [0,h] \\ v = 0 \ \text{ for } t = 0,\ t = T,\ x = 0 \text{ and } x = h. \end{cases} \tag{3.1}$$

We shall use the convention that $v(t,x) = v(t)(x)$.

Let $C_0^2[0,h]$ be the space of all functions $u \in C^2[0,h]$ with $u(0) = u(h) = 0$, endowed with the C^2-norm. If we let A and B be the operators $Au = -\frac{d^2u}{dt^2}$ $(u \in D(A),\, D(A) = \{u \in C^2([0,T]; C_0^2[0,h]) : u(0) = u(T) = 0\})$ and $Bu = -\frac{d^2u}{dx^2}$ $(u \in C_0^2[0,h])$ and we make the substitution $v = (B^{-1} - A^{-1})u$, we can write (3.1) under the form

$$\begin{cases} u_{tt} - u_{xx} = g((B^{-1} - A^{-1})u(t)) \ \text{ on } [0,T] \\ u(0) = u(T) = 0 \\ u \in C^2([0,T]; C_0^2[0,h]). \end{cases} \tag{3.2}$$

In this case B^{-1} is positive. Problem (3.2) is of type (2.9) with $\sigma = 1$ and $f(u)(t) = g(u(t))$. This choice of B is suitable to guarantee the existence of a positive solution to (3.1).

Also, if A is the above operator while $Bu = +\frac{d^2u}{dx^2}$, then (3.1) is equivalent to

$$\begin{cases} u_{tt} + u_{xx} = -g((B^{-1} - A^{-1})u(t)) \ \text{ on } [0,T] \\ u(0) = u(T) = 0 \\ u \in C^2([0,T]; C_0^2[0,h]). \end{cases} \tag{3.3}$$

This time, B^{-1} is negative. Problem (3.3) is of type (2.9) with $\sigma = -1$ and $f(u)(t) = -g(-u(t))$. In this approach we shall be able to guarantee the existence of negative solutions for (3.1).

Before we state and prove two existence and localization results for positive and respectively negative solutions to (3.1), we introduce some notations.

Here the kernel k of A^{-1} is the Green function corresponding to the operator $-\frac{d^2}{dt^2}$, the interval $[0,T]$ and the boundary condition $u(0) = u(T) = 0$, i.e.

$$k(t,s) = \begin{cases} \frac{s(T-t)}{T}, & 0 \le s \le t \le T \\ \frac{t(T-s)}{T}, & 0 \le t \le s \le T. \end{cases}$$

To be more precise we shall denote this function by k^T. Notice that for every subinterval $[a,b]$ of $[0,T]$, $0 < a < b < T$, k^T satisfies the following upper and lower inequalities

$$\begin{aligned} k^T(t,s) &\le k^T(s,s) \quad \text{for } t \in [0,T] \text{ and } s \in [0,T] \\ M_{a,b}^T k^T(s,s) &\le k^T(t,s) \quad \text{for } t \in [a,b] \text{ and } s \in [0,T]. \end{aligned} \tag{3.4}$$

Here
$$M_{a,b}^{T}=\min\left\{\frac{a}{T},\frac{T-b}{T}\right\}.$$
Obviously $0<M_{a,b}^{T}<1$. Also note that $\int_0^T k^T(s,s)\,ds=\frac{T^2}{6}$.

We shall use the notation $|u|_\infty$ to denote the norm of $C[0,h]$ and $\|u\|$ for the norm $\max\limits_{t\in[0,T]}|u(t)|_\infty$ on $C([0,T];C[0,h])$.

Positive solutions to (3.1) are guaranteed as shows the following result.

Theorem 3.1. *Let $g:\mathbf{R}_+\to\mathbf{R}_+$ be continuous and nondecreasing. Assume that there exists $\alpha>0,\beta>0$ with $\beta\neq\alpha$, two intervals $[a,b]$, $[a_0,b_0]$ with $0<a<b<T$, $0<a_0<b_0<h$ and two points $t^*\in[0,T]$, $x^*\in[0,h]$ such that*

$$\frac{(T\,h)^2}{48}\le\frac{\alpha}{g(\alpha)}\tag{3.5}$$

and

$$\frac{\beta}{g\left(M_{a_0,b_0}^{h}\,M_{a,b}^{T}\beta\right)}\le\left|k^h(x^*,.)\right|_{L^1[a_0,b_0]}\left|k^T(t^*,.)\right|_{L^1[a,b]}.\tag{3.6}$$

Then problem (3.1) *has at least one solution v with*
$$0\le M_{a,b}^{T}v(t)\le v(t')\quad\text{for all } t\in[0,T],\ t'\in[a,b];\ \text{and}$$
$$\min\{\alpha,\beta\}\le\|v\|\le\max\{\alpha,\beta\}.$$

Proof. We shall apply Theorem 2.2 with $V=C_0^2[0,h]$, $W=X=Y=C[0,h]$, $W_+=C([0,h];\mathbf{R}_+)$ and $f(u)(x)=g(u(x))$. In this case $\max\limits_{t\in[0,T]}\left|k^T(t,.)\right|_1=\frac{T^2}{8}$.

Also, for a function $u\in B^{-1}(W_+)$ with $|u|_\infty=\alpha$, we have
$$\begin{aligned}\left(B^{-1}f(u)\right)(x)&=\int_0^h k^h(x,y)\,g(u(y))\,dy\le g(\alpha)\int_0^h k^h(x,y)\,dy\\&\le g(\alpha)\frac{x(h-x)}{2}\le\frac{h^2}{8}g(\alpha).\end{aligned}$$
Hence
$$\left|B^{-1}f(u)\right|_\infty\le\frac{h^2}{8}g(\alpha)$$
and so condition (3.5) guarantees (2.11).

Furthermore, if $u\in B^{-1}(W_+)$, then $u=B^{-1}v$ for some $v\in W_+$. Using (3.4) (with $[0,h]$ in the place of $[0,T]$), it follows that

$$\begin{aligned}|u|_\infty&=\max_{x\in[0,h]}B^{-1}v(x)=\max_{x\in[0,h]}\int_0^h k^h(x,y)\,v(y)\,dy\\&\le\int_0^h k^h(y,y)\,v(y)\,dy.\end{aligned}\tag{3.7}$$

On the other hand, for any $x\in[a_0,b_0]$, we have from (3.4)

$$u(x)=\int_0^h k^h(x,y)\,v(y)\,dy\ge M_{a_0,b_0}^{h}\int_0^h k^h(y,y)\,v(y)\,dy.\tag{3.8}$$

Now (3.7) and (3.8) yield
$$u(x)\ge M_{a_0,b_0}^{h}|u|_\infty\quad\text{for } x\in[a_0,b_0].$$

Using this inequality, we obtain for any $u \in B^{-1}(W_+)$ with $|u|_\infty = \beta$ that

$$\begin{aligned} \left(B^{-1} f\left(M_{a,b}^T u\right)\right)(x^*) &= \int_0^h k^h(x^*, y)\, g\left(M_{a,b}^T u(y)\right) dy \\ &\geq \int_{a_0}^{b_0} k^h(x^*, y)\, g\left(M_{a,b}^T u(y)\right) dy \\ &\geq g\left(M_{a_0,b_0}^h\, M_{a,b}\, \beta\right) \int_{a_0}^{b_0} k^h(x^*, y)\, dy. \end{aligned}$$

Hence

$$\left|B^{-1} f\left(M_{a,b}^T u\right)\right|_\infty \geq g\left(M_{a_0,b_0}^h M_{a,b}\, \beta\right) \int_{a_0}^{b_0} k^h(x^*, y)\, dy$$

and so (3.6) guarantees (2.12). ∎

Similarly, negative solutions to (3.1) are guaranteed by the following result.

Theorem 3.2. *Let* $g : \mathbf{R}_- \to \mathbf{R}_-$ *be continuous and nondecreasing. Assume that there exists* $\alpha > 0, \beta > 0$ *with* $\beta \neq \alpha$, *two intervals* $[a, b], [a_0, b_0]$ *with* $0 < a < b < T$, $0 < a_0 < b_0 < h$ *and two points* $t^* \in [0, T], x^* \in [0, h]$ *such that*

$$\frac{(T\, h)^2}{48} \leq \frac{-\alpha}{g(-\alpha)}$$

and

$$\frac{-\beta}{g\left(-M_{a_0,b_0}^h\, M_{a,b}^T\, \beta\right)} \leq \left|k^h(x^*, .)\right|_{L^1[a_0,b_0]} \left|k^T(t^*, .)\right|_{L^1[a,b]}.$$

Then problem (3.1) *has at least one solution* v *with*

$$v(t') \leq M_{a,b}^T v(t) \leq 0 \text{ for all } t \in [0, T],\ t' \in [a, b]; \text{ and}$$

$$\min\{\alpha, \beta\} \leq \|v\| \leq \max\{\alpha, \beta\}.$$

Remark 3.1. The result in Theorem 3.2 also follows from Theorem 3.1 applied to the function $h : \mathbf{R}_+ \to \mathbf{R}_+$ given by $h(\tau) = -g(-\tau)$in the place of g.

Remark 3.2. Multiple (positive and negative) solutions to problem (3.1) are guaranteed by Theorems 3.1 and 3.2 if the assumptions are satisfied for several disjoint intervals $[\alpha, \beta]$ (or $[\beta, \alpha]$).

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