

ON ALMOST SURE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS SEQUENCES FOR THE SOLUTIONS OF SOME INTEGRO-DIFFERENTIAL STOCHASTIC EQUATIONS

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ABSTRACT. We show that the sequence of successive approximations converges almost surely to the unique bounded random solution $x(t, \omega)$ of a nonlinear stochastic integro-differential equation for each $t \in \mathbb{R}_+$.

We will use the renormalization of the space $\mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P))$.

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1. INTRODUCTION

The integro-differential stochastic equations by Volterra type occur in various domains as such of theory of automatic system, theory of turbulent fluid flows, in problems of biological sciences, in physics or engineering.

We deal with the study of stochastic integro-differential equations of the following type

$$(1.1) \quad x'(t, \omega) = h(t, x(t, \omega)) + \int_0^t \kappa(t, s, \omega) f(s, x(s, \omega)) ds, \quad t \geq 0,$$

where:

- i) $\omega \in \Omega$, Ω being the support set of probability field $(\Omega, \mathcal{K}, P)$ and we agree to explicitly write the dependency on ω of stochastic processes;
- ii) $x(t, \cdot)$ is an unknown stochastic process, $t \in \mathbb{R}_+$;
- iii) $h(t, x)$ and $f(t, x)$ are scalar functions on $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$;
- iv) $\kappa(t, s, \cdot)$ is the stochastic kernel defined for t and s satisfying $0 \leq s \leq t < \infty$.

Moreover, we will suppose that for each s and t , $\kappa(t, s, \cdot)$ is essentially bounded, i.e. $\kappa(t, s, \cdot) \in L_\infty(\Omega, \mathcal{K}, P)$ and norm of $\kappa(t, s, \cdot)$ in $L_\infty(\Omega, \mathcal{K}, P)$ will be denoted by

$$\|\kappa(t, s, \cdot)\|_{L_\infty} = P - \text{ess sup}_{\omega \in \Omega} |\kappa(t, s, \omega)|.$$

Also, the map

$$(t, s) \mapsto \kappa(t, s, \cdot) : \Delta = \{(t, s), 0 \leq s \leq t < \infty\} \rightarrow L_\infty(\Omega, \mathcal{K}, P)$$

is continuous and for $s \rightarrow s'$ and we have that $\|\kappa(u, s, \cdot) - \kappa(u, s', \cdot)\|_{L_\infty} \rightarrow 0$ uniform in u for $0 \leq u \leq t < \infty$, $t \in \mathbb{R}_+$ and for each fixed pair t and τ we have that

$$\|\kappa(s, \tau, \cdot)\|_{L_\infty} \leq M_{t, \tau}, \quad \text{uniform for } \tau \leq s \leq t,$$

where $M_{t, \tau} > 0$ is a constant dependent only on t and τ , $0 \leq \tau \leq t < \infty$.

We call the random solution for the equation (1.1), the process $x(t, \cdot)$ (notation for $x(t, \omega)$) which satisfies $P - a.s.$ this equation and is a second order process on $\mathbb{R}_+$.

It is known that the theory of the existence of solutions for the deterministic case of integral equations is based on contraction-type arguments, or on Schauder-type compactness arguments. For stochastic integral equations, results of the Arzela-Ascoli type are not convenient, so in this case the technique of contractions with the preponderance remains.

Our approach allows an extension of these methods. There are several classes of functions that are used to relax Lipschitz-type conditions ([2, 3, 4, 8, 9]) or new types of contractions. Among them we mention the class $\mathcal{M}$ ([2, 3, 4, 11, 12]) for which we impose new conditions for the convergence of the sequence of successive approximations. Thus, we identify a subclass $\mathcal{N} \subset \mathcal{M}$ with the help of which we obtain simplifications of the demonstration regarding the convergence to the solutions of the equations of the type mentioned above.

2. PRELIMINARY

We will introduce a class $\mathcal{M}$ of functions which will prove to be very useful in the investigation of the existence and uniqueness of solution for the stochastic integral equations and to assure the convergence of the successive approximations to solutions.

Certain functions from the class $\mathcal{M}$ are very important in the extending the theory and the applications of stochastic differential equations. In this paper they are used to ensure the almost sure convergence and the mean square convergence of the sequence of successive approximations.

Definition 2.1. Let $\mathcal{M}$ be the class of all continuous nondecreasing functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $x \mapsto x - \varphi(x)$ is nonnegative and strictly increasing on $[0, \infty)$ and satisfies $\sum_{n=1}^{\infty} \varphi^{*n}(x_0) < \infty$ for some $x_0 > 0$, where φ^{*n} is the n^{th} iterate of φ (see [8, 11]).

Note that $\varphi(0) = 0$ if $\varphi \in \mathcal{M}$ while $0 \leq \varphi(x) < \infty$ for all $x > 0$. Moreover, $\sum_{n=1}^{\infty} \varphi^{*n}(x) < \infty$ for all $x \in [0, x_0]$. The condition $\sum_{n=1}^{\infty} \varphi^{*n}(x) < \infty$ for some $x_0 > 0$ is not implied by other properties defining the class $\mathcal{M}$, as one from example of $\varphi(x) = \frac{x}{1+x}$ for $x \geq 0$, in which case formula $\varphi^{*n}(x) = \frac{x}{1+nx}$, for $n \geq 1$ and $x \geq 0$, easily checked by induction, ensures that $\sum_{n=1}^{\infty} \varphi^{*n}(x_0) = \infty$ for all $x_0 > 0$. Indeed, given $x_0 > 0$, if the integer $N \geq 1$ is such that $x_0 \geq \frac{1}{N}$ then

$$\sum_{n=1}^{\infty} \varphi^{*n}(x_0) = \sum_{n=1}^{\infty} \frac{x_0}{1+nx_0} \geq \sum_{n=N}^{\infty} \frac{1}{2n} = \infty.$$

As examples of the functions $\varphi \in \mathcal{M}$ are: $\varphi(x) = \alpha x$, $x \geq 0$, $\alpha \in (0, 1)$; $\eta(x) = \frac{1}{\lambda + x}$ for $x > 0$, $\lambda > 1$, $\eta(0) = 0$ and for the divergence case $\theta(x) = \frac{x}{1+\delta x}$, $x \geq 0$, $\delta > 0$.

Further on, we identify a subclass $\mathcal{N}$ of the class of functions $\mathcal{M}$, $\mathcal{N} \subset \mathcal{M}$, generated by aid of a set of positive strictly decreasing sequences $\{a_n\}_{n \geq 1}$ with the certain properties [8].

This class $\mathcal{N}$ of functions allows to investigate new classes of stochastic integral equations with the important properties regarding different types of convergence of the sequences of successive approximations to solutions. In particular, a relaxation of the classical Lipschitz conditions on their coefficients is given by allowing a suitably controlled growth in the time variable. The class $\mathcal{N}$ can also be used to study the concept of stability of random solution of certain class of stochastic integral equations [9, 12].

Let $\mathcal{S}$ be the set of strictly decreasing positive sequences $\{a_n\}_{n \geq 1}$ with the following properties:

$$a_n - a_{n+1} \geq a_{n+1} - a_{n+2} \text{ for all } n \geq 1, \quad \lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1, \\ \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_{n+2}}{a_n - a_{n+1}} = 1, \quad \text{and} \quad \sum_{n \geq 1} a_n < \infty.$$

Let us consider the sequences

$$\alpha_n = \frac{a_{n+1} - a_{n+2}}{a_n - a_{n+1}} \text{ and } \beta_n = \frac{a_n a_{n+2} - (a_{n+1})^2}{a_n - a_{n+1}}, \quad n \geq 1.$$

We define the function $\varphi : [0, \infty) \rightarrow [0, \infty)$ by $\varphi(0) = 0$ and

$$\varphi(x) = \begin{cases} a_2, & a_1 \leq x \\ \alpha_n x + \beta_n, & a_{n+1} < x \leq a_n, \quad n \geq 1 \end{cases}$$

Also, we denote by $\{\varphi_{\{a_n\}_{n \geq 1}}\}$ the function φ defined by means the sequence $\{a_n\}_{n \geq 1}$. At the same time we observe that the set $\{\varphi_{\{a_n\}_{n \geq 1}} : \text{for all } \{a_n\}_{n \geq 1} \in \mathcal{S}\} \stackrel{\text{def}}{=} \mathcal{N} \subset \mathcal{M}$.

The piecewise linear function φ is continuous and nondecreasing on $[0, \infty)$, satisfies $\varphi(a_n) = a_{n+1}$ for all $n \geq 1$ and $x \in [a_{n+1}, a_n]$ ensures $\varphi(x) \in [a_{n+2}, a_{n+1}]$. Since the sequence $\{a_n\}_{n \geq 1}$

is nonnegative, strictly decreasing and converging to zero, this implies that $x > \varphi(x)$ for all $x > 0$.

On the other hand, $\lim_{x \rightarrow 0} [x - \varphi(x)] = 0$ since $\varphi(x) \leq a_{n+1}$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} a_{n+1} = 0$, while $\lim_{x \rightarrow \infty} [x - \varphi(x)] = \infty$ since $\varphi(x) = a_2$ for all $x \geq a_1$.

Furthermore, since by construction $\varphi(a_n) = a_{n+1}$ for all $n \geq 1$, for the iterate function $\varphi^{*n}(x)$ we obtain that

$$\sum_{n=1}^{\infty} \varphi^{*n}(a_1) = \sum_{n=1}^{\infty} \varphi(a_n) = \sum_{n=1}^{\infty} a_{n+1} < \infty.$$

Moreover, for $x > a_1$, we have that $\varphi^{*n}(x) = a_{n+1}$, $n \geq 1$, and $\sum_{n=1}^{\infty} \varphi^{*n}(x) = \sum_{n=1}^{\infty} a_{n+1} < \infty$.

Also, for the iterate function φ^{*n} and all $x \in (a_{n+1}, a_n]$ we have that

$$\varphi(x) = \varphi^1(x) \in (a_{n+2}, a_{n+1}], \quad \varphi^1(x) \leq a_{n+1}, \quad \varphi^{*2}(x) \in (a_{n+3}, a_{n+2}], \\ \varphi^{*2}(x) \leq a_{n+2}, \quad \dots \quad \varphi^{*n}(x) \in (a_{2n+1}, a_{2n}], \quad \varphi^{*n}(x) \leq a_{2n}.$$

Hence, for each $x \in (0, a_1]$ there exists an $k \geq 1$ such that $x \in (a_{k+1}, a_k]$ with $\varphi(x) \in (a_{k+2}, a_{k+1}]$, and $\varphi^{*k}(x) \in (a_{2k+1}, a_{2k}]$.

Then we have for $m \geq k$

$$\varphi^{*m}(x) \leq a_{2m} \quad \text{and} \quad \sum_{m \geq k} \varphi^{*m}(x) \leq \sum_{m \geq k} a_{2m} < \infty, \quad \text{for all } x \in (0, a_1]$$

and $\sum_{n=1}^{\infty} \varphi^{*n}(x) < \infty$, for all $x \in (0, a_1]$.

On the other hand, it is easily to deduce that the function $\varphi(x)$ is derivable on the complement of the countable subset $\{a_n\}_{n \geq 1}$ of $[0, \infty)$ and $\varphi'(x) = \alpha_n > 0$ on (a_{n+1}, a_n) , $n \geq 1$, hence φ is an increasing function and therefore $\varphi \in \mathcal{N} \subset \mathcal{M}$. Note that there does not exist $\gamma \in (0, 1)$ such that $\varphi(x) \leq \gamma x$, for all $x \geq 0$, since $\varphi(a_n) = a_{n+1} \leq \gamma a_n$, for all $n \geq 1$, and so $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ is impossible.

The fact that the sequence $\{\alpha_n\}_{n \geq 1}$ is strictly increasing implies that $\varphi'(x)$ is a nonincreasing function and that $\varphi(x)$ is concave function (see [6]).

To see this more clearly, one can have a look on Figure 1, which compares the function φ with the identity map for $x \in [a_{20}, a_2 + 0.2]$, Supplementary, the graph of $\varphi'(x)$ is given in Figure 2.

As an example for a sequence with the above properties, we choose the following particular sequence

$$\{a_n\}_{n \geq 1} = \left\{ \frac{1}{2n^2 + 1} \right\}_{n \geq 1}$$

for which we have that $a_n - a_{n+1} \geq a_{n+1} - a_{n+2}$, which is ensured by

$$\frac{12(2n^2 + 4n + 1)}{(2n^2 + 1)(2n^2 + 4n + 3)(2n^2 + 8n + 9)} \geq 0,$$

and $\lim_{n \rightarrow \infty} a_n = 0$, $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$, $\sum_{n=1}^{\infty} a_n < \infty$.

From the calculus we deduce that:

$$\alpha_n = \frac{(2n+3)(2n^2+1)}{(2n+1)(2n^2+8n+9)}, \quad \alpha_{n+1} = \frac{(2n+5)(2n^2+4n+3)}{(2n+3)(2n^2+12n+19)} \\ \beta_n = \frac{4n(n+2)}{(2n+1)(2n^2+4n+3)(2n^2+8n+9)}$$

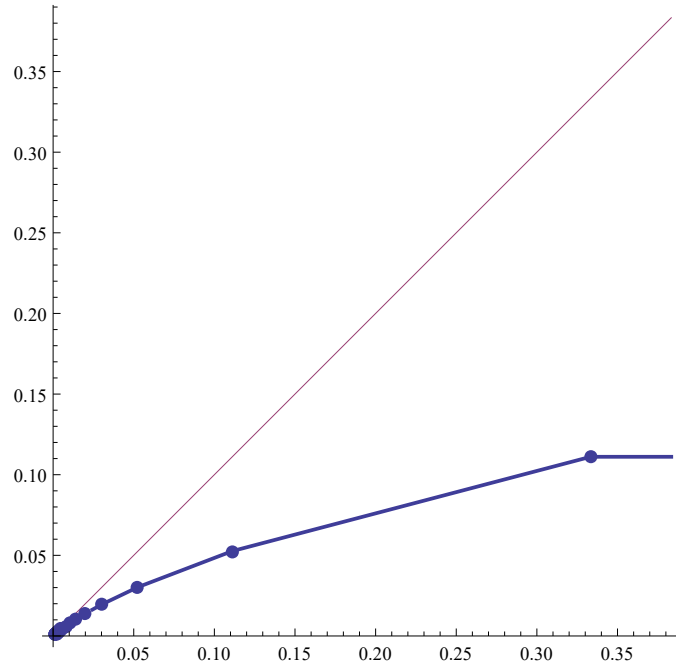


Figure 1: The graphs of $\varphi(x)$ (thick line) and the identity map $f(x) = x$ (dotted line), with some points $(a_n, \varphi(a_n))$ located (dots)

we see that $\lim_{n \rightarrow \infty} \alpha_n = 1$, with $\lim_{n \rightarrow \infty} \beta_n = 0$. Note that

$$\begin{aligned} \alpha_n - \alpha_{n+1} &= -\frac{12(4n^4 + 24n^3 + 44n^2 + 24n - 3)}{(2n+1)(2n+3)(2n^2+8n+9)(2n^2+12n+19)} = \\ &= -\frac{12(4n(n+1)(n+2)(n+3) - 3)}{(2n+1)(2n+3)(2n(n+4)+9)(2n(n+6)+19)} < 0, \end{aligned}$$

i.e. $\alpha_n < \alpha_{n+1}$ for $n \geq 1$.

Other examples of sequences with similarly properties with the sequence $\{a_n\}_{n \geq 1}$ are: $b_n = \sum_{k > n+1} \frac{1}{k^3}$, $c_n = \frac{1}{n^2}$, $d_n = \frac{1}{n^2 - n + 1}$, $e_n = \frac{1}{n\sqrt{n}}$, $n \geq 1$ and others.

3. MAIN RESULT

In this section, we present a theorem regarding almost sure convergence and mean square convergence of the successive approximations sequence to the random solution of the equation (1.1). We will use the renormalization of space $\mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P))$, as the stochastic counterpart of Bielecki approach to deterministic differential equations (see the discussions in [1, 5]).

We recall two Banach spaces of continuous functions used below.

Definition 3.1. We define the space $\mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P))$ of all continuous functions defined on $\mathbb{R}_+$ valued in $L_2(\Omega, \mathcal{K}, P)$ with uniform topology on each interval $[0, a]$, $a > 0$, i.e. $x_n(t, \cdot) \rightarrow x(t, \cdot)$ in $\mathcal{C}$ if and only if

$$\lim_{n \rightarrow \infty} \|x_n(t, \cdot) - x(t, \cdot)\|_{L_2} = 0$$

uniform on each interval $[0, a]$, $a > 0$ ([4, 12, 15, 16]).

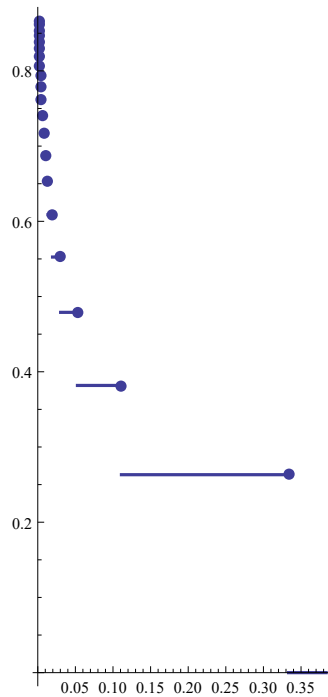


Figure 2: The graphs of $\varphi'(x)$ (thin line) with some points (a_n, α_n) located (dots)

Definition 3.2. We will denote by $\mathcal{C}_g(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P))$ the Banach space of all continuous functions from $\mathbb{R}_+$ to $L_2(\Omega, \mathcal{K}, P)$ with property that there exists a constant $M > 0$ and a continuous and positive function $g : \mathbb{R}_+ \rightarrow (0, \infty)$ such that

$$\left\{ E[|x(t, \cdot)|^2] \right\}^{1/2} = \left\{ \int_{\Omega} |x(t, \cdot)|^2 dP \right\}^{1/2} \leq M g(t), \quad t \in \mathbb{R}_+,$$

The norm of the Banach space $\mathcal{C}_g(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P))$ is

$$\|x(t, \cdot)\|_{\mathcal{C}_g} = \sup_{t \geq 0} \left\{ \frac{\|x(t, \cdot)\|_{L_2}}{g(t)} \right\}$$

Now, integrating the equation (1.1) from 0 to t we obtain the equation

$$(3.1) \quad x(t, \cdot) = x_0(\cdot) + \int_0^t h(s, x(s, \cdot)) ds + \int_0^t K(t, s, \cdot) f(s, x(s, \cdot)) ds$$

where

$$K(t, s, \cdot) = \int_s^t \kappa(u, s, \cdot) du, \quad 0 \leq s \leq t < \infty.$$

In [14], Theorem 3.1, we proved that the equation (1.1) has a unique solution and it is obtained with the help of successive approximation sequence which converges in norm of the space $\mathcal{C}_g(\mathbb{R}_+, L_2(\omega, \mathcal{K}, P))$.

Now, we will investigate almost sure convergence of the sequence

$$(3.2) \quad x_{n+1}(t, \cdot) = x_0(\cdot) + \int_0^t h(s, x_n(s, \cdot)) ds + \int_0^t K(t, s, \cdot) f(s, x_n(s, \cdot)) ds, \quad 0 \leq s \leq t < \infty,$$

where $x_0(\cdot) = x(0, \cdot)$ și $K(t, s) = \int_s^t \kappa(u, s, \cdot) du$, $0 \leq s \leq u \leq t < \infty$.

Theorem 3.1. *We consider the stochastic integro-differential equation (3.1) which fulfills the following conditions:*

- i) *the function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(t, \cdot) \in \mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P))$ and $g : \mathbb{R}_+ \rightarrow (0, \infty)$ is continuous and positive function and for which are satisfy the following conditions*

$$|f(t, x) - f(t, y)| \leq g(t) \varphi \left(\frac{|x - y|}{2g(t)} \right), \quad t \in \mathbb{R}_+, \quad x, y \in \mathbb{R},$$

where g is by the type

$$(3.3) \quad g(t) = \exp \left[2M e^{\int_0^t e^{-\beta s} ds} \right], \quad t \in \mathbb{R}_+,$$

$\varphi \in \mathcal{N}$, $\beta > 0$ and there exists a constant $M_1 > 0$ such that

$$\|f(t, 0)\|_{L_2} \leq M_1 g(t), \quad t \in \mathbb{R}_+;$$

- ii) *there exists a constant $M_2 > 0$ și $\beta > 0$ such that for the continuous and bounded function $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ și $h(t, \cdot) \in \mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{C}, P))$ we have that*

$$|h(t, x) - h(t, y)| \leq M_2 g(t) e^{-\beta t} \cdot \varphi \left(\frac{|x - y|}{2g(t)} \right), \quad t \in \mathbb{R}_+$$

and there exists $M_3 > 0$ such that

$$\|h(t, \cdot)\|_{L_2} \leq M_3 g(t) e^{-\beta t}, \quad t \in \mathbb{R}_+;$$

iii)

$$\int_0^t \|K(t, s, \cdot)\|_{L_\infty} ds \leq M_4 e^{-\beta t}, \quad t \in \mathbb{R}_+, \quad 0 \leq s \leq t < \infty, \quad M_4 > 0;$$

iv)

$$x(0, \cdot) = x_0(\cdot) \in L_2(\Omega, \mathcal{K}, P).$$

Then, for each $t \in \mathbb{R}_+$,

$$x_n(t, \cdot) \xrightarrow{a.s.} \bar{x}(t, \cdot)$$

where $\bar{x}(t, \cdot)$ is the unique bounded solution of the equation (3.1) on $\mathbb{R}_+$.

Proof : First of all, we specify that the existence and uniqueness of the solution $\bar{x}(t, \cdot) \in \mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{C}, P))$ for the equation (3.1) is proved in Theorem 3.1 from [14], with the same condition i) – iv) from above. Moreover, the solution is stable with respect to small perturbations of the stochastic initial data $x_0(\cdot)$.

We define the operator $T : \mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P)) \rightarrow \mathcal{C}(\mathbb{R}_+, L_2(\Omega, \mathcal{K}, P))$ by

$$T(x(t, \cdot)) = x_0(\cdot) + \int_0^t h(s, x(s, \cdot)) ds + \int_0^t K(t, s, \cdot) f(s, x(s, \cdot)) ds, \quad 0 \leq s \leq t < \infty$$

and we denote by $M = M_1 \vee M_2 \vee M_3 \vee M_4$.

We have that

$$\frac{\|T(x(t, \cdot)) - T(y(t, \cdot))\|_{L_2}}{g(t)} \leq \frac{1}{g(t)} \int_0^t \|h(s, x(s, \cdot)) - h(s, y(s, \cdot))\|_{L_2} ds +$$

$$\begin{aligned}
& + \frac{1}{g(t)} \int_0^t \|K(t, s, \cdot)\|_{L_\infty} \cdot \|f(s, x(s, \cdot)) - f(s, y(s, \cdot))\|_{L_2} ds \leq \\
& \leq \frac{M}{g(t)} \int_0^t g(s) e^{-\beta s} \varphi \left(\sup_{s \geq 0} \frac{\|x(s, \cdot) - y(s, \cdot)\|_{L_2}}{2g(s)} \right) ds + \\
& + \frac{M}{g(t)} \int_0^t g(s) e^{-\beta s} \varphi \left(\sup_{s \geq 0} \frac{\|x(s, \cdot) - y(s, \cdot)\|_{L_2}}{2g(s)} \right) ds \leq \\
& \leq \varphi \left(\frac{\|x(t, \cdot) - y(t, \cdot)\|_{C_g}}{2} \right) \frac{2M}{g(t)} \int_0^t g(s) e^{-\beta s} ds \leq \frac{\|x(t, \cdot) - y(t, \cdot)\|_{C_g}}{2} \frac{2M}{g(t)} \int_0^t g(s) e^{-\beta s} ds
\end{aligned}$$

If g is the function given by the relation (3.3), we have that

$$(3.4) \quad \|T(x(t, \cdot)) - T(y(t, \cdot))\|_{C_g} \leq \frac{\|x(t, \cdot) - y(t, \cdot)\|_{C_g}}{2}$$

since $g'(t) = g(t) \left(2M \frac{1-e^{-\beta t}}{\beta} \right)' = 2Mg(t)e^{-\beta t}$ and

$$\int_0^t g'(s) ds = g(t) - 1 < g(t)$$

and $\varphi(x) < x, \forall x > 0$.

We observe that for the choice of g as in the relation (3.3) we obtain

$$2 \leq 2g(t) = 2 \exp \left[2M \int_0^t e^{-\beta s} ds \right] = 2 \exp \left[2M \frac{1 - e^{-\beta t}}{\beta} \right] < 2e^{\frac{2M}{\beta}}$$

Then, for $t \in \mathbb{R}_+$ we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} \left\{ \int_{\Omega} |x_n(t, \cdot) - \bar{x}(t, \cdot)|^2 dP \right\}^{1/2} & \leq \sum_{n=0}^{\infty} \sup_{t \geq 0} \frac{\|x_n(t, \cdot) - \bar{x}(t, \cdot)\|_{L_2}}{2g(t)} 2e^{\frac{2M}{\beta}} \leq \\
& \leq \sum_{n=0}^{\infty} \|x_n(t, \cdot) - \bar{x}(t, \cdot)\|_{C_g} e^{\frac{2M}{\beta}}.
\end{aligned}$$

On the other hand, taking into account the relation (3.4) and Theorem 3.1. ([14]), we obtain that

$$\|x_n(t, \cdot) - \bar{x}(t, \cdot)\|_{C_g} = \|T(x_{n-1}(t, \cdot)) - T(\bar{x}(t, \cdot))\|_{C_g} \leq \frac{1}{2} \|x_{n-1}(t, \cdot) - \bar{x}(t, \cdot)\|_{C_g}$$

where $\bar{x}(t, \cdot)$ is the unique solution of the equation (3.1).

Repeating this procedure by $n - 1$ times we get

$$\|x_n(t, \cdot) - \bar{x}(t, \cdot)\|_{C_g} \leq \frac{1}{2^n} \|x_0(t, \cdot) - \bar{x}(t, \cdot)\|_{C_g} \leq \frac{1}{2^n} \delta$$

where $\delta > 0$ is chosen such that $\|x_0(t, \cdot) - \bar{x}(t, \cdot)\|_{C_g} \leq \delta$.

Thus, we deduce that

$$\sum_{n=0}^{\infty} \left\{ \int_{\Omega} |x_n(t, \cdot) - \bar{x}(t, \cdot)|^2 dP \right\}^{1/2} \leq 2\delta e^{\frac{2M}{\beta}} < \infty.$$

Then, there exists an $N_0 > 0$ such that for $n \geq N_0$ we get

$$\left\{ \int_{\Omega} |x_n(t, \cdot) - \bar{x}(t, \cdot)|^2 dP \right\} < 1.$$

We thus obtain that

$$\begin{aligned} \sum_{n=0}^{N_0} \left\{ \int_{\Omega} |x_n(t, \cdot) - \bar{x}(t, \cdot)|^2 dP \right\}^{1/2} + \sum_{n=N_0+1}^{\infty} \left\{ \int_{\Omega} |x_n(t, \cdot) - \bar{x}(t, \cdot)|^2 dP \right\} &\leq \\ &\leq \sum_{n=0}^{\infty} \left\{ \int_{\Omega} |x_n(t, \cdot) - \bar{x}(t, \cdot)|^2 dP \right\} \leq 2\delta e^{\frac{2M}{\beta}} < \infty, \end{aligned}$$

because $\sqrt{x} > x$ for $x \in (0, 1)$.

Let now be a finite sequence $M_n > 0$, for $n \leq N_0$, then we have that

$$\left\{ \int_{\Omega} |x_n(t, \cdot) - \bar{x}(t, \cdot)|^2 dP \right\}^{1/2} \leq M_n < \infty$$

(this choice being possible because $\bar{x}(t, \cdot), x_n(t, \cdot) \in L_2(\Omega, \mathcal{K}, P)$ what it implies $x_n(t, \cdot) - \bar{x}(t, \cdot) \in L_2(\Omega, \mathcal{K}, P)$).

From the above it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} E [|x_n(t, \cdot) - \bar{x}(t, \cdot)|^2] &= \sum_{n=0}^{N_0} E [|x_n(t, \cdot) - \bar{x}(t, \cdot)|^2] + \\ &+ \sum_{n=N_0+1}^{\infty} E [|x_n(t, \cdot) - \bar{x}(t, \cdot)|^2] \leq \sum_{n=0}^{N_0} M_n^2 + 2\delta e^{\frac{2M}{\beta}} < \infty. \end{aligned}$$

Because

$$\sum_{n=0}^{\infty} E [|x_n(t, \cdot) - \bar{x}(t, \cdot)|^2] < \infty$$

results (from Loève ([7], pp.173) that

$$x_n(t, \cdot) \xrightarrow{a.s.} \bar{x}(t, \cdot)$$

i.e. the sequence of successive approximations converges almost sure to the unique solution (which is ensured by the Theorem 3.1. from [14]).

From this it follows that the sequence $\{x_n(t, \cdot)\}_{n \geq 0}$ converges in probability and in distribution for each $t \in \mathbb{R}_+$.

On the other hand,

$$\sum_{n=0}^{\infty} E [|x_n(t, \cdot) - \bar{x}(t, \cdot)|^2] < \infty$$

results the convergence of the sequence $x_n(t, \cdot)$ to the solution $\bar{x}(t, \cdot)$ in square mean ([4, 5, 9, 16]). $\square$

Remark 3.1. The result from above The result above provides an improvement on a result obtained by Tsokos ([15, 16]) which assumes that $M, \|h\|_{L_2}$ și $\|f\|_{L_2}$ they must be quite small, which is not necessary here.

Also, are improved some results given in [2, 3, 4, 10, 13].

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