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THREE POINTS INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL OF LIPSCHITZIAN OR BOUNDED VARIATION INTEGRANDS AND INTEGRATORS OF r-H HÖLDER TYPE WITH APPLICATIONS

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ABSTRACT. In this paper we obtained some new simple error bounds in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of three points rule

$$[u(b)-u((1-\lambda)x+\lambda b)]f(b)+[u(\upsilon a+(1-\upsilon)x)-u(a)]f(a)$$
$$+[u((1-\lambda)x+\lambda b)-u(\upsilon a+(1-\upsilon)x)]f(x),$$

where $\lambda,\upsilon\in[0,1]$, $x\in[a,b]$ and assuming that the function f is L-Lipschitzian or of bounded variation and u is r-H-Hölder type on $[a,b]$. The important case of weighted integrals is considered, compounding quadrature rules are provided and applications for approximation of Fourier transforms on finite intervals are also given.

Key words and phrases: Riemann-Stieltjes integral, Lipschitz functions, Functions of bounded variation, Quadrature rules, Finite Fourier transform.

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1. INTRODUCTION

The Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ is an essential concept in Mathematics with various applications in different areas of the field. In recent years, the approximations problem of the Riemann-Stieltjes integral $\int_a^b f du$ and several error approximation bounds have been studied. Some of the most interesting approximations have been established using the Generalized Trapezoid and Ostrowski type rule.

Dragomir ([13], [14]) introduced the approximation of the Stieltjes integral $\int_a^b f(t)\,du(t)$ using *Ostrowski type rule*

$$f(x)\,[u(b)-u(a)]$$

where $x \in [a,b]$. He considers the functional for a priory sharp bound on the approximation error defined by

$$\Theta(f,u;a,b,x) := \int_a^b f(t)\,du(t) - f(x)\,[u(b)-u(a)].$$

In [13], Dragomir proved that

$$\begin{aligned}(1.1)\qquad &|\Theta(f,u;a,b,x)| \\ &\leq H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right] \\ &\leq H \times \begin{cases} [(x-a)^r + (b-x)^r]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]; \\ [(x-a)^{qr} + (b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \end{cases}\end{aligned}$$

provided that f is of bounded variation and u is of r-H-*Hölder type*, i.e.,

$$(1.2)\qquad |u(t)-u(s)| \leq H|t-s|^r, \text{ for each } t,\ s \in [a,b],$$

with given $H>0$ and $r \in (0,1]$.

If the integrand f is of q-K-Hölder type and the integrator u is of bounded variation, then [14]

$$(1.3)\qquad |\Theta(f,u;a,b,x)| \leq K\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^q \bigvee_a^b (u),$$

for any $x \in [a,b]$.

For various error bounds on the functional $\Theta(f,u;a,b,x)$ with f and u under different assumptions for which the Stieltjes integral exists, the reader may refer to [2], [3], [6]-[14] and [21] and the references therein.

In addition, the authors of [16], considered the problem of approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the *trapezoid type rule* defined by the quantity

$$(1.4)\qquad [u(b)-u(x)]\,f(b) + [u(x)-u(a)]\,f(a).$$

To provide a priory sharp bounds for the approximation error, the authors introduced the functional:

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]\,f(b)-[u(x)-u(a)]\,f(a),$$

where $x\in[a,b]$ and obtained the result

$$|T(f,u;a,b,x)|\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f),$$

provided that f is of bounded variation while u is of the r-H-*Hölder type*.

In [20], the authors established the following results

$$\begin{aligned}(1.5)\qquad &|T(f,u;a,x,b)|\\ &\leq H\left[(x-a)^r\bigvee_a^x(u)+(b-x)^r\bigvee_x^b(u)\right]\\ &\leq H\times\begin{cases}[(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right];\\ [(x-a)^{\alpha r}+(b-x)^{\alpha r}]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{\beta}}\\ \qquad\qquad\text{if }\alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(u),\end{cases}\end{aligned}$$

provided f is of r-H-*Hölder type* and u is of bounded variation.

For various error bounds on the functional $T(f,u;a,b,x)$ where f and u under different assumptions for which the Stieltjes integral exists, the reader may refer to [2], [3], [6], [22]-[24] and the references therein.

Motivated by the above results and [17], [18], in this paper we point out some new approaches to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of three points formula, namely we establish bounds for the associated error functional

$$\begin{aligned}(1.6)\qquad &T\Theta(f,u;a,b,x,\lambda,\upsilon)\\ &:=[u(b)-u((1-\lambda)x+\lambda b)]\,f(b)+[u(\upsilon a+(1-\upsilon)x)-u(a)]f(a)\\ &+[u((1-\lambda)x+\lambda b)-u(\upsilon a+(1-\upsilon)x)]\,f(x)-\int_a^b f(t)\,du(t),\end{aligned}$$

where $\lambda,\upsilon\in[0,1]$ and $x\in[a,b]$ assuming that the function$f$ is L-Lipschitzian or of bounded variation and u is r-H-Hölder type on $[a,b]$. The important particular case of weighted integrals is considered, compounding quadrature rules are provided and applications for approximation Fourier transforms on finite intervals are also given.

2. MAIN RESULTS

Assume that $u,\ f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $f\in\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $f\in\mathcal{R}(u)$.

We start with the following identity of interest, see also [18].

Lemma 2.1. *Let $f, u : [a,b] \to \mathbb{C}$ and $x \in [a,b]$ such that $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$. Then for any γ, $\mu \in \mathbb{C}$,*

$$[u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + (\mu-\gamma) f(x) - \int_a^b f(t)\, du(t) \tag{2.1}$$
$$= \int_a^x [u(t)-\gamma]\, df(t) + \int_x^b [u(t)-\mu]\, df(t).$$

Proof. For the sake of completeness, we give here a short proof.

Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_a^x [u(t)-\gamma]\, df(t) = [u(x)-\gamma] f(x) - [u(a)-\gamma] f(a) - \int_a^x f(t)\, du(t)$$

and

$$\int_x^b [u(t)-\mu]\, df(t) = [u(b)-\mu] f(b) - [u(x)-\mu] f(x) - \int_x^b f(t)\, du(t)$$

for any $x \in [a,b]$.

If we add these two equalities, we get

$$\begin{aligned}
&\int_a^x [u(t)-\gamma]\, df(t) + \int_x^b [u(t)-\mu]\, df(t) \\
&= [u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + [\mu - u(x)] f(x) \\
&+ [u(x)-\gamma] f(x) - \int_a^x f(t)\, du(t) - \int_x^b f(t)\, du(t) \\
&= [u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + (\mu-\gamma) f(x) - \int_a^b f(t)\, du(t)
\end{aligned}$$

for any $x \in [a,b]$, which proves the desired equality (2.1). ∎

The following corollaries are of interest and will be used in the sequel.

Corollary 2.2. *Let $f, u : [a,b] \to \mathbb{C}$ and $x \in (a,b)$ such that $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$. Then for any $\lambda, \upsilon \in [0,1]$,*

$$T\Theta(f,u;a,b,x,\lambda,\upsilon) \tag{2.2}$$
$$= \int_a^x [u(t) - u(\upsilon a + (1-\upsilon)x)]\, df(t) + \int_x^b [u(t) - u((1-\lambda)x + \lambda b)]\, df(t).$$

The proof follows by (2.1) for $\mu = u((1-\lambda)x + \lambda b)$ and $\gamma = u(\upsilon a + (1-\upsilon)x)$, for any $\lambda, \upsilon \in [0,1]$.

If we take $\lambda = \upsilon$, then we define $T\Theta(f,u;a,b,x,\upsilon) := T\Theta(f,u;a,b,x,\upsilon,\upsilon)$. For , $\lambda = 0$ and $\upsilon = 0$, we have

$$\begin{aligned}
&T\Theta(f,u;a,b,x,0) \\
&:= [u(b)-u(x)] f(b) + (u(x)-u(a)) f(x) - \int_a^b f(t)\, du(t) \\
&= \int_a^x [u(t)-u(a)]\, df(t) + \int_x^b [u(t)-u(x)]\, df(t).
\end{aligned}$$

For $\lambda = 0$ and $\upsilon = 1$, we have that

$$\begin{aligned} &T\Theta\left(f, u; a, b, x, 0, 1\right) \\ &:= \left[u\left(b\right) - u\left(x\right)\right] f\left(b\right) + \left[u\left(x\right) - u\left(a\right)\right] f\left(x\right) - \int_a^b f\left(t\right) du\left(t\right) \\ &= \int_a^x \left[u\left(t\right) - u\left(a\right)\right] df\left(t\right) + \int_x^b \left[u\left(t\right) - u\left(x\right)\right] df\left(t\right), \end{aligned}$$

while for $\lambda = 1$ and $\upsilon = 0$, that

$$\begin{aligned} &T\Theta\left(f, u; a, b, x, 1, 0\right) \\ &:= \left(u\left(x\right) - u\left(a\right)\right) f\left(a\right) + \left(u\left(b\right) - u\left(x\right)\right) f\left(x\right) - \int_a^b f\left(t\right) du\left(t\right) \\ &= \int_a^x \left[u\left(t\right) - u\left(x\right)\right] df\left(t\right) + \int_x^b \left[u\left(t\right) - u\left(b\right)\right] df\left(t\right). \end{aligned}$$

For $\lambda = 1$ and $\upsilon = 1$, we have

$$\begin{aligned} &T\Theta\left(f, u; a, b, x, 1\right) \\ &:= \left[u\left(b\right) - u\left(a\right)\right] f\left(x\right) - \int_a^b f\left(t\right) du\left(t\right) \\ &= \int_a^x \left[u\left(t\right) - u\left(a\right)\right] df\left(t\right) + \int_x^b \left[u\left(t\right) - u\left(b\right)\right] df\left(t\right). \end{aligned}$$

For $\lambda = 1/2$ and $\upsilon = 1/2$, we get

$$\begin{aligned} &T\Theta\left(f, u; a, b, x, 1/2\right) \\ &:= \left[u\left(b\right) - u\left(\frac{x+b}{2}\right)\right] f\left(b\right) + \left[u\left(\frac{a+x}{2}\right) - u\left(a\right)\right] f\left(a\right) \\ &+ \left(u\left(\frac{x+b}{2}\right) - u\left(\frac{a+x}{2}\right)\right) f\left(x\right) - \int_a^b f\left(t\right) du\left(t\right) \\ &= \int_a^x \left[u\left(t\right) - u\left(\frac{a+x}{2}\right)\right] df\left(t\right) + \int_x^b \left[u\left(t\right) - u\left(\frac{x+b}{2}\right)\right] df\left(t\right). \end{aligned}$$

Corollary 2.3. *Let $f, u : [a, b] \to \mathbb{C}$ and such that $f \in \mathcal{R}_{\mathbb{C}}\left(u, [a, b]\right)$. Then for any $\lambda, \upsilon \in [0, 1]$,*

$$\begin{aligned} &T\Theta\left(f, u; a, b, \frac{a+b}{2}, \lambda, \upsilon\right) \\ &:= \left[u\left(b\right) - u\left(\frac{\left(1-\lambda\right)a + \left(1+\lambda\right)b}{2}\right)\right] f\left(b\right) \\ &+ \left[u\left(\frac{\left(1+\upsilon\right)a + \left(1-\upsilon\right)b}{2}\right) - u\left(a\right)\right] f\left(a\right) \\ &+ \left(u\left(\frac{\left(1-\lambda\right)a + \left(1+\lambda\right)b}{2}\right) - u\left(\frac{\left(1+\upsilon\right)a + \left(1-\upsilon\right)b}{2}\right)\right) f\left(x\right) \\ &- \int_a^b f\left(t\right) du\left(t\right) \end{aligned} \tag{2.3}$$

$$= \int_a^{\frac{a+b}{2}} \left[u(t) - u\left(\frac{(1+\upsilon)a + (1-\upsilon)b}{2}\right)\right] df(t)$$
$$+ \int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right)\right] df(t).$$

The proof follows by Corollary 2.2 for $x = \frac{a+b}{2}$. For $\lambda = \upsilon = 1/2$, we obtain

$$T\Theta\left(f, u; a, b, \frac{a+b}{2}, 1/2\right) \tag{2.4}$$
$$:= \left[u(b) - u\left(\frac{a+3b}{4}\right)\right] f(b) + \left[u\left(\frac{3a+b}{4}\right) - u(a)\right] f(a)$$
$$+ \left(u\left(\frac{a+3b}{4}\right) - u\left(\frac{3a+b}{4}\right)\right) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, du(t)$$
$$= \int_a^{\frac{a+b}{2}} \left[u(t) - u\left(\frac{3a+b}{4}\right)\right] df(t) + \int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{a+3b}{4}\right)\right] df(t).$$

If u is an integral, namely $u(t) := \int_a^t w(s)\, ds$, where w is integrable on $[a, b]$, then we can state the following identity as well:

Corollary 2.4. *Assume that w is integrable on $[a, b]$ and f is measurable on $[a, b]$ such that wf is also integrable on $[a, b]$, then we have the equality*

$$T\Theta\left(f, \int_a^{\cdot} w(s)\, ds; a, b, x, \lambda, \upsilon\right) \tag{2.5}$$
$$:= f(b) \int_{(1-\lambda)x+\lambda b}^b w(s)\, ds + f(a) \int_a^{\upsilon a+(1-\upsilon)x} w(s)\, ds$$
$$+ f(x) \int_{\upsilon a+(1-\upsilon)x}^{(1-\lambda)x+\lambda b} w(s)\, ds - \int_a^b f(s)\, w(s)\, ds$$
$$= \int_a^x \left(\int_{\upsilon a+(1-\upsilon)x}^t w(s)\, ds\right) df(t) + \int_x^b \left(\int_{(1-\lambda)x+\lambda b}^t w(s)\, ds\right) df(t),$$

where $\lambda, \upsilon \in [0, 1]$.

For $\lambda = \upsilon$ we put

$$T\Theta\left(f, \int_a^{\cdot} w(s)\, ds; a, b, x, \upsilon\right) := T\Theta\left(f, \int_a^{\cdot} w(s)\, ds; a, b, x, \upsilon, \upsilon\right).$$

By taking here different values of $\lambda, \upsilon \in [0, 1]$ one can get different equalities similar with the above ones. For $\lambda = \upsilon = 1/2$ we obtain

$$
\begin{aligned}
(2.6)\qquad & T\Theta\left(f,\int_a^{\cdot} w(s)\,ds;a,b,x,1/2\right)\\
&:=f(b)\int_{\frac{x+b}{2}}^{b} w(s)\,ds+f(a)\int_a^{\frac{a+x}{2}} w(s)\,ds+f(x)\int_{\frac{a+x}{2}}^{\frac{x+b}{2}} w(s)\,ds\\
&-\int_a^b f(s)\,w(s)\,ds\\
&=\int_a^x\left(\int_{\frac{a+x}{2}}^{t} w(s)\,ds\right)df(t)+\int_x^b\left(\int_{\frac{x+b}{2}}^{t} w(s)\,ds\right)df(t),
\end{aligned}
$$

for $x\in(a,b)$.

If we take $x=\frac{a+b}{2}$, then we obtain

$$
\begin{aligned}
(2.7)\qquad & T\Theta\left(f,\int_a^{\cdot} w(s)\,ds;a,b,\frac{a+b}{2},1/2\right)\\
&:=f(b)\int_{\frac{a+3b}{4}}^{b} w(s)\,ds+f(a)\int_a^{\frac{3a+b}{4}} w(s)\,ds+f(x)\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} w(s)\,ds\\
&-\int_a^b f(s)\,w(s)\,ds\\
&=\int_a^{\frac{a+b}{2}}\left(\int_{\frac{3a+b}{4}}^{t} w(s)\,ds\right)df(t)+\int_{\frac{a+b}{2}}^b\left(\int_{\frac{a+3b}{4}}^{t} w(s)\,ds\right)df(t).
\end{aligned}
$$

The following theorem provides error upper bounds in the case when f is L-Lipschitzian and u is of r-H-Hölder type with the constant H and $r\in(0,1]$.

Theorem 2.5. *Let $f,u:[a,b]\to\mathbb{C}$ and $x\in[a,b]$. If f is Lipschitzian with the constant $L>0$, namely*

$$|f(t)-f(s)|\le L|t-s| \text{ for all } t,\ s\in[a,b]$$

and u is of r-H-Hölder type with the constant H and $r\in(0,1]$, namely

$$|u(t)-u(s)|\le H|t-s|^r \text{ for all } t,\ s\in[a,b],$$

then $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$ and we have the inequality:

$$
\begin{aligned}
(2.8)\qquad & |T\Theta(f,u;a,b,x,\lambda,\upsilon)|\\
&\le\frac{LH}{r+1}\\
&\times\left\{\left[(1-\upsilon)^{r+1}+\upsilon^{r+1}\right](x-a)^{r+1}+\left[(1-\lambda)^{r+1}+\lambda^{r+1}\right](b-x)^{r+1}\right\}
\end{aligned}
$$

for $\lambda,\upsilon\in[0,1]$.

In particular, for $\lambda=\upsilon$, we obtain

$$
\begin{aligned}
(2.9)\qquad & |T\Theta(f,u;a,b,x,\upsilon)|\\
&\le\frac{LH}{r+1}\left[(1-\upsilon)^{r+1}+\upsilon^{r+1}\right]\left[(x-a)^{r+1}+(b-x)^{r+1}\right],
\end{aligned}
$$

where $T\Theta(f,u;a,b,x,\upsilon):=T\Theta(f,u;a,b,x,\lambda,\upsilon)$.

Proof. It is well known that, if $p : [a,b] \to \mathbb{C}$ is Riemann integrable and $v : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and [1]

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b |p(t)|\,dt. \tag{2.10}$$

From the identity (2.2) and the properties of the integral, we then have

$$\begin{aligned}
&|T\Theta(f,u;a,b,x,\lambda,\upsilon)| \\
&= \left|\int_a^x [u(t)-u(\upsilon a+(1-\upsilon)x)]\,df(t)+\int_x^b [u(t)-u((1-\lambda)x+\lambda b)]\,df(t)\right| \\
&\leq \left|\int_a^x [u(t)-u(\upsilon a+(1-\upsilon)x)]\,df(t)\right|+\left|\int_x^b [u(t)-u((1-\lambda)x+\lambda b)]\,df(t)\right| \\
&\leq L\left[\int_a^x |u(t)-u(\upsilon a+(1-\upsilon)x)|\,dt+\int_x^b |[u(t)-u((1-\lambda)x+\lambda b)]|\,dt\right]
\end{aligned} \tag{2.11}$$

for $x \in [a,b]$ and $\lambda, \upsilon \in [0,1]$.

First, we observe that, in general, if $c \in [a,b]$ and $r \in (0,1]$, then

$$\int_a^b |t-c|^r\,dt = \int_a^c (c-t)^r\,dt+\int_c^b (t-c)^r\,dt = \frac{1}{r+1}\left[(c-a)^{r+1}+(b-c)^{r+1}\right].$$

Since u is of r-H-Hölder type with the constant H, then

$$\begin{aligned}
&\int_a^x |u(t)-u(\upsilon a+(1-\upsilon)x)|\,dt \\
&\leq H\int_a^x |t-(\upsilon a+(1-\upsilon)x)|^r\,dt \\
&= \frac{H}{r+1}\left[(\upsilon a+(1-\upsilon)x-a)^{r+1}+(x-(\upsilon a+(1-\upsilon)x))^{r+1}\right] \\
&= \frac{H}{r+1}\left[(1-\upsilon)^{r+1}(x-a)^{r+1}+\upsilon^{r+1}(x-a)^{r+1}\right] \\
&= \frac{H}{r+1}\left[(1-\upsilon)^{r+1}+\upsilon^{r+1}\right](x-a)^{r+1}
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b |[u(t)-u((1-\lambda)x+\lambda b)]|\,dt \\
&\leq H\int_x^b |t-((1-\lambda)x+\lambda b)|^r\,dt \\
&= \frac{H}{r+1}\left[(b-((1-\lambda)x+\lambda b))^{r+1}+(((1-\lambda)x+\lambda b)-x)^{r+1}\right] \\
&= \frac{H}{r+1}\left[(1-\lambda)^{r+1}(b-x)^{r+1}+\lambda^{r+1}(b-x)^{r+1}\right] \\
&= \frac{H}{r+1}\left[(1-\lambda)^{r+1}+\lambda^{r+1}\right](b-x)^{r+1}
\end{aligned}$$

for $x \in [a,b]$ and $\lambda, \upsilon \in [0,1]$.

By making use of (2.11) we obtain the error bound from (2.8). ∎

Corollary 2.6. *With the assumptions of Theorem 2.5 we get*

$$\left|T\Theta(f,u;a,b,\frac{a+b}{2},\lambda,\upsilon)\right| \leq \frac{LH}{2^{r+1}(r+1)}\left[(1-\upsilon)^{r+1}+\upsilon^{r+1}+(1-\lambda)^{r+1}+\lambda^{r+1}\right](b-a)^{r+1}$$

for $\lambda,\upsilon\in[0,1]$.

In particular,

$$\left|T\Theta(f,u;a,b,\frac{a+b}{2},\upsilon)\right| \leq \frac{LH}{2^{r}(r+1)}\left[(1-\upsilon)^{r+1}+\upsilon^{r+1}\right](b-a)^{r+1}$$

for $\upsilon\in[0,1]$.

Assume that f and u are as in Theorem 2.5 for any $\lambda,\upsilon\in[0,1]$. For , $\lambda=0$ and $\upsilon=0$, we get the trapezoid type inequalities

$$|T(f,u;a,b,x,0)| \leq \frac{LH}{r+1}\left\{(x-a)^{r+1}+(b-x)^{r+1}\right\}.$$

For , $\lambda=0$ and $\upsilon=1$, we have

$$|T\Theta(f,u;a,b,x,0,1)| \leq \frac{LH}{r+1}\left\{(x-a)^{r+1}+(b-x)^{r+1}\right\}.$$

while for , $\lambda=1$ and $\upsilon=0$, we get

$$|T\Theta(f,u;a,b,x,1,0)| \leq \frac{LH}{r+1}\left\{(x-a)^{r+1}+(b-x)^{r+1}\right\}.$$

For , $\lambda=1$ and $\upsilon=1$, we obtain the Ostrowski type inequalities

$$|\Theta(f,u;a,b,x,1)| \leq \frac{LH}{r+1}\left\{(x-a)^{r+1}+(b-x)^{r+1}\right\}.$$

For , $\lambda=\frac{1}{2}$ and $\upsilon=\frac{1}{2}$, we derive

$$|T\Theta\left(f,u;a,b,x,1/2\right)| \leq \frac{LH}{2^{r}(r+1)}\left[(x-a)^{r+1}+(b-x)^{r+1}\right].$$

If we take $x=\frac{a+b}{2}$, then we get

$$\left|T\Theta\left(f,u;a,b,\frac{a+b}{2},1/2\right)\right| \leq \frac{LH}{2^{2r}(r+1)}(b-a)^{r+1}.$$

For , $\lambda=\frac{2}{3}$ and $\upsilon=\frac{1}{3}$, we have

$$|T\Theta\left(f,u;a,b,x,2/3,1/3\right)| \leq \frac{LH}{r+1}\left[\left(\frac{2}{3}\right)^{r+1}+\left(\frac{1}{3}\right)^{r+1}\right]\left[(x-a)^{r+1}+(b-x)^{r+1}\right].$$

For , $\lambda=1-\upsilon$, we also get that

$$|T(f,u;a,b,x,1-\upsilon,\upsilon)| \leq \frac{LH}{r+1}\left[(1-\upsilon)^{r+1}+\upsilon^{r+1}\right]\left[(x-a)^{r+1}+(b-x)^{r+1}\right].$$

The case when u is the integral of another function is important for applications since it provides weighted integral rules, in which one can be the kernel of an integral transform like Laplace, Fourier or Mellin transform.

We recall *Hölder's integral inequality*

$$\left|\int_c^d h(t)\,g(t)\,dt\right| \le \left(\int_c^d |h(t)|^\alpha\,dt\right)^{1/\alpha}\left(\int_c^d |g(t)|^\beta\,dt\right)^{1/\beta}, \tag{2.12}$$

where $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and the integrals on the right side are finite.

If we take $u(t)=\int_a^t w(\tau)\,d\tau$ and use (2.12), then we get

$$\begin{aligned}|u(t)-u(s)| &= \left|\int_a^t w(\tau)\,d\tau-\int_a^s w(\tau)\,d\tau\right| = \left|\int_s^t w(\tau)\,d\tau\right| \\ &\le \left|\int_s^t 1d\tau\right|^{1/\alpha}\left|\int_s^t |w(\tau)|^\beta\,d\tau\right|^{1/\beta} = |t-s|^{1/\alpha}\left|\int_s^t |w(\tau)|^\beta\,d\tau\right|^{1/\beta} \\ &\le |t-s|^{1/\alpha}\left(\int_a^b |w(\tau)|^\beta\,d\tau\right)^{1/\beta}\end{aligned}$$

for all $t,s\in[a,b]$.

This shows that the function $u(t)=\int_a^t w(\tau)\,d\tau$ is of r-H-Hölder's type with the exponent $r=\frac{1}{\alpha}<1$ and the positive constant $H=\left(\int_a^b |w(\tau)|^\beta\,d\tau\right)^{1/\beta}$, provided that this integral is finite.

By making use of the inequalities (2.8) and (2.9) for $u(t)=\int_a^t w(\tau)\,d\tau$, $r=\frac{1}{\alpha}<1$ and $H=\left(\int_a^b |w(\tau)|^\beta\,d\tau\right)^{1/\beta}$, where $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we obtain that

$$\begin{aligned}&\left|T\Theta(f,\int_a^{\cdot} w(\tau)\,d\tau;a,b,x,\lambda,\upsilon)\right| \\ &\le \frac{L\alpha}{\alpha+1}\left(\int_a^b |w(\tau)|^\beta\,d\tau\right)^{1/\beta} \\ &\times\left\{\left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right](x-a)^{\frac{\alpha+1}{\alpha}}+\left[(1-\lambda)^{\frac{\alpha+1}{\alpha}}+\lambda^{\frac{\alpha+1}{\alpha}}\right](b-x)^{\frac{\alpha+1}{\alpha}}\right\}\end{aligned} \tag{2.13}$$

for $\lambda,\upsilon\in[0,1]$.

In particular, for $\lambda=\upsilon$, we get

$$\begin{aligned}|T\Theta(f,u;a,b,x,\upsilon)| &\le \frac{L\alpha}{\alpha+1}\left(\int_a^b |w(\tau)|^\beta\,d\tau\right)^{1/\beta}\left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right] \\ &\times\left[(x-a)^{\frac{\alpha+1}{\alpha}}+(b-x)^{\frac{\alpha+1}{\alpha}}\right],\end{aligned} \tag{2.14}$$

where $T\Theta(f,\int_a^{\cdot} w(\tau)\,d\tau;a,b,x,\upsilon):=T\Theta(f,\int_a^{\cdot} w(\tau)\,d\tau;a,b,x,\upsilon,\upsilon)$.

For $\lambda=1/2$ and $\upsilon=1/2$ we get

$$\begin{aligned}&\left|T\Theta\left(f,\int_a^t w(\tau)\,d\tau;a,b,x,1/2\right)\right| \\ &\le \frac{L\alpha}{2^{\frac{1}{\alpha}}(\alpha+1)}\left(\int_a^b |w(\tau)|^\beta\,d\tau\right)^{1/\beta}\left[(x-a)^{\frac{\alpha+1}{\alpha}}+(b-x)^{\frac{\alpha+1}{\alpha}}\right].\end{aligned}$$

for $x\in(a,b)$.

If we take $x = \frac{a+b}{2}$, then we obtain the simpler bound

$$\left|T\Theta\left(f, \int_a^t w(\tau)\,d\tau; a, b, \frac{a+b}{2}, 1/2\right)\right| \leq \frac{L\alpha}{2^{\frac{2}{\alpha}}(\alpha+1)}\left(\int_a^b |w(\tau)|^\beta\,d\tau\right)^{1/\beta}(b-a)^{\frac{\alpha+1}{\alpha}}.$$

The case when f is bounded variation and u is a r-H-Hölder type mapping, then the following error bounds can be obtained as well:

Theorem 2.7. *Let $f : [a,b] \to \mathbb{C}$ be a mapping of bounded variation on $[a,b]$ and $u : [a,b] \to \mathbb{C}$ be a r-H-Hölder type mapping. Then we have the inequality:*

$$\begin{aligned}(2.15)\qquad &|T\Theta(f,u;a,b,x,\lambda,\upsilon)| \\ &\leq H\left[\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^r(x-a)^r\bigvee_a^x(f)+\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^r(b-x)^r\bigvee_x^b(f)\right] \\ &\leq H\left[\max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^r,\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^r\right\}\right] \\ &\times\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \\ &\leq H\left[\max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^r,\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^r\right\}\right] \\ &\times\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^r\bigvee_a^b(f)\end{aligned}$$

for $\lambda, \upsilon \in [0,1]$.

In particular, for $\lambda = \upsilon$, we have

$$\begin{aligned}(2.16)\qquad &|T\Theta(f,u;a,b,x,\upsilon)| \\ &\leq H\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^r\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \\ &\leq H\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^r\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^r\bigvee_a^b(f),\end{aligned}$$

where $T\Theta(f,u;a,b,x,\upsilon) := T\Theta(f,u;a,b,x,\lambda,\upsilon)$.

Proof. It is well known that, if $p : [a,b] \to \mathbb{C}$ is Riemann integrable and $v : [a,b] \to \mathbb{C}$ is bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds:

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v).$$

From the identity (2.2) and the properties of the integral, we have

$$
\begin{aligned}
(2.17)\quad & |T\Theta(f,u;a,b,x,\lambda,\upsilon)| \\
&= \left|\int_a^x [u(t)-u(\upsilon a+(1-\upsilon)x)]\,df(t)+\int_x^b [u(t)-u((1-\lambda)x+\lambda b)]\,df(t)\right| \\
&\le \left|\int_a^x [u(t)-u(\upsilon a+(1-\upsilon)x)]\,df(t)\right| \\
&+ \left|\int_x^b [u(t)-u((1-\lambda)x+\lambda b)]\,df(t)\right| \\
&\le \max_{t\in[a,x]} |u(t)-u(\upsilon a+(1-\upsilon)x)| \bigvee_a^x (f) \\
&+ \max_{t\in[x,b]} |[u(t)-u((1-\lambda)x+\lambda b)]| \bigvee_x^b (f)
\end{aligned}
$$

for $x\in[a,b]$ and $\lambda,\upsilon\in[0,1]$.

Since u is of r-H-Hölder type with the constant H, we can state that

$$|u(t)-u(\upsilon a+(1-\upsilon)x)| \le H\,|t-(\upsilon a+(1-\upsilon)x)|^r$$

and

$$|u(t)-u((1-\lambda)x+\lambda b)| \le H\,|t-((1-\lambda)x+\lambda b)|^r$$

and then

$$\max_{t\in[a,x]} |u(t)-u(\upsilon a+(1-\upsilon)x)| \le H\max\{\upsilon^r,(1-\upsilon)^r\}(x-a)^r$$

and

$$\max_{t\in[x,b]} |u(t)-u((1-\lambda)x+\lambda b)| \le H\max\{\lambda^r,(1-\lambda)^r\}(b-x)^r.$$

Now, by using (2.17) , we can state that

$$
\begin{aligned}
(2.18)\quad & |T\Theta(f,u;a,b,x,\lambda,\upsilon)| \\
&\le H\left[\max\{\upsilon^r,(1-\upsilon)^r\}(x-a)^r\bigvee_a^x(f)+\max\{\lambda^r,(1-\lambda)^r\}(b-x)^r\bigvee_x^b(f)\right] \\
&\le H\max\{\max\{\upsilon^r,(1-\upsilon)^r\},\max\{\lambda^r,(1-\lambda)^r\}\} \\
&\times\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \\
&\le H\max\{\max\{\upsilon^r,(1-\upsilon)^r\},\max\{\lambda^r,(1-\lambda)^r\}\} \\
&\times\max\{(x-a)^r,(b-x)^r\}\bigvee_a^b(f)
\end{aligned}
$$

for $x\in[a,b]$ and $\lambda,\upsilon\in[0,1]$.

Using the well known facts for the maximum, namely

$$\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{X-Y}{2}\right|$$

and

$$\max\{X^r, Y^r\} = (\max\{X, Y\})^r = \left(\frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|\right)^r,\ r > 0$$

then we get

$$\max\{\upsilon^r, (1-\upsilon)^r\} = \left[\frac{1}{2} + \left|\upsilon - \frac{1}{2}\right|\right]^r,$$

$$\max\{\lambda^r, (1-\lambda)^r\} = \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^r$$

and

$$\max\{(x-a)^r, (b-x)^r\} = [\max\{x-a, b-x\}]^r$$

$$= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^r.$$

By making use of (2.18) we obtain the desired result (2.15). ∎

Corollary 2.8. *With the assumptions of Theorem 2.7 we have*

$$\left|T\Theta(f, u; a, b, \frac{a+b}{2}, \lambda, \upsilon)\right|$$

$$\leq H \max\left\{\left(\frac{1}{2} + \left|\upsilon - \frac{1}{2}\right|\right)^r, \left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right)^r\right\}$$

$$\times \left(\frac{b-a}{2}\right)^r \bigvee_a^b (f)$$

for $\lambda, \upsilon \in [0,1]$.

In particular,

$$\left|T\Theta(f, u; a, b, \frac{a+b}{2}, \upsilon)\right| \leq H\left(\frac{1}{2} + \left|\upsilon - \frac{1}{2}\right|\right)^r \left(\frac{b-a}{2}\right)^r \bigvee_a^b (f)$$

for $\upsilon \in [0,1]$.

Assume that f and u are as in Theorem 2.7 for any $\lambda, \upsilon \in [0,1]$. For , $\lambda = 0$ and $\upsilon = 0$, we get the trapezoid type inequalities

$$|T(f, u; a, b, x, 0)| \leq H\left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^r \bigvee_a^b (f).$$

For , $\lambda = 0$ and $\upsilon = 1$, we get

$$|T\Theta(f, u; a, b, x, 0, 1)| \leq H\left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^r \bigvee_a^b (f).$$

while for , $\lambda = 1$ and $\upsilon = 0$, we get

$$|T\Theta(f, u; a, b, x, 1, 0)| \leq H\left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^r \bigvee_a^b (f).$$

For , $\lambda = 1$ and $\upsilon = 1$, we obtain the Ostrowski type inequalities

$$|\Theta(f, u; a, b, x, 1)| \leq H\left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^r \bigvee_a^b (f).$$

For , $\lambda = \frac{1}{2}$ and $\upsilon = \frac{1}{2}$, we get

$$|T\Theta(f,u;a,b,x,1/2)| \leq \frac{1}{2^r} H \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^r \bigvee_a^b (f).$$

If we take $x = \frac{a+b}{2}$, then we derive the simpler inequality

$$\left| T\Theta\left(f,u;a,b,\frac{a+b}{2},1/2\right) \right| \leq \frac{1}{2^{2r}} H (b-a)^r \bigvee_a^b (f).$$

For , $\lambda = \frac{2}{3}$ and $\upsilon = \frac{1}{3}$, we have

$$|T\Theta(f,u;a,b,x,2/3,1/3)| \leq H \left(\frac{2}{3}\right)^r \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^r \bigvee_a^b (f).$$

whils if , $\lambda = 1-\upsilon$, we get

$$\begin{aligned} &|T(f,u;a,b,x,1-\upsilon,\upsilon)| \\ &\leq H \left(\frac{1}{2} + \left| \upsilon - \frac{1}{2} \right| \right)^r \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^r \bigvee_a^b (f). \end{aligned}$$

By making use of the inequalities (2.15) and (2.16) for $u(t) = \int_a^t w(\tau)\, d\tau$, $r = \frac{1}{\alpha} < 1$ and $H = \left(\int_a^b |w(\tau)|^\beta \, d\tau \right)^{1/\beta}$, where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we obtain the following weighted inequalities

$$\begin{aligned} &\left| T\Theta(f, \int_a^{\cdot} w(\tau)\, d\tau; a,b,x,\lambda,\upsilon) \right| \\ &\leq \max\left\{ \left(\frac{1}{2} + \left| \upsilon - \frac{1}{2} \right| \right)^{\frac{1}{\alpha}}, \left(\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right)^{\frac{1}{\alpha}} \right\} \\ &\times \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^{\frac{1}{\alpha}} \left(\int_a^b |w(\tau)|^\beta \, d\tau \right)^{1/\beta} \bigvee_a^b (f) \end{aligned} \tag{2.19}$$

for $\lambda, \upsilon \in [0,1]$.

In particular, for $\lambda = \upsilon$, we have

$$\begin{aligned} &\left| T\Theta(f, \int_a^{\cdot} w(\tau)\, d\tau; a,b,x,\upsilon) \right| \\ &\leq \left(\frac{1}{2} + \left| \upsilon - \frac{1}{2} \right| \right)^{\frac{1}{\alpha}} \\ &\times \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^{\frac{1}{\alpha}} \left(\int_a^b |w(\tau)|^\beta \, d\tau \right)^{1/\beta} \bigvee_a^b (f) \end{aligned} \tag{2.20}$$

where $T\Theta(f, \int_a^{\cdot} w(\tau)\, d\tau; a,b,x,\upsilon) := T\Theta(f, \int_a^{\cdot} w(\tau)\, d\tau; a,b,x,\upsilon,\upsilon)$

For $\lambda = 1/2$ $\upsilon = 1/2$ we get

$$\left|T\Theta\left(f, \int_a^t w(\tau)\, d\tau; a, b, x, 1/2\right)\right|$$
$$\leq \frac{1}{2^{1/\alpha}}\left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^{\frac{1}{\alpha}} \left(\int_a^b |w(\tau)|^\beta\, d\tau\right)^{1/\beta} \bigvee_a^b (f)$$

for $x \in (a, b)$.

If we take $x = \frac{a+b}{2}$, then we get

$$\left|T\Theta\left(f, \int_a^t w(\tau)\, d\tau; a, b, \frac{a+b}{2}, 1/2\right)\right| \leq \frac{1}{2^{2/\alpha}}(b-a)^{\frac{1}{\alpha}} \left(\int_a^b |w(\tau)|^\beta\, d\tau\right)^{1/\beta} \bigvee_a^b (f).$$

3. Some Compounding Rules for the Weighted Integral

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, put $h_k := x_{k+1} - x_k \quad (k = 0, \ldots, n-1)$ and $\nu(h) := \max\{h_k | k = 0, ..., n-1\}$ and any intermediate point vector $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$ such that $\xi_k \in [x_k, x_{k+1}]$ for $k = 0, 1, \ldots, n-1$, where $\lambda, \upsilon \in [0, 1]$. Define the sum related to the division I_n and the intermediate points ξ by

$$\begin{aligned}(3.1)\qquad &\Psi(f, I_n, \xi, \lambda, \upsilon)\\ &:= \sum_{k=0}^{n-1} f(x_{k+1}) \int_{(1-\lambda)\xi_k + \lambda x_{k+1}}^{x_{k+1}} w(s)\, ds + \sum_{k=0}^{n-1} f(x_k) \int_{x_k}^{\upsilon x_k + (1-\upsilon)\xi_k} w(s)\, ds\\ &+ \sum_{k=0}^{n-1} f(\xi_k) \int_{\upsilon x_k + (1-\upsilon)\xi_k}^{(1-\lambda)\xi_k + \lambda x_{k+1}} w(s)\, ds\end{aligned}$$

and the error of approximation for the weighted integral $\int_a^b f(s) w(s)\, ds$ by $\Psi(f, I_n, \xi, \lambda, \upsilon)$, namely

$$(3.2)\qquad \mathcal{R}(f, I_n, \xi, \lambda, \upsilon) := \Psi(f, I_n, \xi, \lambda, \upsilon) - \int_a^b f(s) w(s)\, ds.$$

The following result provides an error bound in the case when f is Lipschitzian and the weight w has the integral $\int_a^b |w(\tau)|^\beta\, d\tau$ finite.

Theorem 3.1. *Assume that f is Lipschitzian with the constant $L > 0$ and the weight w has the integral $\int_a^b |w(\tau)|^\beta\, d\tau$ finite where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let I_n be a division of $[a, b]$ and the intermediate point vector $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$. Then for $\lambda, \upsilon \in [0, 1]$ we have the error*

bounds

$$
\begin{aligned}
|\mathcal{R}(f, I_n, \xi, \lambda, \upsilon)| &\leq \frac{L\alpha}{\alpha+1}\left(\int_a^b |w(\tau)|^\beta d\tau\right)^{1/\beta} \nu(h) \\
&\times \left\{\left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}}\right]\left(\sum_{k=0}^{n-1}(\xi_k - x_k)\right)^{1/\alpha}\right. \\
&\left. + \left[(1-\lambda)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right]\left(\sum_{k=0}^{n-1}(x_{k+1} - \xi_k)\right)^{1/\alpha}\right\} \\
&\leq \frac{L\alpha}{\alpha+1}\left(\int_a^b |w(\tau)|^\beta d\tau\right)^{1/\beta} (b-a)^{1/\alpha} \\
&\times \left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}} + (1-\lambda)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right]\nu(h).
\end{aligned} \tag{3.3}
$$

Proof. By making use of the inequality (2.13) written for the interval $[x_k, x_{k+1}]$ and the intermediate points $\xi_k \in [x_k, x_{k+1}]$ for $k = 0, 1, \ldots, n-1$ we have

$$
\begin{aligned}
&\left| f(x_{k+1}) \int_{(1-\lambda)\xi_k + \lambda b}^{x_{k+1}} w(s)\, ds + f(x_k) \int_{x_k}^{\upsilon x_k + (1-\upsilon)\xi_k} w(s)\, ds \right. \\
&\left. f(\xi_k) \int_{\upsilon x_k + (1-\upsilon)\xi_k}^{(1-\lambda)\xi_k + \lambda x_{k+1}} w(s)\, ds - \int_{x_k}^{x_{k+1}} f(s)\, w(s)\, ds \right| \\
&\leq \frac{L\alpha}{\alpha+1}\left(\int_{x_k}^{x_{k+1}} |w(\tau)|^\beta d\tau\right)^{1/\beta} \\
&\times \left\{\left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}}\right](\xi_k - x_k)^{\frac{\alpha+1}{\alpha}} + \left[(1-\lambda)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right](x_{k+1} - \xi_k)^{\frac{\alpha+1}{\alpha}}\right\},
\end{aligned}
$$

for $k = 0, 1, \ldots, n-1$, which by summation gives

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \left| f(x_{k+1}) \int_{(1-\lambda)\xi_k + \lambda b}^{x_{k+1}} w(s)\, ds + f(x_k) \int_{x_k}^{\upsilon x_k + (1-\upsilon)\xi_k} w(s)\, ds \right. \\
&\left. f(\xi_k) \int_{\upsilon x_k + (1-\upsilon)\xi_k}^{(1-\lambda)\xi_k + \lambda x_{k+1}} w(s)\, ds - \int_{x_k}^{x_{k+1}} f(s)\, w(s)\, ds \right| \\
&\leq \frac{L\alpha}{\alpha+1} \sum_{k=0}^{n-1} \left(\int_{x_k}^{x_{k+1}} |w(\tau)|^\beta d\tau\right)^{1/\beta} \\
&\times \left\{\left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}}\right](\xi_k - x_k)^{\frac{\alpha+1}{\alpha}}\right. \\
&\left. + \left[(1-\lambda)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right](x_{k+1} - \xi_k)^{\frac{\alpha+1}{\alpha}}\right\}.
\end{aligned} \tag{3.4}
$$

By the generalized triangle inequality for modulus, we have

$$
\begin{aligned}
(3.5)\qquad & \sum_{k=0}^{n-1}\left|f\left(x_{k+1}\right)\int_{(1-\lambda)\xi_k+\lambda b}^{x_{k+1}} w\left(s\right)ds + f\left(x_k\right)\int_{x_k}^{\upsilon x_k+(1-\upsilon)\xi_k} w\left(s\right)ds\right. \\
& \left. f\left(\xi_k\right)\int_{\upsilon x_k+(1-\upsilon)\xi_k}^{(1-\lambda)\xi_k+\lambda x_{k+1}} w\left(s\right)ds - \int_{x_k}^{x_{k+1}} f\left(s\right)w\left(s\right)ds\right| \\
& \geq \left|\sum_{k=0}^{n-1} f\left(x_{k+1}\right)\int_{(1-\lambda)\xi_k+\lambda b}^{x_{k+1}} w\left(s\right)ds + \sum_{k=0}^{n-1} f\left(x_k\right)\int_{x_k}^{\upsilon x_k+(1-\upsilon)\xi_k} w\left(s\right)ds\right. \\
& \left.\sum_{k=0}^{n-1} f\left(\xi_k\right)\int_{\upsilon x_k+(1-\upsilon)\xi_k}^{(1-\lambda)\xi_k+\lambda x_{k+1}} w\left(s\right)ds - \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f\left(s\right)w\left(s\right)ds\right| \\
& = \left|\Psi\left(f, I_n, t, \lambda, \upsilon\right) - \int_a^b f\left(s\right)w\left(s\right)ds\right|
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta} \\
& \times\left\{\left[\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right]\left(\xi_k-x_k\right)^{\frac{\alpha+1}{\alpha}}+\left[\left(1-\lambda\right)^{\frac{\alpha+1}{\alpha}}+\lambda^{\frac{\alpha+1}{\alpha}}\right]\left(x_{k+1}-\xi_k\right)^{\frac{\alpha+1}{\alpha}}\right\} \\
& = \left[\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right]\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta}\left(\xi_k-x_k\right)^{\frac{\alpha+1}{\alpha}} \\
& + \left[\left(1-\lambda\right)^{\frac{\alpha+1}{\alpha}}+\lambda^{\frac{\alpha+1}{\alpha}}\right]\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta}\left(x_{k+1}-\xi_k\right)^{\frac{\alpha+1}{\alpha}}.
\end{aligned}
$$

By the Hölder's discrete inequality for $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, namely

$$
\sum_{k=0}^{n-1} a_i b_i \leq \left(\sum_{k=0}^{n-1} a_i^{\beta}\right)^{1/\beta}\left(\sum_{k=0}^{n-1} b_i^{\alpha}\right)^{1/\alpha}
$$

we deduce that

$$
\begin{aligned}
& \sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta}\left(\xi_k-x_k\right)^{\frac{\alpha+1}{\alpha}} \\
& \leq \left(\sum_{k=0}^{n-1}\left[\left(\int_{x_k}^{x_{k+1}}\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta}\right]^{\beta}\right)^{1/\beta}\left(\sum_{k=0}^{n-1}\left[\left(\xi_k-x_k\right)^{\frac{\alpha+1}{\alpha}}\right]^{\alpha}\right)^{1/\alpha} \\
& = \left(\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta}\left(\sum_{k=0}^{n-1}\left(\xi_k-x_k\right)^{\alpha+1}\right)^{1/\alpha} \\
& = \left(\int_a^b\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta}\left(\sum_{k=0}^{n-1}\left(\xi_k-x_k\right)^{\alpha+1}\right)^{1/\alpha}
\end{aligned}
$$

and, similarly

$$\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}d\tau\right)^{1/\beta}(x_{k+1}-\xi_k)^{\frac{\alpha+1}{\alpha}}$$
$$\leq\left(\int_a^b|w(\tau)|^{\beta}d\tau\right)^{1/\beta}\left(\sum_{k=0}^{n-1}(x_{k+1}-\xi_k)^{\alpha+1}\right)^{1/\alpha}.$$

Also observe that

$$\left(\sum_{k=0}^{n-1}(\xi_k-x_k)^{\alpha+1}\right)^{1/\alpha}=\left(\sum_{k=0}^{n-1}(\xi_k-x_k)^{\alpha}(\xi_k-x_k)\right)^{1/\alpha}$$
$$\leq\nu(h)\left(\sum_{k=0}^{n-1}(\xi_k-x_k)\right)^{1/\alpha}$$
$$\leq\nu(h)\left(\sum_{k=0}^{n-1}(x_{k+1}-x_k)\right)^{1/\alpha}=\nu(h)(b-a)^{1/\alpha}$$

and, similarly

$$\left(\sum_{k=0}^{n-1}(x_{k+1}-\xi_k)^{\alpha+1}\right)^{1/\alpha}\leq\nu(h)\left(\sum_{k=0}^{n-1}(x_{k+1}-\xi_k)\right)^{1/\alpha}\leq\nu(h)(b-a)^{1/\alpha}.$$

Therefore

$$\left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right]\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}d\tau\right)^{1/\beta}(\xi_k-x_k)^{\frac{\alpha+1}{\alpha}}$$
$$+\left[(1-\lambda)^{\frac{\alpha+1}{\alpha}}+\lambda^{\frac{\alpha+1}{\alpha}}\right]\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}d\tau\right)^{1/\beta}(x_{k+1}-\xi_k)^{\frac{\alpha+1}{\alpha}}$$
$$\leq\left(\int_a^b|w(\tau)|^{\beta}d\tau\right)^{1/\beta}\left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right]\nu(h)\left(\sum_{k=0}^{n-1}(\xi_k-x_k)\right)^{1/\alpha}$$
$$+\left(\int_a^b|w(\tau)|^{\beta}d\tau\right)^{1/\beta}\left[(1-\lambda)^{\frac{\alpha+1}{\alpha}}+\lambda^{\frac{\alpha+1}{\alpha}}\right]\nu(h)\left(\sum_{k=0}^{n-1}(x_{k+1}-\xi_k)\right)^{1/\alpha}$$

$$
\begin{aligned}
&= \left(\int_a^b |w(\tau)|^\beta \, d\tau\right)^{1/\beta} \nu(h) \\
&\times \left\{ \left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}}\right] \left(\sum_{k=0}^{n-1} (\xi_k - x_k)\right)^{1/\alpha} \right. \\
&\left. + \left[(1-\lambda)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right] \left(\sum_{k=0}^{n-1} (x_{k+1} - \xi_k)\right)^{1/\alpha} \right\} \\
&\leq \left(\int_a^b |w(\tau)|^\beta \, d\tau\right)^{1/\beta} (b-a)^{1/\alpha} \\
&\times \left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}} + (1-\lambda)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right] \nu(h)
\end{aligned}
$$

and by (3.4) the desired bound (3.3) is obtained. ∎

Remark 3.1. One can derive many particular cases from the above Theorem 3.1 by taking particular values of the parameters $\lambda, \upsilon \in [0,1]$ and the intermediate points $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$. If we consider the intermediate points $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ where $\mu_k = \frac{\xi_k + \xi_{k+1}}{2}$ for $k = 0, ..., n-1$ and consider the sum

$$
\begin{aligned}
\Psi(f, I_n, \mu, \lambda, \upsilon) := &\sum_{k=0}^{n-1} f(x_{k+1}) \int_{(1-\lambda)\frac{\xi_k+\xi_{k+1}}{2} + \lambda x_{k+1}}^{x_{k+1}} w(s)\, ds \\
&+ \sum_{k=0}^{n-1} f(x_k) \int_{x_k}^{\upsilon x_k + (1-\upsilon)\frac{\xi_k+\xi_{k+1}}{2}} w(s)\, ds \\
&+ \sum_{k=0}^{n-1} f\left(\frac{\xi_k + \xi_{k+1}}{2}\right) \int_{\upsilon x_k + (1-\upsilon)\frac{\xi_k+\xi_{k+1}}{2}}^{(1-\lambda)\frac{\xi_k+\xi_{k+1}}{2} + \lambda x_{k+1}} w(s)\, ds
\end{aligned}
$$

then from the first inequality in (3.3) we get the error bound

$$
\begin{aligned}
|\mathcal{R}(f, I_n, \mu, \lambda, \upsilon)| \leq &\frac{L\alpha}{(\alpha+1)\, 2^{1/\alpha}} \left(\int_a^b |w(\tau)|^\beta \, d\tau\right)^{1/\beta} (b-a)^{1/\alpha} \\
&\times \left[(1-\upsilon)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}} + (1-\lambda)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right] \nu(h),
\end{aligned} \tag{3.6}
$$

where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

The case when f is of bounded variation is incorporated in the following result.

Theorem 3.2. *Assume that f is of bounded variation on $[a,b]$ and the weight w has the integral $\int_a^b |w(\tau)|^\beta \, d\tau$ finite where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let I_n be a division of $[a,b]$ and the intermediate point vector $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$. Then for $\lambda, \upsilon \in [0,1]$ we have the error*

bounds

$$
\begin{aligned}
(3.7)\qquad |\mathcal{R}(f, I_n, \xi, \lambda, \upsilon)| &\leq \max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \\
&\times \left(\frac{1}{2}\nu(h)+\max_{k=0,\ldots,n-1}\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}} \\
&\times \left(\max_{k=0,\ldots,n-1}\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\bigvee_{a}^{b}(f) \\
&\leq \max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \\
&\times [\nu(h)]^{1/\alpha}\left(\max_{k=0,\ldots,n-1}\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\bigvee_{a}^{b}(f).
\end{aligned}
$$

Proof. By making use of the inequality (2.19) written for the interval $[x_k, x_{k+1}]$ and the intermediate points $\xi_k \in [x_k, x_{k+1}]$ for $k = 0, 1, \ldots, n-1$ we have

$$
\begin{aligned}
&\left|f(x_{k+1})\int_{(1-\lambda)\xi_k+\lambda b}^{x_{k+1}} w(s)\,ds + f(x_k)\int_{x_k}^{\upsilon x_k+(1-\upsilon)\xi_k} w(s)\,ds\right. \\
&\left. f(\xi_k)\int_{\upsilon x_k+(1-\upsilon)\xi_k}^{(1-\lambda)\xi_k+\lambda x_{k+1}} w(s)\,ds - \int_{x_k}^{x_{k+1}} f(s)\,w(s)\,ds\right| \\
&\leq \max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \\
&\times \left(\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\bigvee_{x_k}^{x_{k+1}}(f)
\end{aligned}
$$

for $k = 0, 1, \ldots, n-1$, which by summation gives

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\left|f(x_{k+1})\int_{(1-\lambda)\xi_k+\lambda b}^{x_{k+1}} w(s)\,ds + f(x_k)\int_{x_k}^{\upsilon x_k+(1-\upsilon)\xi_k} w(s)\,ds\right. \\
&\left. f(\xi_k)\int_{\upsilon x_k+(1-\upsilon)\xi_k}^{(1-\lambda)\xi_k+\lambda x_{k+1}} w(s)\,ds - \int_{x_k}^{x_{k+1}} f(s)\,w(s)\,ds\right| \\
&\leq \max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \\
&\times \sum_{k=0}^{n-1}\left(\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\bigvee_{x_k}^{x_{k+1}}(f).
\end{aligned}
$$

Now, observe that

$$\begin{aligned}
&\sum_{k=0}^{n-1}\left(\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\bigvee_{x_k}^{x_{k+1}}(f)\\
\leq\ &\max_{k=0,\ldots,n-1}\left[\left(\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)\right]\\
&\times\sum_{k=0}^{n-1}\bigvee_{x_k}^{x_{k+1}}(f)\\
=\ &\max_{k=0,\ldots,n-1}\left[\left(\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)\right]\\
&\times\bigvee_{a}^{b}(f).
\end{aligned}$$

Observe also that

$$\begin{aligned}
&\max_{k=0,\ldots,n-1}\left[\left(\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\right]\\
&\leq\max_{k=0,\ldots,n-1}\left(\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\\
&\times\max_{k=0,\ldots,n-1}\left(\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\\
&\leq\left(\max_{k=0,\ldots,n-1}\frac{x_{k+1}-x_k}{2}+\max_{k=0,\ldots,n-1}\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right)^{\frac{1}{\alpha}}\\
&\times\left(\max_{k=0,\ldots,n-1}\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta},
\end{aligned}$$

which proves the first part of (3.7).

The second part follows by the fact that

$$\max_{k=0,\ldots,n-1}\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\leq\max_{k=0,\ldots,n-1}\frac{x_{k+1}-x_k}{2}=\frac{1}{2}\nu(h).$$

∎

Remark 3.2. As noticed above, one can derive many particular cases from the above Theorem 3.2 by taking particular values of the parameters $\lambda,\upsilon\in[0,1]$ and the intermediate points $\xi=(\xi_0,\xi_1,\ldots,\xi_{n-1})$. If we consider the intermediate points $\mu=(\mu_0,\mu_1,\ldots,\mu_{n-1})$ where $\mu_k=\frac{\xi_k+\xi_{k+1}}{2}$ for $k=0,...,n-1$ then from the first inequality in (3.7) we get the error bound

$$\begin{aligned}
&|\mathcal{R}(f,I_n,\mu,\lambda,\upsilon)|\\
&\leq\frac{1}{2^{1/\alpha}}\max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}},\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\}\\
&\times[\nu(h)]^{\frac{1}{\alpha}}\left(\max_{k=0,\ldots,n-1}\int_{x_k}^{x_{k+1}}|w(\tau)|^{\beta}\,d\tau\right)^{1/\beta}\bigvee_{a}^{b}(f),
\end{aligned}\tag{3.8}$$

where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

4. APPLICATIONS FOR FOURIER TRANSFORM ON FINITE INTERVALS

The Fourier transform is one of the most essential tools of the mathematical sciences. It appears in unexpected applications to fields as diverse as differential equations, numerical analysis, probability theory, number theory, quantum mechanics, optics, medical imaging, and signal processing [25].

Let $g : [a, b] \to \mathbb{R}$ be a Lebesgue integrable mapping defined on the finite interval $[a, b]$ and $\mathcal{F}(g)$ its finite Fourier transform on the interval $[a, b]$, i.e.

$$\mathcal{F}(g)(t) := \int_a^b g(s)e^{-2\pi its}ds,\ t \in \mathbb{R}.$$

The following inequality was obtained in [26]:

Theorem 4.1. *Let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$. Then we have the inequality*

$$\left|\mathcal{F}(g)(t) - E(-2\pi ita, -2\pi itb)\int_a^b g(s)ds\right|$$
$$\leq \begin{cases} \frac{1}{3}\|g'\|_\infty\ (b-a)^2, & \text{if } g' \in L_\infty[a,b], \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}\,(b-a)^{1+\frac{1}{q}}\,\|g'\|_p, & \text{if } g' \in L_p[a,b]; \frac{1}{p}+\frac{1}{q}=1, p>1,; \\ (b-a)\,\|g'\|_1, \end{cases}$$

for all $t \in [a, b]$, $t \neq 0$, where E is the exponential mean of two complex numbers, that is,

$$E(z, w) := \begin{cases} \frac{e^z - e^w}{z-w} & \text{if } z \neq w, \\ exp(w) & \text{if } z = w, \end{cases} \quad , z, \omega \in \mathbb{C}.$$

The following inequality for the Fourier transform of functions of bounded variation was obtained in [27]:

Theorem 4.2. *Let $g : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$, then we have the inequality*

$$\left|\mathcal{F}(g)(t) - E(-2\pi ita, -2\pi itb)\int_a^b g(t)dt\right| \leq \frac{3}{4}\,(b-a)\bigvee_a^b(g)$$

for all $t \in [a, b]$, $t \neq 0$, where $\bigvee_a^b(g)$ is the total variation of g on $[a, b]$.

Now, for $w(s) = e^{-2\pi its}$, $t, s \in \mathbb{R}$ we have that

$$\int_p^q e^{-2\pi its}ds = (q-p)\,E(-2\pi itp, -2\pi itq)$$

for any p, q, t real numbers.

Using this formula, we then have by (2.5) that

$$\begin{aligned}
&T\Theta\left(f,\int_a^{\cdot}e^{-2\pi its}ds;a,b,x,\lambda,\upsilon\right)\\
&=f\left(b\right)\int_{(1-\lambda)x+\lambda b}^{b}e^{-2\pi its}ds+f\left(a\right)\int_a^{\upsilon a+(1-\upsilon)x}e^{-2\pi its}ds\\
&+f\left(x\right)\int_{\upsilon a+(1-\upsilon)x}^{(1-\lambda)x+\lambda b}e^{-2\pi its}ds-\int_a^b f\left(s\right)e^{-2\pi its}ds\\
&=f\left(b\right)\left(b-\left(1-\lambda\right)x-\lambda b\right)E(-2\pi itb,-2\pi it\left(\left(1-\lambda\right)x+\lambda b\right))\\
&+f\left(a\right)\left(\upsilon a+\left(1-\upsilon\right)x-a\right)E\left(-2\pi it\left(\upsilon a+\left(1-\upsilon\right)x\right),-2\pi ita\right)\\
&+f\left(x\right)\left(\left(1-\lambda\right)x+\lambda b-\upsilon a-(1-\upsilon)x\right)\\
&\times E(-2\pi it\left(\left(1-\lambda\right)x+\lambda b\right),-2\pi it\left(\upsilon a+(1-\upsilon)x\right))\\
&-\mathcal{F}(f)(t)\\
&=f\left(b\right)(1-\lambda)(b-x)E(-2\pi itb,-2\pi it((1-\lambda)x+\lambda b)\\
&+f\left(a\right)(1-\upsilon)(x-a)E\left(-2\pi it\left(\upsilon a+\left(1-\upsilon\right)x\right),-2\pi ita\right)\\
&+f\left(x\right)\left(\lambda(b-x)+\upsilon(x-a)\right)\\
&\times E(-2\pi it\left(\left(1-\lambda\right)x+\lambda b\right),-2\pi it\left(\upsilon a+(1-\upsilon)x\right))-\mathcal{F}(f)(t)
\end{aligned}$$

where $\lambda,\upsilon\in[0,1]$ and $x\in[a,b]$.

Now, by making use of the inequality (2.13) for $w(s)=e^{-2\pi its}$, which gives that $\left(\int_a^b\left|w\left(\tau\right)\right|^{\beta}d\tau\right)^{1/\beta}=(b-a)^{1/\beta}$, where $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we get the following error bound in approximating the Fourier transform

$$\begin{aligned}
&|(1-\lambda)f\left(b\right)(b-x)E(-2\pi itb,-2\pi it((1-\lambda)x+\lambda b)\\
&+(1-\upsilon)f\left(a\right)(x-a)E(-2\pi it(\upsilon a+(1-\upsilon)x),-2\pi ita)\\
&+f\left(x\right)\left(\lambda(b-x)+\upsilon(x-a)\right)\\
&\times E(-2\pi it((1-\lambda)x+\lambda b),-2\pi it(\upsilon a+(1-\upsilon)x))\\
&-\mathcal{F}(f)(t)|\\
&\leq\frac{L\alpha}{\alpha+1}\left(b-a\right)^{1/\beta}\\
&\times\left\{\left[\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right]\left(x-a\right)^{\frac{\alpha+1}{\alpha}}+\left[\left(1-\lambda\right)^{\frac{\alpha+1}{\alpha}}+\lambda^{\frac{\alpha+1}{\alpha}}\right]\left(b-x\right)^{\frac{\alpha+1}{\alpha}}\right\}
\end{aligned}\tag{4.1}$$

for $t,x\in[a,b]$, $t\neq0$ and $\lambda,\upsilon\in[0,1]$.

In particular, for $\lambda=\upsilon$, we derive

$$\begin{aligned}
&\Big|(1-\upsilon)f\left(b\right)(b-x)E(-2\pi itb,-2\pi it((1-\upsilon)x+\upsilon b)\\
&+(1-\upsilon)f\left(a\right)(x-a)E(-2\pi it(\upsilon a+(1-\upsilon)x),-2\pi ita)\\
&+\upsilon f\left(x\right)(b-a)E(-2\pi it((1-\upsilon)x+\upsilon b),-2\pi it(\upsilon a+(1-\upsilon)x))\\
&-\mathcal{F}(f)(t)\Big|\\
&\leq\frac{L\alpha}{\alpha+1}\left(b-a\right)^{1/\beta}\left[\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}}+\upsilon^{\frac{\alpha+1}{\alpha}}\right]\left[\left(x-a\right)^{\frac{\alpha+1}{\alpha}}+\left(b-x\right)^{\frac{\alpha+1}{\alpha}}\right],
\end{aligned}$$

for $t,x\in[a,b]$, $t\neq0$ and $\upsilon\in[0,1]$.

For $\lambda = 1/2$ and $\upsilon = 1/2$ we get

$$\begin{aligned}
&\Big|\frac{1}{2}f\left(b\right)\left(b-x\right)E(-2\pi itb,-2\pi it\left(\frac{b+x}{2}\right))\\
&+\frac{1}{2}f\left(a\right)\left(x-a\right)E(-2\pi it\left(\frac{a+x}{2}\right),-2\pi ita)\\
&+\frac{1}{2}f\left(x\right)\left(b-a\right)E(-2\pi it\left(\frac{b+x}{2}\right),-2\pi it\left(\frac{a+x}{2}\right))-\mathcal{F}(f)(t)\Big|\\
&\leq\frac{L\alpha}{2^{\frac{1}{\alpha}}\left(\alpha+1\right)}\left(b-a\right)^{1/\beta}\left[\left(x-a\right)^{\frac{\alpha+1}{\alpha}}+\left(b-x\right)^{\frac{\alpha+1}{\alpha}}\right].
\end{aligned}$$

for $t,x\in[a,b]$, $t\neq 0$.

If we take $x=\frac{a+b}{2}$, then we obtain

$$\begin{aligned}
&\Big|\left(\frac{b-a}{4}\right)f\left(b\right)E(-2\pi itb,-2\pi it\left(\frac{a+3b}{4}\right))\\
&+\left(\frac{b-a}{4}\right)f\left(a\right)E(-2\pi it\left(\frac{3a+b}{4}\right),-2\pi ita)\\
&+\left(\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)E(-2\pi it\left(\frac{a+3b}{4}\right),-2\pi it\left(\frac{3a+b}{4}\right))-\mathcal{F}(f)(t)\Big|\\
&\leq\frac{L\alpha}{2^{\frac{2}{\alpha}}\left(\alpha+1\right)}\left(b-a\right)^{1/\beta}\left(b-a\right)^{\frac{\alpha+1}{\alpha}}=\frac{L\alpha}{2^{\frac{2}{\alpha}}\left(\alpha+1\right)}\left(b-a\right)^{2}.
\end{aligned}$$

In addition, by making use the inequality (2.19) for $w(s)=e^{-2\pi its}$, $t,x\in[a,b]$, $t\neq 0$, we get the following error bound in approximating the Fourier transform in the case when f is of bounded variation

$$\begin{aligned}
&\Big|(1-\lambda)f\left(b\right)\left(b-x\right)E(-2\pi itb,-2\pi it((1-\lambda)\,x+\lambda b)\\
&+(1-\upsilon)f\left(a\right)\left(x-a\right)E(-2\pi it(\upsilon a+(1-\upsilon)x),-2\pi ita)\\
&+f\left(x\right)\left(\lambda(b-x)+\upsilon(x-a)\right)\\
&\times E(-2\pi it((1-\lambda)\,x+\lambda b),-2\pi it(\upsilon a+(1-\upsilon)x))-\mathcal{F}(f)(t)\Big|\\
&\leq\max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}},\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\}\\
&\times\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^{\frac{1}{\alpha}}\left(b-a\right)^{1/\beta}\bigvee_{a}^{b}\left(f\right)
\end{aligned}\tag{4.2}$$

for $t,x\in[a,b]$, $t\neq 0$, $\lambda,\upsilon\in[0,1]$ and $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

In particular, for $\lambda = \upsilon$, we derive that

$$\begin{aligned}
&\Big|(1-\upsilon)f(b)(b-x)E(-2\pi itb,-2\pi it((1-\upsilon)x+\upsilon b)\\
&+(1-\upsilon)f(a)(x-a)E(-2\pi it(\upsilon a+(1-\upsilon)x),-2\pi ita)\\
&+\upsilon f(x)(b-a)E(-2\pi it((1-\upsilon)x+\upsilon b),-2\pi it(\upsilon a+(1-\upsilon)x))\\
&-\mathcal{F}(f)(t)\Big|\\
&\leq\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^{\frac{1}{\alpha}}(b-a)^{1/\beta}\bigvee_a^b(f)
\end{aligned}$$

for $t,x\in[a,b]$, $t\neq 0$ and $\upsilon\in[0,1]$.

For $\lambda=1/2$ and $\upsilon=1/2$ we get

$$\begin{aligned}
&\Big|\frac{1}{2}f(b)(b-x)E(-2\pi itb,-2\pi it\left(\frac{b+x}{2}\right))\\
&+\frac{1}{2}f(a)(x-a)E(-2\pi it\left(\frac{a+x}{2}\right),-2\pi ita)\\
&+\frac{1}{2}f(x)(b-a)E(-2\pi it\left(\frac{b+x}{2}\right),-2\pi it\left(\frac{a+x}{2}\right))-\mathcal{F}(f)(t)\Big|\\
&\leq\frac{1}{2^{1/\alpha}}\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^{\frac{1}{\alpha}}(b-a)^{1/\beta}\bigvee_a^b(f)
\end{aligned}$$

for $t,x\in[a,b]$, $t\neq 0$.

If we take $x=\frac{a+b}{2}$, then we obtain

$$\begin{aligned}
&\Big|\left(\frac{b-a}{4}\right)f(b)E(-2\pi itb,-2\pi it\left(\frac{a+3b}{4}\right))\\
&+\left(\frac{b-a}{4}\right)f(a)E(-2\pi it\left(\frac{3a+b}{4}\right),-2\pi ita)\\
&+\left(\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)E(-2\pi it\left(\frac{a+3b}{4}\right),-2\pi it\left(\frac{3a+b}{4}\right))-\mathcal{F}(f)(t)\Big|\\
&\leq\frac{1}{2^{2/\alpha}}(b-a)\bigvee_a^b(f)
\end{aligned}$$

for $t\in[a,b]$, $t\neq 0$.

5. A Numerical Quadrature Formula

Let $I_n : a=x_0<x_1<...<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$, put $h_k:=x_{k+1}-x_k\quad(k=0,\ldots,n-1)$ and $\nu(h):=\max\{h_k|k=0,...,n-1\}$ and any intermediate point vector $\xi=(\xi_0,\xi_1,\ldots,\xi_{n-1})$ such that $\xi_k\in[x_k,x_{k+1}]$ for $k=0,1,\ldots,n-1$,

where $\lambda, \upsilon \in [0,1]$. Define the sum

$$\begin{aligned}(5.1)\qquad &\mathcal{E}\left(f, I_n, \xi, t, \lambda, \upsilon\right)\\ &:= \sum_{k=0}^{n-1}\Big[(1-\lambda) f\left(x_{k+1}\right)\left(x_{k+1}-\xi_k\right) E(-2\pi i t x_{k+1}, -2\pi i t((1-\lambda)\xi_k + \lambda x_{k+1})\\ &+ (1-\upsilon) f\left(x_k\right)\left(\xi_k - x_k\right) E(-2\pi i t(\upsilon x_k + (1-\upsilon)\xi_k), -2\pi i t x_k)\\ &+ f\left(\xi_k\right)\left(\lambda(x_{k+1}-\xi_k) + \upsilon(\xi_k - x_k)\right)\\ &\times E(-2\pi i t((1-\lambda)\xi_k + \lambda x_{k+1}), -2\pi i t(\upsilon x_k + (1-\upsilon)\xi_k))\Big]\end{aligned}$$

for $t, \xi_k \in [x_k, x_{k+1}]$, $t \neq 0$, $\lambda, \upsilon \in [0,1]$. In particular, for $\lambda = \upsilon$, we have

$$\begin{aligned}(5.2)\qquad &\mathcal{E}\left(f, I_n, \xi, t, \upsilon\right)\\ &:= \sum_{k=0}^{n-1}\Big[(1-\upsilon) f\left(x_{k+1}\right)\left(x_{k+1}-\xi_k\right) E(-2\pi i t x_{k+1}, -2\pi i t((1-\upsilon)\xi_k + \upsilon x_{k+1})\\ &+ (1-\upsilon) f\left(x_k\right)\left(\xi_k - x_k\right) E(-2\pi i t(\upsilon x_k + (1-\upsilon)\xi_k), -2\pi i t x_k)\\ &+ \upsilon f\left(\xi_k\right)\left(x_{k+1} - x_k\right)\\ &\times E(-2\pi i t((1-\upsilon)\xi_k + \upsilon x_{k+1}), -2\pi i t(\upsilon x_k + (1-\upsilon)\xi_k))\Big]\end{aligned}$$

for $t, \xi_k \in [x_k, x_{k+1}]$, $t \neq 0$, $\upsilon \in [0,1]$.

For $\lambda = 1/2$ and $\upsilon = 1/2$ we get

$$\begin{aligned}(5.3)\qquad &\mathcal{E}\left(f, I_n, \xi, t, 1/2\right)\\ &:= \sum_{k=0}^{n-1}\Big[\frac{1}{2} f\left(x_{k+1}\right)\left(x_{k+1}-\xi_k\right) E(-2\pi i t x_{k+1}, -2\pi i t\left(\frac{x_{k+1}+\xi_k}{2}\right))\\ &+ \frac{1}{2} f\left(x_k\right)\left(\xi_k - x_k\right) E(-2\pi i t\left(\frac{x_k+\xi_k}{2}\right), -2\pi i t x_k)\\ &+ \frac{1}{2} f\left(\xi_k\right)\left(x_{k+1} - x_k\right) E(-2\pi i t\left(\frac{x_{k+1}+\xi_k}{2}\right), -2\pi i t\left(\frac{x_k+\xi_k}{2}\right))\Big]\end{aligned}$$

for $t, \xi_k \in [x_k, x_{k+1}]$, $t \neq 0$.

If we take $\xi_k = \frac{x_k + x_{k+1}}{2}$, $k = 0, 1, \ldots, n-1$, then we get

$$\begin{aligned}(5.4)\qquad &\mathcal{M}\left(f, I_n, t\right)\\ &:= \sum_{k=0}^{n-1}\Big[\left(\frac{x_{k+1}-x_k}{4}\right) f\left(x_{k+1}\right) E(-2\pi i t x_{k+1}, -2\pi i t\left(\frac{x_k + 3x_{k+1}}{4}\right))\\ &+ \left(\frac{x_{k+1}-x_k}{4}\right) f\left(x_k\right) E(-2\pi i t\left(\frac{3x_k + x_{k+1}}{4}\right), -2\pi i t x_k)\\ &+ \left(\frac{x_{k+1}-x_k}{2}\right) f\left(\frac{x_{k+1}+x_k}{2}\right)\\ &\times E(-2\pi i t\left(\frac{x_k + 3x_{k+1}}{4}\right), -2\pi i t\left(\frac{3x_k + x_{k+1}}{4}\right))\Big],\end{aligned}$$

where $t \in [a,b]$ and $t \neq 0$.

The following approximation theorem holds.

Theorem 5.1. *Let $f : [a,b] \to \mathbb{C}$ be a Lipschitzian with the constant $L > 0$ on $[a,b]$. Then we have the representation*

$$\mathcal{F}(f)(t) = \mathcal{E}\left(f, I_n, \xi, t, \lambda, \upsilon\right) + R\left(f, I_n, \xi, t, \lambda, \upsilon\right); \tag{5.5}$$

where $\lambda, \upsilon \in [0,1]$ and $\mathcal{E}\left(f, I_n, \xi, t, \lambda, \upsilon\right)$, as defined in (5.1), approximates the Fourier transform $\mathcal{F}(f)(t)$ at every point $t \in [a,b]$. The error of approximation $R\left(f, I_n, \xi, t, \lambda, \upsilon\right)$ satisfies the bound

$$\begin{aligned} \left|R\left(f, I_n, \xi, t, \lambda, \upsilon\right)\right| &\leq \frac{L\alpha}{\alpha+1}\left(b-a\right) \\ &\quad \times \left\{\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}} + \left(1-\lambda\right)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right\} \nu(h) \end{aligned} \tag{5.6}$$

for $\alpha > 1$.

Proof. The proof follows by Theorem 3.1 by observing that the bound

$$\begin{aligned} &\frac{L\alpha}{\alpha+1}\left(\int_a^b \left|w\left(\tau\right)\right|^{\beta} d\tau\right)^{1/\beta} \left(b-a\right)^{1/\alpha} \\ &\times \left[\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}} + \left(1-\lambda\right)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right] \nu(h) \end{aligned}$$

becomes for $w(s) = e^{-2\pi its}$ that

$$\begin{aligned} &\frac{L\alpha}{\alpha+1}\left(b-a\right)^{1/\beta}\left(b-a\right)^{1/\alpha} \\ &\times \left[\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}} + \left(1-\lambda\right)^{\frac{\alpha+1}{\alpha}} + \lambda^{\frac{\alpha+1}{\alpha}}\right] \nu(h), \end{aligned}$$

namely the right hand side of (5.6). ∎

In particular, for $\lambda = \upsilon$, we have

$$\left|R\left(f, I_n, \xi, t, \upsilon\right)\right| \leq \frac{2L\alpha}{\alpha+1}(b-a)\left[\left(1-\upsilon\right)^{\frac{\alpha+1}{\alpha}} + \upsilon^{\frac{\alpha+1}{\alpha}}\right]\nu(h).$$

For $\lambda = 1/2$ and $\upsilon = 1/2$ we get

$$\left|R\left(f, I_n, \xi, t, 1/2\right)\right| \leq \frac{2L\alpha}{2^{\frac{1}{\alpha}}\left(\alpha+1\right)}(b-a)\nu(h).$$

In the case when $\xi_k = \frac{x_k + x_{k+1}}{2}$, then we get in a similar way that

$$\mathcal{F}(f)(t) = \mathcal{M}\left(f, I_n, t\right) + \widetilde{R}\left(f, I_n, t\right)$$

and the reminder $\widetilde{R}\left(f, I_n, t\right)$ satisfies the bound

$$\left|\widetilde{R}\left(f, I_n, t\right)\right| \leq \frac{L\alpha}{2^{\frac{2}{\alpha}}\left(\alpha+1\right)}\left(b-a\right)\nu(h).$$

Theorem 5.2. *Let $f : [a,b] \to \mathbb{C}$ be a mapping of bounded variation on $[a,b]$. Then we have the representation*

$$\mathcal{F}(f)(t) = \mathcal{E}\left(f, I_n, \xi, t, \lambda, \upsilon\right) + R\left(f, I_n, \xi, t, \lambda, \upsilon\right); \tag{5.7}$$

where $\lambda, \upsilon \in [0,1]$, $\mathcal{E}(f, I_n, \xi, t, \lambda, \upsilon)$*, as defined in (5.1), approximates the Fourier transform* $\mathcal{F}(f)$ *at every point* $t \in [a,b]$*. The error of approximation* $R(f, I_n, \xi, t, \lambda, \upsilon)$ *satisfies the bound*

$$|R(f, I_n, \xi, t, \lambda, \upsilon)| \leq \max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \nu(h) \bigvee_a^b (f) \tag{5.8}$$

for $\lambda, \upsilon \in [0,1]$.

Proof. The proof follows by Theorem 3.2 by observing that the bound

$$\max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \times [\nu(h)]^{1/\alpha} \left(\max_{k=0,\dots,n-1} \int_{x_k}^{x_{k+1}} |w(\tau)|^{\beta} d\tau\right)^{1/\beta} \bigvee_a^b (f).$$

becomes for $w(s) = e^{-2\pi i t s}$ that

$$\begin{aligned}
&\max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \\
&\times [\nu(h)]^{1/\alpha} \left(\max_{k=0,\dots,n-1} (x_{k+1}-x_k)\right)^{1/\beta} \bigvee_a^b (f) \\
&= \max\left\{\left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}}\right\} \\
&\times [\nu(h)]^{1/\alpha} (\nu(h))^{1/\beta} \bigvee_a^b (f),
\end{aligned}$$

which is the right hand side of (5.8). ∎

In particular, for $\lambda = \upsilon$, we get

$$|R(f, I_n, \xi, t, \upsilon)| \leq \left(\frac{1}{2}+\left|\upsilon-\frac{1}{2}\right|\right)^{\frac{1}{\alpha}} \nu(h) \bigvee_a^b (f).$$

while for $\lambda = 1/2$ and $\upsilon = 1/2$ we have

$$|R(f, I_n, \xi, t, 1/2)| \leq \frac{1}{2^{1/\alpha}} \nu(h) \bigvee_a^b (f).$$

If we take $\xi_k = \frac{x_k + x_{k+1}}{2}$, then we get in a similar way that

$$\mathcal{F}(f)(t) = \mathcal{M}(f, I_n, t) + \widetilde{R}(f, I_n, t)$$

and the reminder $\widetilde{R}(f, I_n, t)$ satisfies the bound

$$\left|\widetilde{R}(f, I_n, t)\right| \leq \frac{1}{2^{\frac{2}{\alpha}}} \nu(h) \bigvee_a^b (f).$$

In practical applications, it is more convenient to consider the equidistant partitioning of the interval $[a,b]$. Thus, let

$$I_n : x_k = a + k \cdot \frac{b-a}{n},\ k = 0,1,\ldots,n;$$

with $h_k = \frac{b-a}{n}$ be an equidistant partition of $[a,b]$.

Observe that

$$x_{k+1} - x_k = \frac{b-a}{n},\ \frac{x_{k+1}+x_k}{2} = a + \frac{2k+1}{2}\cdot\frac{b-a}{n},$$

$$\frac{3x_k + x_{k+1}}{4} = \frac{3\left(a + k\cdot\frac{b-a}{n}\right) + a + (k+1)\cdot\frac{b-a}{n}}{4} = a + \frac{4k+1}{4}\cdot\frac{b-a}{n},$$

$$\frac{x_k + 3x_{k+1}}{4} = \frac{a + k\cdot\frac{b-a}{n} + 3\left(a + (k+1)\cdot\frac{b-a}{n}\right)}{4} = a + \frac{4k+3}{4}\cdot\frac{b-a}{n}$$

and by employing (5.4) we can consider the quadrature rule

$$\begin{aligned}
(5.9)\qquad \mathcal{M}_n(f,t) := \frac{b-a}{4n}\sum_{k=0}^{n-1}\Big[& f\left(a + (k+1)\cdot\frac{b-a}{n}\right) \\
& \times E(-2\pi i t\left(a + (k+1)\cdot\frac{b-a}{n}\right), -2\pi i t\left(a + \frac{4k+3}{4}\cdot\frac{b-a}{n}\right)) \\
& + f\left(a + k\cdot\frac{b-a}{n}\right) \\
& \times E(-2\pi i t\left(a + \frac{4k+1}{4}\cdot\frac{b-a}{n}\right), -2\pi i t\left(a + k\cdot\frac{b-a}{n}\right)) \\
& + 2f\left(a + \frac{2k+1}{2}\cdot\frac{b-a}{n}\right) \\
& \times E(-2\pi i t\left(a + \frac{4k+3}{4}\cdot\frac{b-a}{n}\right), -2\pi i t\left(a + \frac{4k+1}{4}\cdot\frac{b-a}{n}\right))\Big].
\end{aligned}$$

The following corollary of Theorem 5.1 holds:

Corollary 5.3. *Let f be as defined in Theorem 5.1. Then we have the quadrature rule*

$$(5.10)\qquad \mathcal{F}(f)(t) = \mathcal{M}_n(f,t) + \mathcal{R}_n(f,t);$$

where $\mathcal{M}_n(f,t)$ approximates the Fourier transform at every point $t \in [a,b]$ while the error of approximation $\mathcal{R}_n(f,t)$ satisfies the bound

$$|\mathcal{R}_n(f,t)| \le \frac{L\alpha}{2^{\frac{2}{\alpha}}(\alpha+1)}\frac{(b-a)^2}{n}.$$

Remark 5.1. If we wish to approximate the Fourier transform $\mathcal{F}(f)(t)$ using the quadrature formula $\mathcal{M}_n(f,t)$ and (5.10) with an error less than a given $\epsilon > 0$, we have to divide the interval $[a,b]$ into at least $n_\epsilon \in N$ points, with

$$n_\epsilon := \left\lceil \frac{L\alpha}{2^{\frac{2}{\alpha}}(\alpha+1)}\frac{(b-a)^2}{\epsilon}\right\rceil + 1,$$

where $\lceil x\rceil$ is the integer part of the real number x.

The following corollary of Theorem 5.2 holds:

Corollary 5.4. *Let f be as defined in Theorem 5.2. Then we have*

$$\mathcal{F}(f)(t) = \mathcal{M}_n(f,t) + \mathcal{R}_n(f,t); \tag{5.11}$$

The error of approximation $\mathcal{R}_n(f,t)$ satisfies the bound

$$|\mathcal{R}_n(f,t)| \leq \frac{1}{2^{\frac{2}{\alpha}}} \frac{b-a}{n} \bigvee_a^b (f).$$

Remark 5.2. If we know the total variation $\bigvee_a^b(f)$ of f and would like to approximate the Fourier transform $\mathcal{F}(f)(t)$ by the adaptive quadrature formula $\mathcal{M}_n(f,t)$ with an error less than a given $\epsilon > 0$, we have to divide the interval $[a,b]$ into at least $n_\epsilon \in N$ points, where

$$n_\epsilon := \left\lceil \frac{1}{2^{2/\alpha}} \frac{b-a}{\epsilon} \bigvee_a^b (f) \right\rceil + 1.$$

6. NUMERICAL EXAMPLES

In the following section, we investigate some numerical examples to approximate the finite Fourier transform for $f(s) = s^2$ and $f(s) = cos(2\pi s)$.

The approximation formula is given below.

$$\begin{aligned}
\mathcal{M}_n(f,t) := \frac{b-a}{4n} \sum_{k=0}^{n-1} \Big[& f\left(a + (k+1)\cdot\frac{b-a}{n}\right) \\
& \times E(-2\pi i t\left(a + (k+1)\cdot\frac{b-a}{n}\right), -2\pi i t\left(a + \frac{4k+3}{4}\cdot\frac{b-a}{n}\right)) \\
& + f\left(a + k\cdot\frac{b-a}{n}\right) \\
& \times E(-2\pi i t\left(a + \frac{4k+1}{4}\cdot\frac{b-a}{n}\right), -2\pi i t\left(a + k\cdot\frac{b-a}{n}\right)) \\
& + 2f\left(a + \frac{2k+1}{2}\cdot\frac{b-a}{n}\right) \\
& \times E(-2\pi i t\left(a + \frac{4k+3}{4}\cdot\frac{b-a}{n}\right), -2\pi i t\left(a + \frac{4k+1}{4}\cdot\frac{b-a}{n}\right)) \Big].
\end{aligned}$$

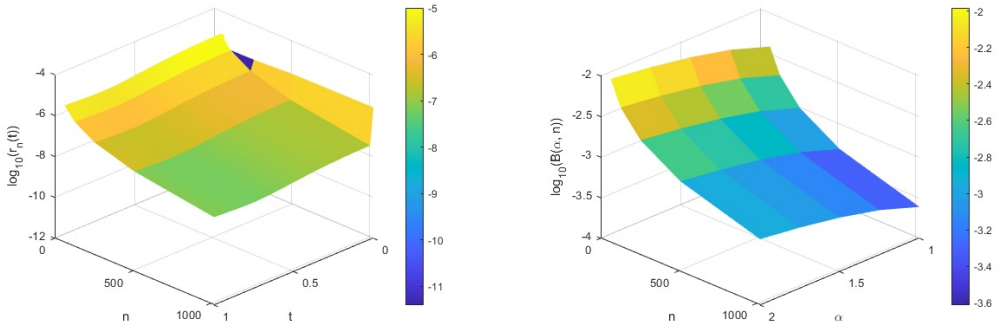
And the bound formula is defined as follows.

$$B(\alpha,n) = \frac{L\alpha}{2^{\frac{2}{\alpha}}(\alpha+1)} \frac{(b-a)^2}{n}. \tag{6.1}$$

6.1. $\mathbf{f(s) = s^2}$**.** The analytic form of the Fourier transform for $f(s) = s^2$ is given by,

$$\frac{\mathrm{i}}{4\,t^3\,\pi^3} - \frac{\mathrm{e}^{-\pi\,t\,2\mathrm{i}}\,(-4\,\pi^2\,t^2 + 4\mathrm{i}\,\pi\,t + 2)\,\mathrm{i}}{8\,t^3\,\pi^3}.$$

The following is a graphical representation of the accuracy of the approximation for the Fourier transform for various n, and $t \in (0,1]$. Also presented is the bound (6.1) for the same various n, and $\alpha \in (1,2]$.



(a) 3D-plot for $r_n(t)$ **(b)** 3D-plot for$B(1.5, n)$

Figure 1: Plots of the error $r_n(t) := |R_n(f,t)|$ *and* $B(\alpha = 1.5, n)$ *for the function* $f(s) = s^2$.

We can see in figure 3 that the approximation improves as n increases. Also, we note that $B(\alpha, n) \geq |R_n(f,t)|$ as predicted by the theoretical appoach. Below are some numerical observations of the approximation (and bound).

	$\mathbf{t = 0.2}$	$\mathbf{t = 0.4}$	$\mathbf{t = 0.6}$	$\mathbf{t = 0.8}$	$\mathbf{t = 1}$	$\mathbf{B(\alpha = 1.5, n)}$
$\mathbf{r_{64}}$	9.3632e-06	7.2657e-06	4.9387e-06	4.1829e-06	5.0866e-06	0.0074409
$\mathbf{r_{128}}$	2.3408e-06	1.8164e-06	1.2346e-06	1.0457e-06	1.2716e-06	0.0037205
$\mathbf{r_{256}}$	5.852e-07	4.541e-07	3.0866e-07	2.6142e-07	3.1789e-07	0.0018602
$\mathbf{r_{512}}$	1.463e-07	1.1352e-07	7.7164e-08	6.5356e-08	7.9473e-08	0.00093012
$\mathbf{r_{1024}}$	3.6575e-08	2.8381e-08	1.9291e-08	1.6339e-08	1.9868e-08	0.00046506

Table 6.1: Numerical results for r_n *and* $B(\alpha = 1.5, n)$ *for the function* $f(s) = s^2$.

We now present the approximation based on the real and imaginary part, compared to the analytic solution. Here is the real part of the analytic solution,

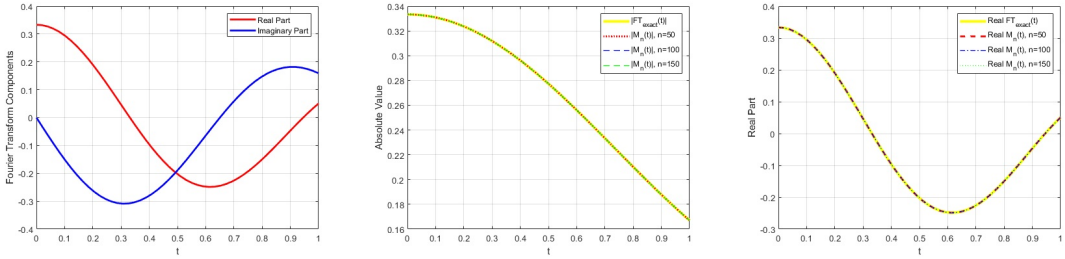
$$\frac{2\,t^2\,\pi^2\,\sin(2\,\pi\,t) - \sin(2\,\pi\,t) + 2\,\pi\,t\,\cos(2\,\pi\,t)}{4\,t^3\,\pi^3} + \frac{\left(\frac{1}{t^3}\right)}{4\,\pi^3},$$

and the imaginary part,

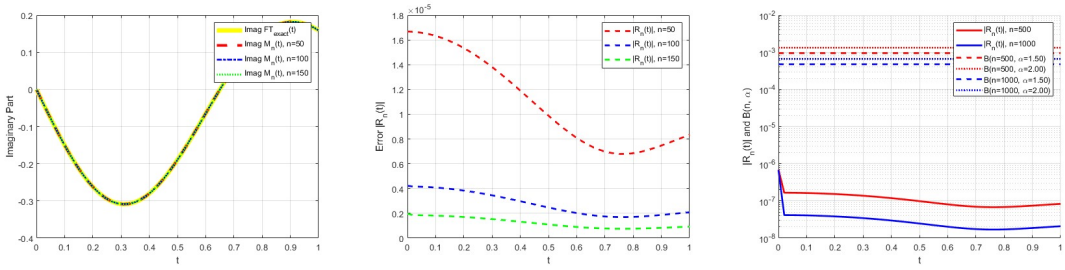
$$-\frac{\mathrm{e}^{-\pi\,t\,2\mathrm{i}}\,\left(-4\,\pi^2\,t^2 + 4\mathrm{i}\,\pi\,t + 2\right)\,\mathrm{i}}{8\,t^3\,\pi^3}$$

We can see in these plots that the approximation of both the real and imaginary part of the Fourier transform is good.

Remark 6.1. The plots and table presented in this section clearly show that the absolute error, $|R_n(f,t)|$ decreases when n increases. We also observed that the bound decreases as n increases. It is also evident that $|R_n(f,t)|$ is always smaller than the upper bound $B(\alpha, n)$ when applying equation (5.9) to approximate the Fourier transform for $f(s) = s^2$.



(a) Plot 1 - real and imaginary parts of the Fourier transform, Plot 2 - the FT and approximation and Plot 3 - real part of the FT and approximation.



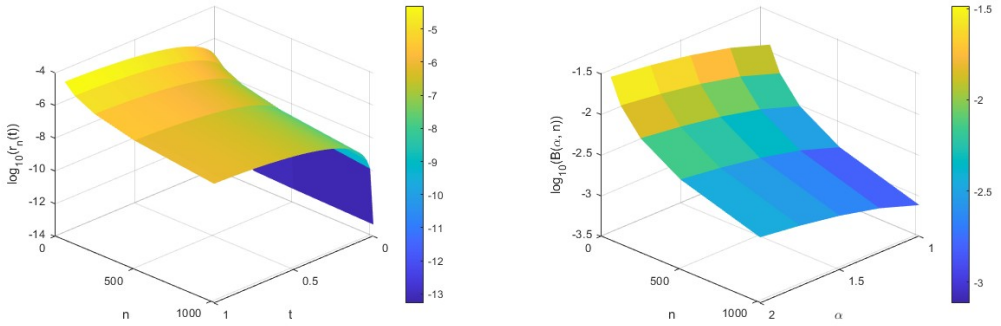
(b) Plot 1 - imaginary part of the FT and approximation, Plot 2 - error between the analytic solution and the approximation and Plot 3 - the bound and the effect of α and n on it.

Figure 2: Various plots for $f(s) = s^2$.

6.2. $\mathbf{f(s) = \cos(2\pi s)}$**.** The analytic expression of the Fourier transform for $f(s) = cos(2\pi s)$ is

$$\frac{t\,\left(\mathrm{e}^{-\pi\,t\,2\mathrm{i}}-1\right)\,\mathrm{i}}{2\,\pi\,\left(t-1\right)\,\left(t+1\right)}.$$

The following provides a graphical representation of the accuracy of the approximation for the Fourier transform for various n, and $t \in (0, 1)$. In addition, the corresponding error bound (6.1) is presented for the same various n, and $\alpha \in (1, 2]$.



(a) 3D-plot for $r_n(t)$ **(b)** 3D-plot for$B(1.5, n)$

Figure 3: Plots of the error $r_n(t) := |R_n(f, t)|$ *and* $B(\alpha = 1.5, n)$ *for the function* $f(s) = cos(2\pi s)$.

As shown in figure 3 the approximation improves with increasing values of n . Also, we observe that $B(\alpha, n) \geq |R_n(f,t)|$. Presented below are some numerical observations of the approximation (and bound).

	$\mathbf{t=0.1}$	$\mathbf{t=0.3}$	$\mathbf{t=0.5}$	$\mathbf{t=0.7}$	$\mathbf{t=0.9}$	$\mathbf{B(\alpha=1.5,n)}$
$\mathbf{r_{64}}$	9.9729e-07	8.5216e-06	2.1301e-05	3.5482e-05	4.6776e-05	0.023376
$\mathbf{r_{128}}$	2.4937e-07	2.1307e-06	5.3261e-06	8.8713e-06	1.1695e-05	0.011688
$\mathbf{r_{256}}$	6.2345e-08	5.3271e-07	1.3316e-06	2.2179e-06	2.9237e-06	0.0058441
$\mathbf{r_{512}}$	1.5586e-08	1.3318e-07	3.3289e-07	5.5448e-07	7.3092e-07	0.0029221
$\mathbf{r_{1024}}$	3.8966e-09	3.3295e-08	8.3224e-08	1.3862e-07	1.8273e-07	0.001461

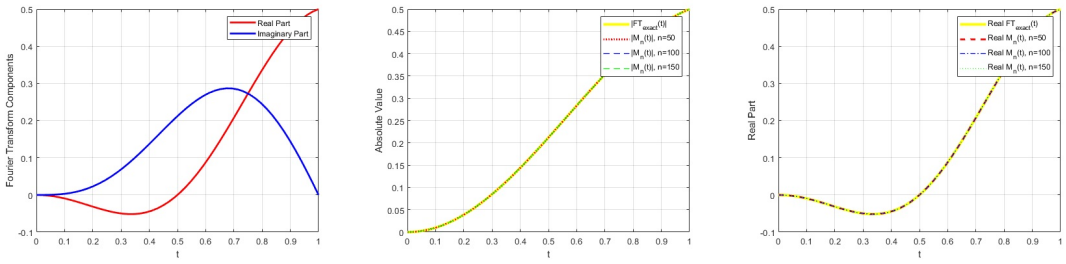
Table 6.2: Numerical results for r_n and $B(\alpha = 1.5, n)$ for the function $f(s) = cos(2\pi s)$.

We now present the approximation based on the real and imaginary parts, compared to the analytic solution.
The real part of the Fourier transforms for the given function is as follow:

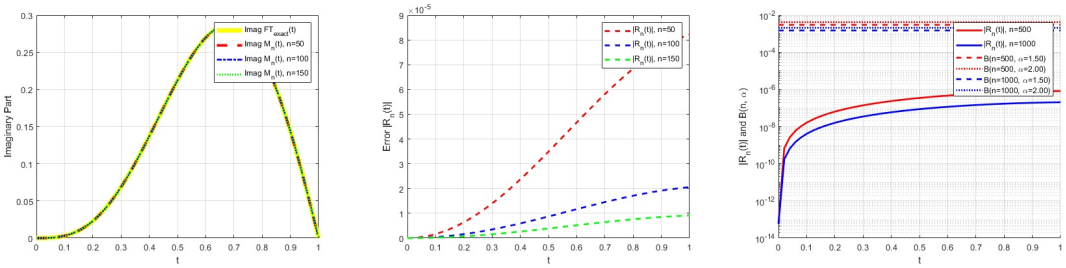
$$\begin{cases} \frac{1}{2} & \text{if } t = -1 \vee t = 1 \\ \frac{t\,\sin(2\,\pi\,t)}{2\,\pi\,(t^2-1)} & \text{if } t \neq -1 \wedge t \neq 1 \end{cases}$$

and the imaginary part of the Fourier fransforms is ,

$$\frac{t\,\left(\mathrm{e}^{-\pi\,t\,2\mathrm{i}}-1\right)\,\mathrm{i}}{2\,\pi\,\left(t-1\right)\,\left(t+1\right)}$$



(a) Plot 1 - real and imaginary parts of the Fourier transform, Plot 2 - the FT and approximation and Plot 3 - real part of the FT and approximation.



(b) Plot 1 - imaginary part of the FT and approximation, Plot 2 - error between the analytic solution and the approximation and Plot 3 - the bound and the effect of α and n on it.

Figure 4: Various plots for $f(s) = cos(2\pi s)$.

These plots illustrate that the approximation based on the real and imaginary parts of the Fourier transform is accurate.

Remark 6.2. The plots and table presented in this section clearly demonstrate that the absolute error, $|R_n(f,t)|$ decreases with increasing n. Additionally, we observed that the bound decreases as n increases. It is also evident that the absolute error is always smaller than the upper bound $B(\alpha,n)$ when applying equation (5.9) to approximate the Fourier transform for $f(s)=cos(2\pi s)$.

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