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DIVERGENCE CONDITIONS FOR CONTINUED RECIPROCAL POWERS

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ABSTRACT. We derive conditions on a continued reciprocal power's (strictly positive) terms sufficient for its divergence. The work involves reviewing and extending convergence criteria for continued supraunitary powers. We close with a brief comment on the history of a convergence condition for continued roots.

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1. INTRODUCTION

For $p > 1$, the infinite expression

$$\mathrm{K}_{i=0}^{\infty}(a_i)^{-p} = a_0 + \cfrac{1}{\left(a_1 + \cfrac{1}{\left(a_2 + \cfrac{1}{\left(\ddots\right)^p}\right)^p}\right)^p} , \tag{1.1}$$

is a continued reciprocal pth power, a name modelled on the German *reziproke Kettenpotenzen* coined by Laugwitz [11]. Using left-aligned exponent notation ${}^{y}(x) = x^y$, this can be written

$$\mathrm{K}_{i=0}^{\infty}(a_i)^{-p} = a_0 + {}^{-p}\big(a_1 + {}^{-p}\big(a_2 + {}^{-p}\big(\,\ldots\big)\big)\big) , \tag{1.2}$$

which in turn is a special case of the general continued power

$$\mathrm{K}_{i=0}^{\infty}(a_i)^{\rho} = \lim_{i\to\infty} a_0 + {}^{\rho}\big(a_1 + {}^{\rho}\big(a_2 + {}^{\rho}\big(\,\ldots + {}^{\rho}\big(a_i\big)\big)\big)\big) , \tag{1.3}$$

where we take ρ to be a real number. (An historical aspect of (1.3) for $0 < \rho < 1$ is discussed in Note 8.2 below.) We call expression (1.3) a continued supraunitary power when $\rho > 1$.

The earliest known appearance of (1.1) is in an 1832 paper by Doppler [3]. Apart from passing mentions in 1837 [13] and 1878 [2], to the best of our knowledge it is not seriously treated again until 1990, when Laugwitz considered continued operations, expressions of the form

$$f(a_0 + f(a_1 + f(a_2 + \ldots))) , \tag{1.4}$$

and used $f(x) = x^p$ and $f(x) = x^{-p}$ as principal examples. In carrying Laugwitz's work forward, Schönefuss [18] gave a sufficient convergence condition for (1.1), stated below in §2.4.

This paper presents conditions sufficient for the *divergence* of continued reciprocal powers of positive terms. After reviewing and extending convergence results for continued supraunitary pth powers, we show how the convergence of a pair of continued supraunitary p^2 powers is sufficient to cause divergence of a continued reciprocal pth power. We then develop boundedness conditions on the terms a_n which are sufficient for divergence. These conditions involve real-valued zeroes of certain trinomial equations; series expansions for these zeroes are given in Note 8.1.

2. PRELIMINARIES

We take $\mathbb{N}$ to be the nonnegative integers. For $p > 1$ and $n \in \mathbb{N}$, the nth approximant of the continued reciprocal pth power (1.2) is

$$\mathrm{K}_{i=0}^{n}(a_i)^{-p} = a_0 + {}^{-p}\big(a_1 + {}^{-p}\big(a_2 + {}^{-p}\big(\,\ldots + {}^{-p}\big(a_n\big)\big)\big)\big) .$$

For $m \in \mathbb{N}$ and $0 \le m \le n$, the (m, n)th tail (or simply a finite tail) is

$$\mathrm{K}_{i=m}^{n}(a_i)^{-p} = a_m + {}^{-p}\big(a_{m+1} + {}^{-p}\big(a_{m+2} + {}^{-p}\big(\,\ldots + {}^{-p}\big(a_n\big)\big)\big)\big) .$$

When p and the a_i are understood, we abbreviate these as K_0^n, and K_m^n, respectively. We restrict our attention to terms $a_n \in (0, \infty)$.

An nth approximant is merely a $(0, n)$th tail. The following result from [6] shows that the behaviour of an approximant is directly dependent on its tails.

Proposition 2.1. *For any $p \neq 0$, a continued pth power converges (resp. diverges) if, and only if, one of its tails converges (resp. diverges).*

2.1. **The sequence of tails of continued reciprocal powers.** Like continued fraction convergents, the even approximants K_0^{2k} of a continued reciprocal power form an increasing sequence bounded above by any odd approximant K_0^{2k+1}, and the odd approximants form a decreasing sequence bounded below by any even approximant. The following applies to continued reciprocal roots, continued fractions, and continued reciprocal powers.

Proposition 2.2 ([9], Proposition 3)**.** *Given $p > 0$ and $m, n \in \mathbb{N}$ with $0 \leq m < n$, the finite tails of the continued $-p$th power with terms $a_i > 0$, $i \in \mathbb{N}$, satisfy*

$$0 < \mathrm{K}_m^m < \mathrm{K}_m^{m+2} < \cdots < \mathrm{K}_m^{n-2} < \mathrm{K}_m^n < \mathrm{K}_m^{n-1} < \cdots < \mathrm{K}_m^{m+3} < \mathrm{K}_m^{m+1}$$

for $m \equiv n \pmod 2$, and

$$0 < \mathrm{K}_m^m < \mathrm{K}_m^{m+2} < \cdots < \mathrm{K}_m^{n-1} < \mathrm{K}_m^n < \mathrm{K}_m^{n-2} < \cdots < \mathrm{K}_m^{m+3} < \mathrm{K}_m^{m+1}$$

for $m \not\equiv n \pmod 2$.

Thus, when $m = 0$, the even and odd approximants comprise distinct, convergent sequences; a continued reciprocal power converges if the two limits are identical. But it is often easier to prove that the even and odd approximants converge to different limits. The following is adapted from Definition 1 in [9].

Definition 2.1 (divergence)**.** For $p > 1$, a continued reciprocal pth power of positive terms diverges if there exist real numbers d_0 and d_1 such that

$$\lim_{k\to\infty} \mathrm{K}_0^{2k} = d_0, \quad \lim_{k\to\infty} \mathrm{K}_0^{2k+1} = d_1, \text{ and } \ d_1 - d_0 > 0.$$

The closed interval $[d_0, d_1]$ is the continued reciprocal power's divergence interval.

2.2. **The sequence of tails for continued supraunitary powers.** When $p > 0$, the sequence of (m, n)th tails is nondecreasing as n increases without bound (cf. [6, Section 4]).

Proposition 2.3. *Given $p > 1$ and terms $a_i \geq 0$, $i \in \mathbb{N}$, a continued supraunitary pth power's finite tails form a nondecreasing sequence. That is, for any $m, k \in \mathbb{N}$,*

$$\mathrm{K}_m^m \leq \mathrm{K}_m^{m+1} \leq \mathrm{K}_m^{m+2} \leq \cdots \leq \mathrm{K}_m^{m+k} .$$

2.3. **Iterated functions and fixed points.** Let f map $(0, \infty)$ to itself. For n a positive integer, the nth iterate of f is

$$f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ terms}}(x) ,$$

and $f^0(x) = x$. Any $x_0 \in (0, \infty)$ satisfying $f(x_0) = x_0$ is a fixed point of f. For our purposes, x_0 is

(i) *strongly attracting* if $f'(x_0) < 1$,
(ii) *left-attracting* if $f'(x_0) = 1$, $f^n(x) \to x_0$ for $x \in (0, x_0]$, and $f^n(x) \not\to x_0$ for $x \in (x_0, \infty)$,
(iii) *right-attracting* if $f'(x_0) = 1$, $f^n(x) \to x_0$ for $x \in [x_0, \infty)$, and $f^n(x) \not\to x_0$ for $x \in (0, x_0)$,
(iv) *weakly attracting* if $f'(x_0) = 1$ and $f^n(x) \to x_0$ for $x \in (0, \infty)$, or
(v) *repelling* if $f'(x_0) > 1$.

2.4. **Previous work by Laugwitz and Schönefuss.** We will use the following two results from [11] and [18]. The first was originally stated for a monotonically decreasing, differentiable function whose derivative tends to 0 as $x \to \infty$.

Proposition 2.4 (Laugwitz 1990, Corollary 1). *For $p > 1$, if $a_n \to \infty$ as $n \to \infty$, then $\mathrm{K}_{i=0}^{\infty}(a_i)^{-p}$ converges.*

Proposition 2.5 (Schönefuss 1992, Corollary 2.4). *The continued reciprocal pth power of constant terms $a > 0$ converges if, and only if,*

$$a \geq Q_p = \frac{p-1}{p^{\frac{p}{p+1}}} \ .$$

Laugwitz and Schönefuss both wrote admiringly of a famous theorem from the 1840s (cf. [14, §11]) connecting continued fraction convergence to infinite series divergence, and lamented that an analogous result for continued pth powers seemed elusive.

Theorem (Seidel-Stern). *Given terms $a_n > 0$, the continued fraction $\mathrm{K}_{i=0}^{\infty}(a_i)^{-1}$ converges if, and only if, the series $\sum_{i=0}^{\infty} a_i$ diverges.*

Perhaps influenced by Seidel-Stern, Schönefuss stated his main convergence theorem in terms of an infinite series.

Theorem (Schönefuss 1992, Theorem 2.6). *If the series $\sum_{i=0}^{\infty} a_i a_{i+1}^p$ diverges such that*

$$\lim_{n\to\infty} p^{n+1} \left[\prod_{i=0}^{n} (a_i a_{i+1}^p + 1) \right]^{-1} = 0 \ ,$$

then $\mathrm{K}_{i=0}^{\infty} \ (a_i)^{-p}$ converges.

However, continued fraction convergence can also be linked to series divergence via the terms' alternating subsequences, a relationship which we will see is more germane to the present case.

3. CONTINUED SUPRAUNITARY POWERS

Continued supraunitary pth powers will be employed in §4 and §5; here we review and extend some of their known convergence criteria, and derive new ones. The constants

$$R_p = \frac{p-1}{p^{\frac{p}{p-1}}} \quad \text{and} \quad L_p = \frac{1}{p^{\frac{1}{p-1}}}$$

appear frequently; they originate in the following composite of [6, Theorem 1] and [18, Corollary 1.3].

Proposition 3.1. *The continued supraunitary pth power with constant terms $a_n = a$ converges if, and only if, $a \in [0, R_p]$. Furthermore,*

$$\mathrm{K}_{i=0}^{\infty}(a)^p \leq L_p \ ,$$

with equality if, and only if, $a = R_p$.

It has been remarked (e.g. [5]) that a continued first power is merely the infinite series $\sum_{i=0}^{\infty} a_i$. For $p > 1$, however, the terms of a continued supraunitary power may increase, and boundedness tests apply.

Proposition 3.2 ([6], Theorem 3)**.** *Given $p > 1$ and terms $a_n \geq 0$, if*

$$\limsup_{n\to\infty} a_n < R_p \ , \tag{3.1}$$

then $\mathrm{K}_{i=0}^{\infty}(a_i)^p$ converges to a limit not exceeding L_p. For $p \geq 1$, $\mathrm{K}_{i=0}^{\infty}(a_i)^p$ diverges if $\liminf_{n\to\infty} a_n > R_p$.

Proof. Most of this result is proved in [6]; what remains to be shown is that $\mathrm{K}_{i=0}^{\infty}(a_i)^p$ does not exceed L_p when (3.1) holds. If $\limsup_{n\to\infty} a_n = M < R_p$, then for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $a_n < M + \epsilon$ for all $n \geq N$. Specifically, choose $\epsilon = \epsilon_0 > 0$ such that $a_n < M + \epsilon_0 < R_p$. Set $u = M + \epsilon_0$, and use $a_n < u$ for $n \geq N$ to construct

$$\begin{aligned} \mathrm{K}_{i=N}^{n}(a_i)^p &= a_N + {}^p\big(a_{N+1} + {}^p\big(a_{N+2} + {}^p\big(\ldots + {}^p\big(a_n\big)\big)\big)\big) \\ &< u + {}^p\big(u + {}^p\big(u + {}^p\big(\ldots + {}^p\big(u\big)\big)\big)\big) = \mathrm{K}_{i=N}^{n}(u)^p \ . \end{aligned} \tag{3.2}$$

By Proposition 3.1, the right-hand side of (3.2) converges if, and only if, $u \in [0, R_p]$, whereby

$$\lim_{n\to\infty} \mathrm{K}_{i=N}^{n} \ (a_i)^p \leq L_p \ . \quad \blacksquare$$

Proposition 3.3. *Given $p > 1$ and terms $a_n \geq 0$, if there exists an $M > 0$ and an $N \in \mathbb{N}$ such that*

$$\left(\frac{a_n}{R_p}\right)^{p^n} \leq M \tag{3.3}$$

for all $n \geq N$, then $\mathrm{K}_{i=N}^{\infty}(a_i)^p$ converges, and

$$\mathrm{K}_{i=N}^{\infty} \ (a_i)^p \leq L_p M^{1/p^N} \ . \tag{3.4}$$

Proof. Condition (3.3) is proved in [7] (and see also [8]). To prove (3.4), set $b = R_p$, rewrite (3.3) as $a_n \leq bM^{1/p^n}$, and build approximants from right to left as follows:

$$\begin{aligned} {}^p(a_n) &\leq {}^p(b)M^{1/p^{n-1}} \\ \mathrm{K}_{n-1}^{n} = a_{n-1} + {}^p(a_n) &\leq bM^{1/p^{n-1}} + {}^p(b)M^{1/p^{n-1}} \\ &= M^{1/p^{n-1}}\big(b + {}^p(b)\big) \\ \mathrm{K}_{n-2}^{n} = a_{n-2} + {}^p(a_{n-1} + {}^p(a_n)) &\leq bM^{1/p^{n-2}} + M^{1/p^{n-2}}\big(b + {}^p(b)\big) \\ &= M^{1/p^{n-2}}\big(b + {}^p(b + {}^p(b))\big) \ . \end{aligned}$$

By induction on the index j, we have in general

$$\mathrm{K}_{i=n-j}^{n}(a_i)^p \leq M^{1/p^{n-j}} \, \mathrm{K}_{i=n-j}^{n}(b)^p \ .$$

Setting $N = n - j$, and replacing b with R_p, produces

$$\mathrm{K}_{i=N}^{n}(a_i)^p \leq M^{1/p^N} \, \mathrm{K}_{i=0}^{n}(R_p)^p \ ,$$

and by Proposition 3.1, as n grows without bound,

$$\mathrm{K}_{i=N}^{\infty}(a_i)^p \leq M^{1/p^N} L_p \ . \quad \blacksquare$$

As with infinite series, there is a ratio test of consecutive terms for continued supraunitary powers. The following is extended from [6, Theorem 5].

Proposition 3.4. *For $p > 1$ and terms $a_n > 0$, if there exists an $N \in \mathbb{N}$ such that*

$$\frac{a_{n+1}^p}{a_n} \leq (R_p)^{p-1} = \frac{(p-1)^{p-1}}{p^p} \ ,$$

for all $n \geq N$, *then* $\mathrm{K}_{i=N}^{\infty}(a_i)^p$ *converges, and its limit is bounded above by*

$$a_N \cdot \frac{p}{p-1} \ . \tag{3.5}$$

Although the bound (3.5) is not stated in [6, Theorem 5], the proof there includes it.

To close this section, we address a case (not covered by Proposition 3.2) in which the terms approach R_p from above. This requires a lemma, which for $p \in (0,1)$ is sketched in [17, Appendix 1], proved in [11, Theorem 1], and is implicit in [7, Example 2].

Lemma 3.5. *Given* $p > 1$ *and a constant* $c > 0$,

$$\mathrm{K}_{i=0}^{\infty}\left(\frac{R_p}{c^{1/p^i}}\right)^p = \frac{1}{c}\cdot L_p \quad \textit{and} \quad \mathrm{K}_{i=0}^{\infty}\big(R_p c^{1/p^i}\big)^p = c\cdot L_p \ .$$

Proof sketch. We claim that

$$L_p = \mathrm{K}_{i=0}^{\infty}\left(R_p\right)^p = c\cdot \mathrm{K}_{i=0}^{\infty}\left(\frac{R_p}{c^{1/p^i}}\right)^p = \frac{1}{c}\cdot \mathrm{K}_{i=0}^{\infty}(R_p c^{1/p^i})^p \ , \tag{3.6}$$

from which the lemma follows. The first equality in (3.6) is from Proposition 3.1. The second arises from

$$L_p = \frac{c}{c}\mathrm{K}_0^n\left(R_p\right)^p = \frac{c}{c}\big(R_p + {}^p\big(R_p + {}^p\big(R_p + {}^p\big(\ldots + {}^p\big(R_p\big)\big)\big)\big)\big) \ ,$$

where there are $n+1$ terms on the right. Distributing the denominator to the right produces

$$L_p = c\cdot\left(R_p c^{-1} + {}^p\big(R_p c^{-1/p} + {}^p\big(R_p c^{-1/p^2} + {}^p\big(\ldots + {}^p\big(R_p c^{-1/p^n}\big)\big)\big)\big)\right) \ .$$

Likewise, the third equality follows by distributing the numerator to the right. ∎

Proposition 3.6. *Given* $p > 1$ *and* $f_p : \mathbb{N} \to \mathbb{R}$, *let*

$$a_n = R_p + \frac{1}{f_p(n)} \ . \tag{3.7}$$

If there exists an $N \in \mathbb{N}$ *such that* $f_p(n) \geq p^n$ *for all* $n \geq N$, *then* $\mathrm{K}_{i=0}^{\infty}(a_i)^p$ *converges.*

Proof. We compare (3.7)'s a_n with $b_n = R_p c^{1/p^n}$ from Lemma 3.5, and show that the constant c can be chosen so that $\mathrm{K}_{i=0}^{\infty}(a_i)^p \leq \mathrm{K}_{i=0}^{\infty}(b_i)^p$. Introducing the quantity A_n, we have

$$a_n = R_p\left(1 + \frac{R_p^{-1}}{f_p(n)}\right) \leq R_p c^{1/p^n} = b_n$$

$$A_n := \left(1 + \frac{R_p^{-1}}{f_p(n)}\right)^{p^n} \leq c \ .$$

In the limit,

$$\lim_{n\to\infty} A_n = \begin{cases} 1 & \text{if } \exists\, N_1 \text{ such that } f_p(n) > p^n \text{ for } n \geq N_1 \ , \\ e^{1/R_p} & \text{if } \exists\, N_2 \text{ such that } f_p(n) = p^n \text{ for } n \geq N_2 \ , \\ \infty & \text{if } \exists\, N_3 \text{ such that } f_p(n) < p^n \text{ for } n \geq N_3 \ . \end{cases}$$

Note that $e^{1/R_p} > e > 1$, because $1/R_p > 1$. Consequently, $A_n \leq e^{1/R_p}$ if there exists $N = \max(N_1, N_2)$ such that $f_p(n) \geq p^n$ for $n \geq N$. When

$$c \geq e^{1/R_p} \ ,$$

we have $a_n \leq b_n$, and thus $\mathrm{K}_{i=0}^{\infty}(a_i)^p \leq \mathrm{K}_{i=0}^{\infty}(b_i)^p$; that is, $\mathrm{K}_{i=0}^{\infty}(a_i)^p$ is increasing, bounded above, and therefore convergent. ∎

4. SPINES

A theorem of Scott and Wall [19] demonstrates the roles of the alternating subsequences of terms in a convergent continued fraction.

Theorem (Scott and Wall 1947). *Let $\{a_n\}$ be a sequence of complex numbers. A necessary condition for the continued fraction $\mathrm{K}_{i=1}^{\infty}(a_i)^{-1}$ to converge is that one of the following statements holds.*

(i) $\sum |a_{2i+1}|$ diverges.
(ii) $\sum |a_{2i+1}(a_2 + a_4 + \cdots + a_{2i})^2|$ diverges.
(iii) $\lim_{i\to\infty} |a_2 + a_4 + \cdots + a_{2i}| = \infty$.

The alternating subsequences are similarly influential for continued reciprocal pth powers; but more structure is revealed when $p > 1$. Instead of infinite series, two continued supraunitary p^2 powers are required.

Definition 4.1 (Spines). Given $p \neq 0$, $k, m, n \in \mathbb{N}$ with $k \leq m$, and terms $a_n > 0$, the (k, m)th even spine is the continued supraunitary p^2 power

$$\mathrm{S}_{i=k}^{m}(a_{2i})^{p^2} = a_{2k} + {}^{p^2}\!\Big(a_{2k+2} + {}^{p^2}\!\Big(a_{2k+4} + {}^{p^2}\!\Big(\ldots + {}^{p^2}\!\Big(a_{2m}\Big)\Big)\Big)\Big) \ ,$$

while the (k, m)th odd spine is

$$\mathrm{S}_{i=k}^{m}(a_{2i+1})^{p^2} = a_{2k+1} + {}^{p^2}\!\Big(a_{2k+3} + {}^{p^2}\!\Big(a_{2k+5} + {}^{p^2}\!\Big(\ldots + {}^{p^2}\!\Big(a_{2m+1}\Big)\Big)\Big)\Big) \ .$$

Loosely speaking, spines are the remnants of a general continued ρth power (and in this paragraph, ρ is any non-zero real number) whose alternating terms have vanished to zero. For instance, when $k = 0$ (and ignoring the possibility of division by 0 when ρ is negative), one can write

$$\mathrm{S}_{i=0}^{m}(a_{2i})^{\rho^2} = a_0 + {}^{\rho}\!\Big(0 + {}^{\rho}\!\Big(a_2 + {}^{\rho}\!\Big(0 + {}^{\rho}\!\Big(\ldots + {}^{\rho}\!\Big(a_{2m}\Big)\Big)\Big)\Big)\Big)$$
$${}^{\rho}\!\Big(\mathrm{S}_{i=0}^{m}(a_{2i+1})^{\rho^2}\Big) = 0 + {}^{\rho}\!\Big(a_1 + {}^{\rho}\!\Big(0 + {}^{\rho}\!\Big(a_3 + {}^{\rho}\!\Big(0 + {}^{\rho}\!\Big(\ldots + {}^{\rho}\!\Big(a_{2m+1}\Big)\Big)\Big)\Big)\Big)\Big) \ .$$

While these formal expressions hold whether ρ is positive or negative, we emphasise that, for $|\rho| > 1$, spines are always continued supraunitary ρ^2 powers. A key property of a continued reciprocal power's spines is

Lemma 4.1. *Given $p > 1$, $k, m, n \in \mathbb{N}$ with $k \leq m$, and terms $a_n > 0$,*

$$\mathrm{K}_{i=2k}^{2m}(a_i)^{-p} < \mathrm{S}_{i=k}^{m}(a_{2i})^{p^2} \quad \textit{and} \quad {}^{-p}\!\Big(\mathrm{S}_{i=k}^{m}(a_{2i+1})^{p^2}\Big) < \mathrm{K}_{i=2k}^{2m+1}(a_i)^{-p} \ .$$

Proof. To prove the first inequality, begin with

$$a_{2m-1} + {}^{-p}(a_{2m}) >{}^{-p}(a_{2m})$$
$${}^{-p}(a_{2m-1} + {}^{-p}(a_{2m})) < {}^{p^2}(a_{2m})$$
$$a_{2m-2} + {}^{-p}(a_{2m-1} + {}^{-p}(a_{2m})) < a_{2m-2} + {}^{p^2}(a_{2m})$$
$$\mathrm{K}_{i=2m-2}^{2m}(a_i)^{-p} < \mathrm{S}_{i=m-1}^{m}(a_{2i})^{p^2} \ .$$

Induction on the index $j \in \mathbb{N}$ shows that

$$\mathrm{K}_{2m-2j}^{2m} < \mathrm{S}_{m-j}^{m} \ ,$$

and the substitution $k = m - j$ yields

$$\mathrm{K}_{i=2k}^{2m}(a_i)^{p} < \mathrm{S}_{i=k}^{m}(a_{2i})^{p} \ .$$

For the second inequality, begin with

$$ {}^{-p}\Big(\mathrm{S}_{i=m}^{m}(a_{2i+1})^{p^2}\Big) = {}^{-p}(a_{2m+1}) < a_{2m} + {}^{-p}(a_{2m+1}) = \mathrm{K}_{i=2m}^{2m+1}(a_i)^{-p} $$

$$ \mathrm{S}_{m-1}^{m} = a_{2m-1} + {}^{p^2}\Big(\mathrm{S}_{m}^{m}\Big) > a_{2m-1} + {}^{-p}\Big(\mathrm{K}_{2m}^{2m+1}\Big) = \mathrm{K}_{2m-1}^{2m+1} $$

$$ {}^{-p}\Big(\mathrm{S}_{m-1}^{m}\Big) < {}^{-p}\Big(\mathrm{K}_{2m-1}^{2m+1}\Big) < a_{2m-2} + {}^{-p}\Big(\mathrm{K}_{2m-1}^{2m+1}\Big) = \mathrm{K}_{2m-2}^{2m+1} $$

By induction on j,

$$ {}^{-p}\Big(\mathrm{S}_{m-j}^{m}\Big) < \mathrm{K}_{2m-2j}^{2m+1} \ , $$

and setting $k = m - j$ we have

$$ {}^{-p}\Big(\mathrm{S}_{i=k}^{m}(a_{2i+1})^{p^2}\Big) < \mathrm{K}_{i=2k}^{2m+1}(a_i)^{-p} \ . \tag{4.1} $$ ∎

5. DIVERGENCE VIA SPINES

A continued reciprocal power's spines can be used to create a divergence interval. The principal result is a straightforward application of Lemma 4.1.

Theorem 5.1. *Given $p > 1$ and terms $a_n > 0$, if there exists an $N \in \mathbb{N}$ such that*

$$ \mathrm{S}_{i=N}^{\infty}(a_{2i})^{p^2} < {}^{-p}\Big(\mathrm{S}_{i=N}^{\infty}(a_{2i+1})^{p^2}\Big) \ , \tag{5.1} $$

then $\mathrm{K}_{i=0}^{\infty}\ (a_i)^{-p}$ diverges.

Proof. The given inequality and Lemma 4.1 imply

$$ \mathrm{K}_{i=2N}^{\infty}(a_i)^{-p} \le \mathrm{S}_{i=N}^{\infty}(a_{2i})^{p^2} < {}^{-p}\Big(\mathrm{S}_{i=N}^{\infty}(a_{2i+1})^{p^2}\Big) \le \mathrm{K}_{i=2N}^{\infty}(a_i)^{-p} \ . $$

Setting $d_0 = \mathrm{S}_{i=N}^{\infty}(a_{2i})^{p^2}$ and $d_1 = {}^{-p}\big(\mathrm{S}_{i=N}^{\infty}(a_{2i+1})^{p^2}\big)$, the difference $d_1 - d_0 > 0$ comprises a divergence interval for K_{N}^{∞}, and therefore K_{0}^{∞} diverges as well. ∎

When does Theorem 5.1 apply? We present three cases, beginning with spines satisfying Proposition 3.2.

Corollary 5.2. *Given $p > 1$ and terms $a_n > 0$, if*

$$ \limsup_{n\to\infty}\ a_n < R_{p^2} = \frac{p^2 - 1}{(p^2)^{\frac{p^2}{p^2-1}}} \ , \tag{5.2} $$

then $\mathrm{K}_{i=0}^{\infty}\ (a_i)^{-p}$ diverges.

Proof. If $\limsup\ a_n < R_{p^2}$, then also $\limsup\ a_{2k} < R_{p^2}$ and $\limsup\ a_{2k+1} < R_{p^2}$, $k \in \mathbb{N}$. By Proposition 3.2, there exist $N_0, N_1 \in \mathbb{N}$ such that

$$ \mathrm{S}_{i=N_0}^{\infty}(a_{2i})^{p^2} \quad \text{and} \quad \mathrm{S}_{i=N_1}^{\infty}(a_{2i+1})^{p^2} $$

both converge. Set $N = \max\{N_0, N_1\}$; by Proposition 3.2,

$$ \mathrm{S}_{i=N}^{\infty}(a_{2i})^{p^2} \le \frac{1}{(p^2)^{\frac{1}{p^2-1}}} < 1 < (p^2)^{\frac{p}{p^2-1}} \le {}^{-p}\Big(\mathrm{S}_{i=N}^{\infty}(a_{2i+1})^{p^2}\Big) \ . $$

By Theorem 5.1, the even and odd spines form a divergence interval, and $\mathrm{K}_{i=2N}^{\infty}(a_i)^{-p}$ diverges. ∎

This is a template for subsequent proofs: 1) show that the even and odd spines converge to limits strictly less than 1, so that 2) the $-p$th power of the odd spine will have a lower bound greater than 1, resulting in 3) a non-empty divergence interval containing an open interval around 1. When a continued reciprocal power's spines satisfy Proposition 3.3, for instance, we have

Corollary 5.3. *Given $p > 1$ and terms $a_n > 0$, if there exists $N \in \mathbb{N}$ and $M > 0$ such that*

$$\left(\frac{a_n}{R_{p^2}}\right)^{p^{2n}} \leq M < \frac{1}{(L_{p^2})^{p^{2N}}} \tag{5.3}$$

for all $n \geq N$, then $\mathrm{K}_{i=N}^{\infty}(a_i)^{-p}$ diverges.

Proof. By Proposition 3.3, the bound on M in (5.3) along with $L_{p^2} < 1$ imply that the even and odd spines converge to limits less than $L_{p^2} M^{1/p^{2N}} < 1$. Hence, the $-p$th power of the odd spine limit is greater than 1, and Theorem 5.1 applies. ∎

Spines satisfying the ratio test of Proposition 3.4 also cause a continued reciprocal power's divergence. For convenience, define

$$r_{p^2} := (R_{p^2})^{p^2-1} = \frac{(p^2-1)^{p^2-1}}{(p^2)^{p^2}} \ .$$

Corollary 5.4. *Suppose there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$,*

$$0 < a_n < \frac{p^2-1}{p^2}$$

and

$$\frac{(a_{2n+2})^{p^2}}{a_{2n}} \leq r_{p^2} \quad \textit{and} \quad \frac{(a_{2n+3})^{p^2}}{a_{2n+1}} \leq r_{p^2} \ .$$

Then $\mathrm{K}_{i=N}^{\infty}(a_i)^{-p}$ diverges.

Proof. By Proposition 3.4, the even and odd spines converge, and

$$\mathrm{S}_N^{\infty}(a_{2i})^{p^2} \leq a_{2N}\frac{p^2}{p^2-1}$$
$$\mathrm{S}_N^{\infty}(a_{2i+1})^{p^2} \leq a_{2N+1}\frac{p^2}{p^2-1} \ .$$

Since $a_n < (p^2-1)/p^2 < 1$, the even and odd spines have upper bounds less than 1. Thus the $-p$th power of the odd spine is greater than 1, and $\mathrm{K}_{i=N}^{\infty}(a_i)^{-p}$ diverges by Theorem 5.1. ∎

6. A PERIOD-2 CONTINUED RECIPROCAL POWER

We now take up the question of boundedness conditions on the terms a_n sufficient for divergence of a continued reciprocal power. In previous work (e.g. [6], [9]), we used continued powers of constant or periodic terms to construct bounds on continued powers of arbitrary terms. We initiate a similar approach here, using the smooth function

$$\Psi(x) = \Psi_{u,v}(x) = u + (v + x^{-p})^{-p} \ ,$$

where $x \in (0, \infty)$; $u, v \in [0, \infty)$ are constants; $\lim_{x\to 0} \Psi(x) = u$; and the unique horizontal asymptote is

$$\lim_{x\to\infty} \Psi(x) = u + v^{-p} \ .$$

Examples of $\Psi(x)$ are shown in Figure 1.

Define a period-2 sequence $\{b_n\}_{n\in\mathbb{N}}$ by $b_{2j} = u$ and $b_{2j+1} = v$ for $j \in \mathbb{N}$. The approximants $\mathrm{K}_{i=0}^{2k}(b_i)^{-p}$ and $\mathrm{K}_{i=0}^{2k+1}(b_i)^{-p}$ are identical to iterates of Ψ using the initial values u and $u + v^{-p}$, respectively:

$$\mathrm{K}_{i=0}^{2k}(b_i)^{-p} = \underbrace{u +{}^{-p}\big(v +{}^{-p}\big(u +{}^{-p}\big(\ldots +{}^{-p}(u)\big)\big)\big)}_{2k+1 \text{ terms}} = \Psi^k(u)\ , \tag{6.1}$$

$$\mathrm{K}_{i=0}^{2k+1}(b_i)^{-p} = \underbrace{u +{}^{-p}\big(v +{}^{-p}\big(u +{}^{-p}\big(\ldots +{}^{-p}(v)\big)\big)\big)}_{2k+2 \text{ terms}} = \Psi^k(u + v^{-p})\ . \tag{6.2}$$

By ensuring that $\Psi^k(u)$ and $\Psi^k(u + v^{-p})$ converge to different limits in the proper order, we can use (6.1) and (6.2) to construct a divergence interval for the continued reciprocal power of arbitrary terms a_n. We therefore turn to the existence and properties of Ψ's fixed points.

6.1. **Fixed points of $\Psi(x)$.** The function $\Psi(x)$ is increasing and has no critical points, because its derivative

$$\Psi'(x) = \frac{p^2 x^{p^2}}{x\,(vx^p + 1)^{p+1}}\ , \tag{6.3}$$

is strictly greater than zero on $(0, \infty)$. It has a single inflection point, because

$$\Psi''(x) = \frac{p^2(p+1)\,(v + x^{-p})^{-p}\,(-vx^p + p - 1)}{x^2\,(vx^p + 1)^2}$$

has a unique zero in $(0, \infty)$ at $x = \lambda$, where

$$\lambda = \left(\frac{p-1}{v}\right)^{\frac{1}{p}}\ . \tag{6.4}$$

With a negative Schwartzian derivative

$$S\ \Psi = -\frac{(p^2-1)(p^2 + (vx^p+1)^2)}{2x^2(vx^p+1)^2}$$

for $p > 1$ and $v, x \in (0, \infty)$, Ψ has at most two attracting fixed points (cf. [1], Theorem 11.4). The number of attracting fixed points, we now show, is related to the slope of the curve at its inflection point λ.

6.2. **Knee and shoulder points.** The slope of Ψ is greater than 1 at λ when

$$\Psi'(\lambda) = \frac{p^2\left(\left(\frac{p-1}{v}\right)^{\frac{1}{p}}\right)^{p^2}}{\left(\frac{p-1}{v}\right)^{\frac{1}{p}}(p^{p+1})} > 1$$

$$\left(\frac{p-1}{v}\right)^{\frac{p^2-1}{p}} > p^{p-1}$$

$$v < \frac{p-1}{p^{\frac{p}{p+1}}} = Q_p\ . \tag{6.5}$$

The constant Q_p appeared in Proposition 2.5. The following lemma implicates Q_p in the existence of two special points at which $\Psi'(x) = 1$ when $\Psi'(\lambda) > 1$.

Lemma 6.1. *When $0 < v < Q_p$, there exist positive real numbers $\kappa < \lambda$ and $\sigma > \lambda$ for which $\Psi'(\kappa) = \Psi'(\sigma) = 1$.*

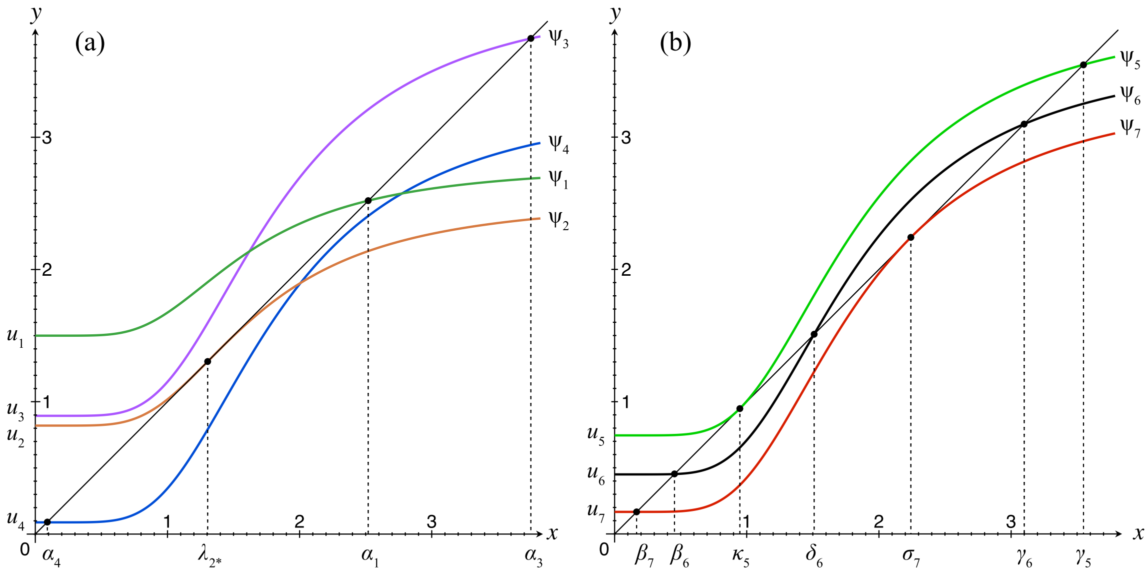

Figure 1: Graphs of the functions $\Psi_i(x) = u_i + (v_i + x^{-p})^{-p}$, $i = 1, \dots, 7$, for $p = 2.7$. In (a) on the left: $(u_1, v_1) = (1.5, 0.91)$, $(u_2, v_2) = (Q_p, Q_p) \approx (0.82, 0.82)$, $(u_3, v_3) = (0.9, 0.65)$, and $(u_4, v_4) = (0.09, 0.65)$; Ψ_2 is tangent to $y = x$ at its inflection point λ_{2*}. In (b) on the right: $v_5 = v_6 = v_7 = 0.65 < Q_p$, and $u_5 = \theta_5(\kappa_5) \approx 0.75$, $u_6 = 0.45$, $u_7 = \theta(\sigma_7) \approx 0.17$. Ψ_5 is tangent to $y = x$ at the knee point κ_5; Ψ_7 is tangent at the shoulder point σ_7. Proposition 6.5(i) is illustrated by Ψ_1, (ii) by Ψ_2, (iii) by Ψ_3 and Ψ_4, (iv) by Ψ_5, (v) by Ψ_7, and (vi) by Ψ_6.

Proof sketch. The work culminating in eq. (6.5) showed that $\Psi'(\lambda) > 1$ if, and only if, $v < Q_p$. Consider the function

$$w(x) = \Psi(x) - x \ .$$

Inspection of $w'(x) = \Psi'(x) - 1$ and $w''(x) = \Psi''(x)$ shows that w is convex on $[0, \lambda]$ and concave on $[\lambda, \infty)$, and that $w'(\lambda) > 0$. Therefore w has a minimum at the unique point $\kappa \in [0, \lambda]$, and a maximum at the unique point $\sigma \in [\lambda, \infty)$. It follows that $\Psi'(\kappa) = \Psi'(\sigma) = 1$. ∎

Thus κ and σ are the two solutions in $(0, \infty)$ of

$$\Psi'(x) = \frac{p^2 x^{p^2}}{x\,(vx^p + 1)^{p+1}} = 1 \ ,$$

and also satisfy the trinomial equation

$$\phi_v(x) = x^p - \frac{p^{\frac{2}{p+1}}}{v} x^{p-1} + \frac{1}{v} = 0 \ , \quad v \neq 0 \ . \tag{6.6}$$

We call κ the *knee point*, and σ the *shoulder point*, of $\Psi(x)$.

We next determine the values of u for which Ψ is tangent to $y = x$ at the knee and shoulder points.

Lemma 6.2. *Suppose $0 < v < Q_p$, and let*

$$\theta(x) = x - \frac{x^p}{p^{\frac{2p}{p+1}}} \ , \quad x \in (0, \infty) \ .$$

Then

$$\Psi(\kappa) \lesseqqgtr \kappa \text{ if } u \lesseqqgtr \theta(\kappa) \quad \text{and} \quad \Psi(\sigma) \lesseqqgtr \sigma \text{ if } u \lesseqqgtr \theta(\sigma) \ .$$

Proof. The bounds on v imply that the knee point κ exists and that it satisfies eq. (6.6). We use $\phi_v(\kappa) = 0$ to express v in terms of κ:

$$v = \zeta(\kappa) = \frac{p^{\frac{2}{p+1}}}{\kappa} - \frac{1}{\kappa^p} \ . \tag{6.7}$$

When $u \lesseqqgtr \theta(\kappa)$, we have

$$\begin{aligned}\Psi(\kappa) &= u + (v_\kappa + \kappa^{-p})^{-p} \\ &\lesseqqgtr \kappa - \frac{\kappa^p}{p^{\frac{2p}{p+1}}} + \left(\frac{p^{\frac{2}{p+1}}}{\kappa} - \frac{1}{\kappa^p} + \kappa^{-p}\right)^{-p} = \kappa \ .\end{aligned} \tag{6.8}$$

The argument is identical for the shoulder point. ∎

Equation (6.6) and Lemmas 6.1 and 6.2 assume $v > 0$. The case $v = 0$ is not precluded, however; indeed, using $\Psi_{u,0}(x) = u + x^{p^2}$ we may calculate the value of u corresponding to the minimal knee point.

Corollary 6.3. *If the knee point exists, then* $u \geq R_{p^2}$.

Proof. In (6.7) we saw v expressed as a function ζ of the knee point value. It may be shown that $\zeta(x)$ is strictly increasing on the interval $[0, p^{\frac{1}{p+1}}]$. We find the value of κ when $v = 0$ as follows:

$$0 = \frac{p^{\frac{2}{p+1}}}{\kappa} - \frac{1}{\kappa^p} \ \Rightarrow \ \kappa = \kappa_* = \frac{1}{(p^2)^{\frac{1}{p^2-1}}} \ .$$

Lemma 6.2's $\theta(x)$ is also strictly increasing on $[0, p^{\frac{1}{p+1}}]$. Since κ_* is minimal on $[0, p^{\frac{1}{p+1}}]$, $u = \theta(\kappa_*)$ is also minimal on that interval, and thus

$$\begin{aligned}u \geq \theta(\kappa_*) = \kappa_* - \frac{\kappa_*^p}{p^{\frac{2p}{p+1}}} &= \frac{1}{(p^2)^{\frac{1}{p^2-1}}} - \frac{1}{(p^2)^{\frac{p}{p^2-1}}} \cdot \frac{1}{(p^2)^{\frac{p}{p+1}}} \\ &= \frac{1}{(p^2)^{\frac{1}{p^2-1}}}\left(1 - \frac{1}{(p^2)^{\frac{p^2-1}{p^2-1}}}\right) \\ &= \frac{p^2-1}{(p^2)^{\frac{p^2}{p^2-1}}} = R_{p^2} \ . \ ∎\end{aligned}$$

Lemma 6.4. *The value of* Ψ *at its inflection point is*

$$\Psi(\lambda) = u + \left(\frac{p-1}{vp}\right)^p \ .$$

Proof. From eq. (6.4) we have

$$\lambda^{-p} = \frac{v}{p-1} \ ,$$

and therefore

$$\Psi(\lambda) = u + \left(v + \frac{v}{p-1}\right)^{-p} = u + \left(v \cdot \frac{p}{p-1}\right)^{-p} = u + \left(\frac{p-1}{vp}\right)^p \ . \ ∎$$

Proposition 6.5. *With* κ *and* σ *as defined in Lemma 6.1, and* $\theta(x)$ *as defined in Lemma 6.2,* $\Psi(x)$ *has the following unique fixed points:*

(i) *one strongly attracting point* $(x = \alpha)$ *for* $v > Q_p$ *and for all* $u \in [0, \infty)$*;*
(ii) *one weakly attracting point* $(x = \lambda_*)$ *for* $u = v = Q_p$*;*

(iii) *one strongly attracting point* $(x = \alpha)$ *for* $v < Q_p$ *and either* $u > \theta(\kappa)$ *or* $u < \theta(\sigma)$*;*
(iv) *one left-attracting* $(x = \kappa)$ *and one strongly attracting point* $(x = \gamma)$*,* $\kappa < \lambda < \gamma$*, for* $v < Q_p$ *and* $u = \theta(\kappa)$*;*
(v) *one strongly attracting* $(x = \beta)$ *and one right-attracting point* $(x = \sigma)$*,* $\beta < \lambda < \sigma$*, for* $v < Q_p$ *and* $u = \theta(\sigma)$*;*
(vi) *one repelling* $(x = \delta)$ *and two strongly attracting points* $(x = \beta$ *and* $x = \gamma)$*, for* $v < Q_p$ *and* $u \in (\theta(\kappa), \theta(\sigma))$.

These points are shown in Figure 1 for $p = 2.7$ and various values of u and v.

Proof. (i) Ψ is continuous, and $v > Q_p$ implies $\Psi'(x) < 1$ on $(0, \infty)$, so Ψ is a contraction, and has a unique, strongly attracting fixed point $(x = \alpha)$. When $u = 0$, $\alpha = (0, 0)$ trivially.

(ii) When $\Psi' = 1$ and Ψ is tangent to $y = x$ at $x = \lambda_*$, from

$$v = Q_p = \frac{p-1}{p^{\frac{p}{p+1}}} \quad \text{and } \lambda_* = \left(\frac{p-1}{v}\right)^{\frac{1}{p}}$$

we have

$$\lambda_* = \left(\frac{p-1}{1} \cdot \frac{p^{\frac{p}{p+1}}}{p-1}\right)^{\frac{1}{p}} = p^{\frac{1}{p+1}} \ ,$$

and hence

$$\begin{aligned}\lambda_* = \Psi(\lambda_*) &= u + (v + \lambda_*^{-p})^{-p} \\ &= u + \left(\frac{p-1}{p^{\frac{p}{p+1}}} + \frac{1}{p^{\frac{p}{p+1}}}\right)^{-p} = u + p^{\frac{-p}{p+1}} = u + \lambda_*^{-p} \ .\end{aligned}$$

Consequently,

$$u = \lambda_* - \frac{1}{\lambda_*^p} = \frac{p-1}{p^{\frac{p}{p+1}}} = Q_p \ .$$

The form of Ψ in this special case is therefore

$$\Psi(x) = Q_p + (Q_p + x^{-p})^{-p} \ ,$$

which, by Proposition 2.5, converges to (λ_*, λ_*). Because $\Psi'(\lambda_*) = 1$, λ_* is weakly attracting.

For parts (iii)–(vi), the knee and shoulder points exist when $v < Q_p$, by Lemma 6.1.

(iii) We prove the claim for $u > \theta(\kappa)$; the proof for $u < \theta(\sigma)$ is similar.

On $(0, \lambda]$: we have, first, that $\Psi(x) > u$ for all $x \in (0, \infty)$; and secondly, that $\Psi(\kappa) > \kappa$ when $u > \theta(\kappa)$ by Lemma 6.2. Thirdly, from the condition $v < Q_p$, we calculate

$$v^{-p} > \frac{p^{\frac{p^2}{p+1}}}{(p-1)^p} \ ,$$

so by Lemma 6.4,

$$\Psi(\lambda) > u + \frac{(p-1)^p}{p^p} \cdot \frac{p^{\frac{p^2}{p+1}}}{(p-1)^p} = u + p^{\frac{-p}{p+1}} > \theta(\kappa) + p^{\frac{-p}{p+1}} \ .$$

Being smooth and strictly increasing, therefore, $\Psi(x)$ is strictly greater than x and has no fixed points on $(0, \lambda]$.

On (λ, ∞): Ψ is a contraction because $\Psi'(x) < 1$, and thus has a unique strongly attracting fixed point in that interval $(x = \alpha)$.

(iv) On $(0,\lambda) = (0,\kappa) \cup \{\kappa\} \cup (\kappa,\lambda)$: By Lemma 6.2, when $u = \theta(\kappa)$ the fixed point (κ,κ) exists. One easily shows that $\Psi^n(x) < \Psi^{n+1}(x) < \kappa$ for $n \in \mathbb{N}$ and $x \in (0,\kappa)$; being strictly increasing and bounded above by $\Psi(\kappa) = \kappa$, $\Psi^n(x)$ converges to $x = \kappa$. On (κ,λ), however, $\Psi(x) > x$, so there are no fixed points in this interval.

On (λ,∞): $\Psi'(x) < 1$, so Ψ is a contraction and has a unique strongly attracting fixed point $(x = \gamma)$.

(v) On $(0,\lambda)$: Ψ is a contraction and has a unique, strongly attracting fixed point $(x = \beta)$.

On $(\lambda,\infty) = (\lambda,\sigma) \cup \{\sigma\} \cup (\sigma,\infty)$: By Lemma 6.2, (σ,σ) is a fixed point when $u = \theta(\sigma)$. Because $\Psi(x) < x$ on (λ,σ), there are no fixed points in this interval. On (σ,∞), $\sigma < \Psi^{n+1}(x) < \Psi^n(x)$ for $n \in \mathbb{N}$; being strictly decreasing and bounded below by $\Psi(\sigma) = \sigma$, $\Psi^n(x)$ converges to $x = \sigma$.

(vi) The strongly attracting fixed points $x = \beta$ and $x = \gamma$ occur in the intervals $(0,\kappa)$ and (σ,∞), where $\Psi'(x) < 1$. Noting that $u < \theta(\kappa)$ implies $\Psi(\kappa) < \kappa$; that $u > \theta(\sigma)$ implies $\Psi(\sigma) > \sigma$; and that Ψ is smooth with derivative exceeding 1, it follows that Ψ has a single repelling fixed point $x = \delta$ on (κ,σ). ∎

7. A BOUNDEDNESS CONDITION FOR DIVERGENCE

We now use the fixed points of $\Psi(x)$ as bounds for the approximants of the general continued reciprocal power.

Theorem 7.1. *Given $p > 1$ and a sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive reals for which $0 < \liminf a_{2i}$ and $0 < \liminf a_{2i+1}$ as $i \to \infty$, let*

$$S = \limsup_{i\to\infty} a_{2i} \quad \textit{and} \quad T = \limsup_{i\to\infty} a_{2i+1}\ .$$

The continued reciprocal power $\mathrm{K}_{i=0}^{\infty}(a_i)^{-p}$ diverges if all of the following hold:

(i) $0 < \limsup a_{2i+1} < Q_p\,$,
(ii) $\liminf a_{2i} \geq \theta(\kappa)$
(iii) $\limsup a_{2i} \leq \theta(\sigma)\,$,

where for $x, v \in (0,\infty)$,

$$\theta(x) = x - \frac{x^p}{p^{\frac{2}{p+1}}}$$

$$\phi_v(x) = x^p - \frac{p^{\frac{2}{p+1}}}{v} x^{p-1} + \frac{1}{v}\ ,$$

κ is the smaller of the two real solutions to $\phi_S(x) = 0$, and σ is the larger of the two real solutions to $\phi_T(x) = 0$.

Proof. In light of Proposition 2.4, we assume that $\limsup a_n < +\infty$. By choosing an appropriate $\epsilon > 0$ and a sufficiently large nonnegative integer M, we may define positive reals a, b, c, d such that, for $i \geq M$,

$$0 < a = \liminf a_{2i} + \epsilon < a_{2i} < \limsup a_{2i} - \epsilon = b\ , \tag{7.1}$$

$$0 < c = \liminf a_{2i+1} + \epsilon < a_{2i+1} < \limsup a_{2i+1} - \epsilon = d\ . \tag{7.2}$$

To prove the theorem, we use relations (7.1) and (7.2) to form an inequality comprising, from left to right and from least to greatest, a continued reciprocal power's $2k$th approximant; an iterated function of b and c; an iterated function of a and d; and finally the $2k+1$st approximant. By adjusting the parameters a, b, c, and d, we cause the limit of the first iterated function to be strictly less than the limit of the second, creating a divergence interval.

To construct an even approximant, begin with eq. (7.1) and $a_{2m} = \mathrm{K}_{2m}^{2m}$, $m \geq 0$:

$$a < \mathrm{K}_{2m}^{2m} < b$$
$$^{-p}(a) > {}^{-p}\big(\mathrm{K}_{2m}^{2m}\big) > {}^{-p}(b)\ .$$

Next, apply $c < a_{2m-1} < d$ from (7.2), to obtain

$$d + {}^{-p}(a) > \mathrm{K}_{2m-1}^{2m} > c + {}^{-p}(b)\ .$$

Repeating these steps gives

$$^{-p}(d + {}^{-p}(a)) < {}^{-p}\big(\mathrm{K}_{2m-1}^{2m}\big) < {}^{-p}(c + {}^{-p}(b))$$
$$a + {}^{-p}(d + {}^{-p}(a)) < \mathrm{K}_{2m-2}^{2m} < b + {}^{-p}(c + {}^{-p}(b))\ . \tag{7.3}$$

Define functions f and g on $(0, \infty)$ by

$$f(x) = b + (c + x^{-p})^{-p}$$
$$g(x) = a + (d + x^{-p})^{-p}\ ,$$

then write (7.3) as

$$g(a) < \mathrm{K}_{2m-2}^{2m} < f(b)\ .$$

By induction on the index j, we see that

$$g^j(a) < \mathrm{K}_{2m-2j}^{2m} < f^j(b)\ . \tag{7.4}$$

For the odd approximants, repeat this construction, beginning with (7.2) and $a_{2m+1} = \mathrm{K}_{2m+1}^{2m+1}$:

$$c < \mathrm{K}_{2m+1}^{2m+1} < d$$
$$^{-p}(c) > {}^{-p}\big(\mathrm{K}_{2m+1}^{2m+1}\big) > {}^{-p}(d)\ .$$

Apply (7.1) and (7.2) alternately to get

$$b + {}^{-p}(c) > \mathrm{K}_{2m}^{2m+1} > a + {}^{-p}(d)$$
$$^{-p}(b + {}^{-p}(c)) < {}^{-p}\big(\mathrm{K}_{2m}^{2m+1}\big) < {}^{-p}(a + {}^{-p}(d))$$
$$c + {}^{-p}(b + {}^{-p}(c)) < \mathrm{K}_{2m-1}^{2m+1} < d + {}^{-p}(a + {}^{-p}(d))$$
$$^{-p}(c + {}^{-p}(b + {}^{-p}(c))) > {}^{-p}\big(\mathrm{K}_{2m-1}^{2m+1}\big) > {}^{-p}(d + {}^{-p}(a + {}^{-p}(d)))$$
$$b + {}^{-p}(c + {}^{-p}(b + {}^{-p}(c))) > \mathrm{K}_{2m-2}^{2m+1} > a + {}^{-p}(d + {}^{-p}(a + {}^{-p}(d)))$$

Using the functions f and g, this may be expressed as

$$f(b + c^{-p}) > \mathrm{K}_{2m-2}^{2m+1} > g(a + d^{-p})\ .$$

Again by induction on the index j, this leads to

$$f^j(b + c^{-p}) > \mathrm{K}_{2m-2j}^{2m+1} > g^j(a + d^{-p})\ . \tag{7.5}$$

Set $k = m - j$. From (7.4) and (7.5), we see that, for k fixed, K_{2k}^{∞} will be divergent if the central inequality in

$$\lim_{m\to\infty} \mathrm{K}_{2k}^{2m} \leq \lim_{m\to\infty} f^{m-k}(b) < \lim_{m\to\infty} g^{m-k}(a + d^{-p}) \leq \lim_{m\to\infty} \mathrm{K}_{2k}^{2m+1}$$

is strict. That strict inequality, in turn, will be guaranteed if there exists a fixed point ω_f for f, and ω_g for g, such that

$$\lim_{m\to\infty} f^{m-k}(b) \leq f(\omega_f) = \omega_f < \omega_g = g(\omega_g) \leq \lim_{m\to\infty} g^{m-k}(a + d^{-p})\ . \tag{7.6}$$

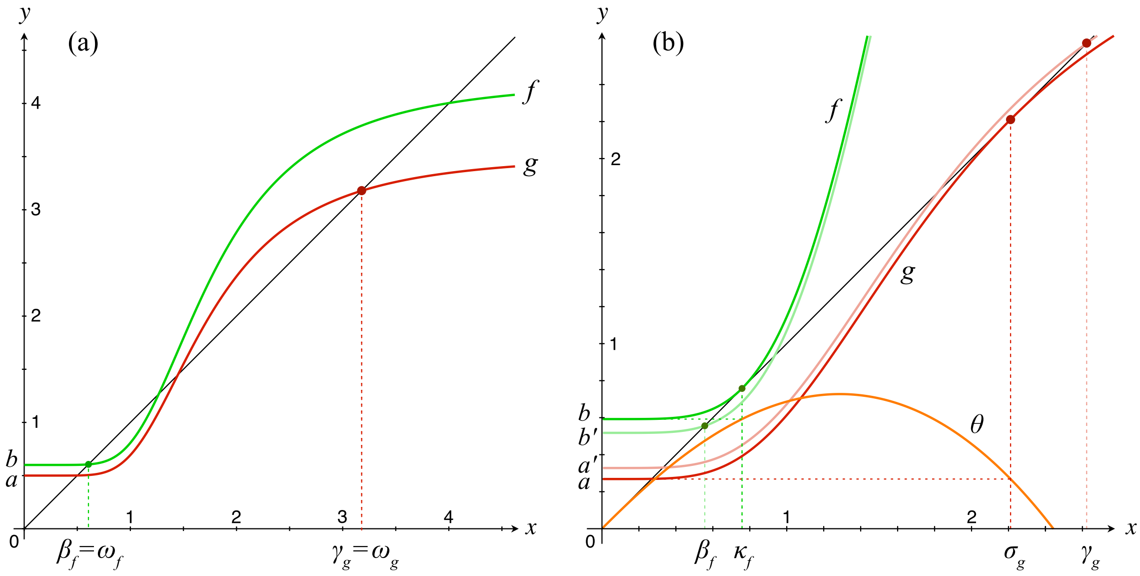

Figure 2: Examples of the functions $f(x) = b + (c + x^{-p})^{-p}$ and $g(x) = a + (d + x^{-p})^{-p}$ used in the proof of Theorem 7.1. Strongly attracting fixed points approached from $x_0 = b$ are smaller and green, those approached from $x_0 = a + d^{-p}$ are larger and red. In (a) on the left: $p = 3.1$, $a = 0.5$, $b = 0.6$, $c = 0.66$, and $d = 0.7$. The points $\beta_f = \omega_f$ and $\gamma_g = \omega_g$ are the bounds used in eq. (7.6). In (b) on the right: $p = 2.3$, $c = 0.294$, and $d = 0.588$. The knee point $\kappa_f \approx 0.757915$ is the smaller of the two numerical solutions to $\phi_c(x) = 0$, and the shoulder point $\sigma_g \approx 2.21125$ is the larger of the two solutions to $\phi_d(x) = 0$. The function θ gives $a = \theta(\sigma_g)$ and $b = \theta(\kappa_f)$; for these values of a and b, respectively, g and f are tangent to $y = x$. A slight vertical translation of a upward to a' results in a strongly attracting fixed point $\gamma_g > \sigma_g$ for $g(x)$; a translation of b downward to b' yields $\beta_f < \kappa_g$ for $f(x)$.

However, this plan must reckon with the following fact.

Lemma 7.2. *Given $p > 1$, $0 < a < b$, and $0 < c < d$, we have $g(x) < f(x)$ for all $x \in [0, \infty)$.*

Proof. Suppose $x > 0$. From $c < d$ we have $(d + x^{-p})^{-p} < (c + x^{-p})^{-p}$, and from $a < b$, $g(x) = a + (d + x^{-p})^{-p} < b + (c + x^{-p})^{-p} = f(x)$. On the other hand, $\lim_{x \to 0} g(x) = a < b = \lim_{x \to 0} f(x)$. ∎

The implication is that invariably $\omega_g < \omega_f$, which is the opposite order from the one desired in (7.6). Indeed, $f(x) = f_{b,c}(x)$ and $g(x) = g_{a,d}(x)$ are instances of $\Psi(x) = \Psi_{u,v}(x)$ from §6, and parts (i)–(iii) of Proposition 6.5 imply that f has a unique fixed point $\alpha_f > \lambda_f$ when 1) $c > Q_p$, or when 2) $0 < c \leq Q_p$ and $b > \theta(\kappa)$. In each case, Lemma 7.2 implies that α_f is greater than the largest fixed point of g; and this is irrespective of the number of g's fixed points. Thus, no interval $[\chi_g, \alpha_f]$, where χ_g is any fixed point of g, can be used as the divergence interval $[\omega_f, \omega_g]$ sought in (7.6).

Nonetheless, it is possible to choose a, b, c, and d so that f and g each have two or more fixed points, and such that β_f lies on or below f's knee point and γ_g lies on or above g's shoulder point (Figure 2a). If β_f (approached from below by iteration from the initial value $x_0 = b$) is less than γ_g (approached from above with $x_0 = a + d^{-p}$), then $[\beta_f, \gamma_g]$ will constitute a divergence interval. Condition (i), per Lemma 6.1, ensures the existence of at least two fixed points each for f and for g. Parts (iv)–(vi) of Proposition 6.5 imply that β_f is less than γ_g. Condition (ii) guarantees that the point $\beta_f < \lambda_f$ exists for $0 < b \leq \theta(\kappa)$; when $b = \theta(\kappa)$, f is tangent to $y = x$ from above at $(\kappa, f(\kappa))$. Condition (iii) means that the point $\gamma_g > \lambda_g$ exists for $a \geq \theta(\sigma)$; when $a = \theta(\sigma)$, g is tangent to $y = x$ from below at $(\sigma, g(\sigma))$.

Thus, the desired divergence interval is created, with left endpoint $\omega_f = \beta_f$ and right endpoint $\omega_g = \gamma_g$. We have shown that

$$\lim_{m\to\infty} \mathrm{K}_{2k}^{2m} \leq f(\omega_f) = \omega_f = \beta_f < \gamma_g = \omega_g = g(\omega_g) \leq \lim_{m\to\infty} \mathrm{K}_{2k}^{2m+1} \,. \tag{7.7}$$

Since the even and odd tails K_{2k}^{2m} and K_{2k}^{2m+1} diverge, K_0^{∞} diverges by Proposition 2.1 ∎.

Figure 2b shows a more detailed illustration of the theorem and its proof. The knee point κ_f is the smaller of the two real solutions to $\phi_c(x) = 0$; the shoulder point σ_g is the larger of the two solutions to $\phi_d(x) = 0$. The function θ produces the values $a = \theta(\sigma_g)$ and $b = \theta(\kappa_f)$; these cause f and g to be tangent to the line $y = x$ at $x = \kappa_f$ and $x = \sigma_g$, respectively. The interval $D = [\kappa_f, \sigma_g]$ comprises a divergence interval in this situation. If a is translated upward vertically to a', and b is translated downward to b', and the relation $a < a' < b' < b$ is maintained, then the divergence interval becomes $D' = [\beta_f, \gamma_g]$, with $D \subset D'$.

8. NOTES

8.1. Series expansions for the knee and shoulder points. Series solutions to trinomial equations have long been known, and are also readily generated by Lagrange inversion using symbolic math software. The series given here derive from formulas in [4]. In §6.2 we saw that $\Psi'(x) = 1$ produced the trinomial

$$\phi_v(x) = x^p - \frac{p^{\frac{2}{p+1}}}{v} x^{p-1} + \frac{1}{v} \,, \quad v \neq 0 \,.$$

When $v < Q_p$, the knee point of $\Psi(x)$ is the smaller of the two real solutions to $\phi_v(x) = 0$; per [4, Eq. 9], its series representation is

$$\kappa = \frac{1}{p^{\frac{2}{p^2-1}}} \sum_{i=0}^{\infty} \binom{\frac{pi-p+2}{p-1}}{i} \left(\frac{v}{p^{\frac{2p}{p^2-1}}} \right)^i ,$$

while the shoulder point σ is the larger solution, per [4, Eq. 7]:

$$\sigma = \frac{p^{\frac{2}{p+1}}}{v} \left(1 - \sum_{i=1}^{n-1} \frac{\binom{ip-1}{i}}{ip-1} \left(\frac{v^{p-1}}{p^{\frac{2p}{p+1}}} \right)^i \right) \,.$$

8.2. Historical note on continued roots. Let $q = 1/p$. When $0 < p < 1$, the general continued pth power (1.3) is called a continued qth root, while $-1 < p < 0$ generates a continued reciprocal qth root.

A result which we have called Herschfeld's Convergence Theorem (see e.g. [7] and [9]) states that a continued qth root of terms $a_n \geq 0$ converges if, and only if,

$$\limsup_{n\to\infty} \; a_n^{p^n} < +\infty \,. \tag{8.1}$$

Herschfeld's 1935 paper [5] cited a version of (8.1) (expressed using logarithms) by Pólya and Szegö from 1916 [15] and 1925 [16]; and Vijayaraghavan [17] gave a condition similar to that of Pólya and Szegö in 1927. Herschfeld believed his statement of (8.1) to be new; but ongoing archival research has revealed that condition (8.1) appeared as early as 1904, in Wiernsberger's paper [21] and dissertation [22]. Thus, Herschfeld's Convergence Theorem joins Pell's equation, Hölder's inequality, and many others in the list of misattributions (cf. [20]).

For an annotated bibliography of continued general powers and related forms, see [10].

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