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RATIONAL FUNCTIONS OF CHEBYSHEV POLYNOMIALS FOR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT. In this work, we treat a new numerical method for solving Volterra-Fredholm integral equations of the second kind. This method is based on orthogonal basis of rational functions derived from Chebyshev polynomials of the first kind. The approximate solution by this series converges to the exact solution. Numerical examples are presented and compared with other methods, in the goal to show the applicability and the efficiency of this method.

Key words and phrases: Chebyshev polynomials rst kind, Rational functions, Volterra-Fredholm integral equation, Collocation method, Numerical method.

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1. INTRODUCTION

Fredholm, Volterra and Volterra-Fredholm integral equations have become a rapidly growing area of research. The study of these types of problems is driven by the fact that several phenomena in engineering, physics, and the life sciences can be modeled in this way. Volterra-Fredholm integral equation describe many phenomena in the applied sciences such as the nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, and concentration in chemical or biological problems. Various problems arising in heat conduction, underground water flow, thermoelectricity, and plasma physics can be reduced Volterra-Fredholm integro-differential equation with integral boundary conditions and so to Volterra-Fredholm integral equation of the form

$$u(x)-\int_a^x k_1(x,t)u(t)dt-\int_a^b k_2(x,t)u(t)dt=f(x), \tag{1.1}$$

with a given function $k(x,t)$ and a function $f(x)$, the kernel $k(x,t)$ is bounded in $a \leq x,t \leq b$, and value 1 is not an eigenvalue of (1.1), the function $u(x)$ is the unknown function to be determined.

As it is known we can't solve analytically Fredholm or Volterra integral equations, certainly it is almost impossible of solving Volterra-Fredholm integral equations, so it is preferable to choose the way of the approximation of the solution. Many studies use the approximate methods such as moving least square method and Chebyshev polynomials [1], [10] Adomian decomposition method [3], application of the four Chebyshev polynomials to the integral equations [6], application of the four Chebyshev polynomials to the Volterra-Fredholm integral equations [2], Lucas polynomials series, Fibonacci polynomials series and Euler series solutions for linear integral equations [7], [8], [9].

2. CHEBYSHEV POLYNOMIALS OF FIRST KIND

Let $T_n(x)$ denote the Chebyshev polynomials of the first kind, classically defined on the interval $[-1,1]$ by the explicit relation

$$T_n(x)=\cos n\theta \quad \text{when} \;\; x=\cos\theta, \tag{2.1}$$

where $x\in[-1,1]$, this involves that the corresponding variable $\theta\in[0,\pi]$. It is easy to see that $T_0(x)=1,\;\; T_1(x)=x$ and by the recurrence formula satisfied by Chebyshev polynomials

$$\cos n\theta+\cos(n-2)\theta=2\cos\theta\cos(n-1)\theta,$$

we obtain the fundamental relation

$$T_n(x)=2xT_{n-1}(x)-T_{n-2}(x), \quad n=2,3,.... \tag{2.2}$$

Noting that the functions $\{T_n(x),\;\; n=0,1,2,....\}$ form an orthogonal system on the interval $[-1,1]$ with respect to the weight $w(x)=\dfrac{1}{\sqrt{1-x^2}}$ and so the polynomial system $S_n(x)$ given by

$$\left\{S_0(x)=\sqrt{\frac{1}{\pi}}T_0(x),\; S_1(x)=\sqrt{\frac{2}{\pi}}T_1(x),\; S_2(x)=\sqrt{\frac{2}{\pi}}T_2(x),...S_n(x)=\sqrt{\frac{2}{\pi}}T_n(x)...\right\},$$

form an orthonormal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \dfrac{1}{\sqrt{1-x^2}}$. In other words

$$\langle S_k(x), S_l(x)\rangle = \int_{-1}^{1} \frac{S_k(x)S_l(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if} \quad k \neq l \\ 1 & \text{if} \quad k = l \end{cases} .$$

We define the functions $Q_n(x)$ as follows

$$Q_n(x) = \frac{x+2}{(1-x^2)^{\frac{1}{4}}} S_n(x). \tag{2.3}$$

Noting that, the rational functions $\dfrac{Q_n(x)}{(x+2)}$ are orthogonal on the interval $[-1, 1]$, say

$$\int_{-1}^{1} \frac{Q_m(x)Q_n(x)}{(x+2)^2}dx = \int_{-1}^{1} \frac{S_m(x)S_n(x)}{\sqrt{(1-x^2)}}dx.$$

Define the orthogonal rational functions $N_n(x)$ as composition

$$N_n(x) = (-1)^n \frac{1}{\sqrt[4]{3}} Q_n\left(\frac{-2x-1}{x+2}\right), \tag{2.4}$$

that is to say

$$\begin{aligned} \int_{-1}^{1} N_m(x)N_n(x)dx &= (-1)^{m+n} \frac{1}{\sqrt{3}} \int_{-1}^{1} Q_m\left(\frac{-2x-1}{x+2}\right) Q_n\left(\frac{-2x-1}{x+2}\right) dx \\ &= (-1)^{m+n} \sqrt{3} \int_{-1}^{1} \frac{Q_m(x)Q_n(x)}{(x+2)^2}dx, \end{aligned}$$

we find

$$\begin{aligned} N_0(x) &= \frac{1}{(x+2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{4}}} \\ N_1(x) &= \frac{2x+1}{(x+2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{4}}} \\ N_2(x) &= \frac{1}{(x+2)^{\frac{5}{2}}(1-x^2)^{\frac{1}{4}}}(7x^2+4x-2) \\ N_3(x) &= \frac{1}{(x+2)^{\frac{7}{2}}(1-x^2)^{\frac{1}{4}}}(26x^3+21x^2-12x-8) \end{aligned}$$

3. DISCRETIZATION OF INTEGRAL EQUATION

Applying a collocation method to the equation (1.1) in order to discredit and convert this equation to a system of linear equations. For this latter, supposing that $a = -1$ and $b = 1$ and approximate the unknown function $u(x)$ by a finite sum of the form

$$u(x) \simeq u_N(x) = \sum_{k=0}^{N} c_k N_k(x), \tag{3.1}$$

where $N_n(x)$ denotes the nth rational function of Chebyshev polynomial of the first kind. After substitution of the expansion (3.1) into the equation (1.1) this latter becomes an approximate equation as

$$\sum_{k=0}^{N}\alpha_k N_k(x)-\int_a^x k_1(x,t)\sum_{k=0}^{N}\alpha_k N_k(t)-\int_a^b k_2(x,t)\sum_{k=0}^{N}\alpha_k N_k(t)=f(x). \tag{3.2}$$

Choosing the Fourier's coefficients α_k such that (3.2) is satisfied on the interval $[-1,1]$. For this technical we take the equidistant collocation points as follows

$$t_j=-1+\frac{2j}{N},\ \ j=0,1,...N, \tag{3.3}$$

and define the residual as

$$R_N(x)=\sum_{k=0}^{N}\alpha_k N_k(x)-\int_a^x k_1(x,t)\sum_{k=0}^{N}\alpha_k N_k(t)-\int_a^b k_2(x,t)\sum_{k=0}^{N}\alpha_k N_k(t)-f(x) \tag{3.4}$$

Then, by imposing conditions at collocation points

$$R_N(x_j)=0,\quad j=0,1,....N, \tag{3.5}$$

the integral equation (3.2) is converted to a system of linear equations.

Theorem 3.1.

Let $A:X\to X$ be compact and the equation

$$(I-A)u=f, \tag{3.6}$$

admit a unique solution.Assume that the projections $P_n\ X\to X_n$ satisfy to $\|P_nA-A\|\to 0,\ n\to\infty$. Then, for sufficiently large n, the approximate equation

$$u_n-P_nAu_n=P_nf, \tag{3.7}$$

has a unique solution for all $f\in X$ and there holds an error estimate

$$\|u-u_n\|\le M\,\|u-P_nu\|\,, \tag{3.8}$$

with some positive constant M depending on A.

Proof. As it is known for all sufficiently large n the inverse operators $(I-P_nA)^{-1}$ exist and are uniformly bounded, see [4], [5]. To verify the error bound, we apply the projection operator P_n to the equation (3.6) and get

$$P_nu-P_nAu=P_nf, \tag{3.9}$$

or again

$$u-P_nAu=P_nf+u-P_nu. \tag{3.10}$$

Subtracting (3.10) from (3.7) we find

$$(I-P_nA)(u-u_n)=(I-P_n)u.$$

Hence the estimate (3.8) follows. ∎

4. NUMERICAL EXAMPLES

Example 1
Consider the Volterra-Fredholm integral equation

$$u(x) - \int_0^x \left(x^2 + t^2\right) u(t)dt - \int_0^1 (xt)\, u(t)dt = f(x),$$

$$\begin{aligned} f(x) &= \frac{2}{2x+3} + \left(x^2 + \frac{9}{4}\right)\left(\ln 3 - \ln 2 - \ln\left(x + \frac{3}{2}\right)\right) \\ &\quad -\frac{1}{2}x^2 - \frac{1}{2}x\left(3\ln 3 - 3\ln 5 - 1\right), \end{aligned}$$

where the function $f(x)$ is chosen so that the solution $u(x)$ is given by

$$u(x) = \frac{2}{2x+2}$$

Applying the rational Chebyshev polynomial $N_n(x)$ to approximate the solution $u(x)$, that is to say $u_N(x)$ solution of the system of linear equations for $n = 8$

Pts of x	Exact sol	Approx sol	Error N=8
0.000e+00	6.666e-01	6.666e-01	3.130e-09
2.500e-01	5.714e-01	5.714e-01	1.679e-09
3.750e-01	5.333e-01	5.333e-01	1.249e-09
5.000e-01	5.000e-01	5.000e-01	1.930e-10
7.500e-01	4.444e-01	4.444e-01	1.016e-09
8.750e-01	4.210e-01	4.210e-01	6.778e-10
1.000e+00	4.076e-01	4.076e-01	1.738e-09

Table 1. The exact and approximate solutions of example 1
in some arbitrary points, using the rational Legendre polynomial $N_8(x)$

Example 2
Consider the Volterra-Fredholm integral equation

$$u(x) - \int_0^x e^{(x-t)} u(t)dt - \int_0^1 \left(xe^{-t}\right) u(t)dt = f(x),$$

$$f(x) = x\left(3e^{-x} + \frac{3}{2}e^{-2} - \frac{1}{2}\right) - \frac{1}{2}\left(e^x - e^{-x}\right),$$

where the function $f(x)$ is chosen so that the solution $u(x)$ is given by

$$u(x) = 2xe^{-x}$$

Applying the rational Chebyshev polynomial $N_n(x)$ to approximate the solution $u(x)$, that is to say $u_N(x)$ solution of the system of linear equations for $n = 8$

Pts of x	Exact sol	Approx sol	Error N=8
0.000e+00	0.000e+00	1.001e-08	1.001e-08
2.500e-01	3.894e-01	3.894e-01	1.633e-09
3.750e-01	5.154e-01	5.154e-01	1.312e-09
5.000e-01	6.065e-01	6.065e-01	8.690e-09
7.500e-01	7.085e-01	7.085e-01	6.136e-09
8.750e-01	7.295e-01	7.295e-01	8.483e-09
1.000e+00	7.357e-01	7.357e-01	2.120e-08

Table 2. The exact and approximate solutions of example 2
in some arbitrary points, using the rational Legendre polynomial $N_8(x)$

Example 3

Consider the Volterra-Fredholm integral equation

$$u(x) - \int_0^x (\sin t \cos x)\, u(t)dt - \int_0^1 (\sin x \cos t)\, u(t)dt = f(x),$$

$$f(x) = \frac{7}{8}\cos x + \frac{1}{8}\cos 3x - (\sin x)\left(\frac{1}{4}\sin 2 + \frac{1}{2}\right),$$

where the function $f(x)$ is chosen so that the solution $u(x)$ is given by

$$u(x) = \cos x$$

Applying the rational Chebyshev polynomial $N_n(x)$ to approximate the solution $u(x)$, that is to say $u_N(x)$ solution of the system of linear equations for $n = 8$

Pts of x	Exact sol	Approx sol	Error N=8
0.000e+00	1.000e+00	1.000e+00	2.329e-07
2.500e-01	9.689e-01	9.689e-01	2.933e-08
3.750e-01	9.305e-01	9.305e-01	9.446e-08
5.000e-01	8.775e-01	8.775e-01	1.582e-07
7.500e-01	7.316e-01	7.316e-01	1.592e-08
8.750e-01	6.409e-01	6.409e-01	7.878e-08
1.000e+00	5.403e-01	5.403e-01	3.665e-07

Table 3. The exact and approximate solutions of example 3
in some arbitrary points, using the rational Legendre polynomial $N_8(x)$

Example 4

Consider the Volterra-Fredholm integral equation

$$u(x) - \int_0^x (x - t)\, u(t)dt - \int_0^1 (x + t)\, u(t)dt = f(x),$$

$$f(x) = x\,(2\ln 2 - \ln 3 - \ln(x + 2) + 1) + \frac{1}{x + 2} + 2\ln 3 - 2\ln(x + 2) - 1,$$

where the function $f(x)$ is chosen so that the solution $u(x)$ is given by

$$u(x) = \frac{1}{x + 2}$$

Applying the rational Chebyshev polynomial $N_n(x)$ to approximate the solution $u(x)$, that is to say $u_N(x)$ solution of the system of linear equations for $n = 8$

Pts of x	Exact sol	Approx sol	Error N=8
0.000e+00	5.000e-01	5.000e-01	2.289e-10
2.500e-01	4.444e-01	4.444e-01	7.998e-11
3.750e-01	4.210e-01	4.210e-01	1.218e-10
5.000e-01	4.000e-01	4.000e-01	2.023e-11
7.500e-01	3.636e-01	3.636e-01	1.235e-10
8.750e-01	3.478e-01	3.478e-01	8.831e-12
1.000e+00	3.333e-01	3.333e-01	2.423e-10

Table 4. The exact and approximate solutions of example 4
in some arbitrary points, using the rational Legendre polynomial $N_8(x$

Example 5
Consider the Volterra-Fredholm integral equation

$$u(x) - \int_0^x \frac{x}{(t+1)} u(t)dt - \int_0^1 \frac{t}{(x+1)} u(t)dt = f(x),$$

$$f(x) = \ln(x+1) - \frac{1}{4x+4} - \frac{1}{2} x \ln^2(x+1),$$

where the function $f(x)$ is chosen so that the solution $u(x)$ is given by

$$u(x) = \ln(x+1)$$

Applying the rational Chebyshev polynomial $N_n(x)$ to approximate the solution $u(x)$, that is to say $u_N(x)$ solution of the system of linear equations for $n = 8$

Pts of x	Exact sol	Approx sol	Error N=8
0.000e+00	0.000e+00	9.440e-09	9.440e-09
2.500e-01	2.231e-01	2.231e-01	6.131e-10
3.750e-01	3.184e-01	3.184e-01	3.321e-09
5.000e-01	4.054e-01	4.054e-01	6.222e-09
7.500e-01	5.596e-01	5.596e-01	1.255e-09
8.750e-01	6.286e-01	6.286e-01	2.870e-09
1.000e+00	6.931e-01	6.931e-01	1.444e-08

Table 5. The exact and approximate solutions of example 5
in some arbitrary points, using the rational Legendre polynomial $N_8(x)$

5. CONCLUSION

In this work, we suppose that the unknown the density $u(x)$ may be approximated by a finite sum of rational functions of Chebyshev polynomials to determine the approximate solution of Volterra-Fredholm integral in order to obtain a system of linear equations with $N+1$ unknowns. The solution of the algebraic system represents the approximate solution $u_N(x)$ of the given equation.

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