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VISCOSITY APPROXIMATION METHODS FOR SPLIT MONOTONE VARIATIONAL INCLUSIONS AND FIXED POINT PROBLEMS OF A FINITE FAMILY OF ξ-DEMIMETRIC MAPPINGS

PRASHANT PATEL AND RAHUL SHUKLA*

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DEPARTMENT OF MATHEMATICS, SCHOOL OF ADVANCED SCIENCES, VIT-AP UNIVERSITY, INAVOLU, BESIDE AP SECRETARIAT, AMARAVATI, 522237, ANDHRA PRADESH, INDIA.
prashant.patel9999@gmail.com,prashant.p@vitap.ac.in

DEPARTMENT OF MATHEMATICAL SCIENCES & COMPUTING, WALTER SISULU UNIVERSITY, MTHATHA 5117, SOUTH AFRICA.
rshukla@wsu.ac.za

ABSTRACT. In this paper, we study viscosity approximation methods and present a new algorithm to find a common element of the fixed point of a finite family of ξ-demimetric mappings and the set of solutions of split monotone variational inclusion problem in Hilbert spaces. Under some conditions, we prove a strong convergence theorem which converges to this common solution.

Key words and phrases: Split monotone variational inclusion problem; Variational inequality; Demimetric mapping.

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1. INTRODUCTION

In nonlinear analysis, and especially in fixed point theory, the existence and convergence of fixed points of nonexpansive mappings are fascinating and significant topics. Generally, a non-expansive mapping's iteration sequence does not have to converge to a fixed point in a Banach space. Moreover, only weak convergence is obtained by certain well-known classical iteration techniques. Viscosity approximation methods are novel techniques proposed by Moudafi [6] to provide strong convergence for a sequence of iterates with nonexpansive mapping. These techniques are useful and efficient tools for solving many other nonlinear analysis problems, such as convex optimization, split and common split feasibility, and variational inequality.

The variational inequality problem is to find a point $\zeta^\dagger \in \mathcal{E}$ such that

$$\langle F(\zeta^\dagger), \zeta^\dagger - \zeta\rangle \geq 0, \text{ for all } \zeta \in \mathcal{E}.$$

The problem of finding common elements of the set of fixed points for mappings and the set of solutions for variational inequalities is a closely related subject of current interest. Over the years, due to its practical uses in various fields such as science, engineering, management, and social sciences, the variational inequality problem has been extended and generalized in many ways, as seen in references [1, 3, 4, 9, 10].

Censor et. al. [3] presented a new variational problem and they named it split variational inequality problem. It entails finding a solution of one Variational Inequality Problem (VIP), the image of which under a given bounded linear transformation is a solution of another VIP. Suppose Σ_1, Σ_2 be two real Hilbert spaces and $F : \Sigma_1 \to \Sigma_1$ and $\Lambda : \Sigma_2 \to \Sigma_2$ are two mappings and $\Psi : \Sigma_1 \to \Sigma_2$ a bounded linear mapping, $\mathcal{E}_1 \subseteq \Sigma_1$ and $\mathcal{E}_2 \subseteq \Sigma_2$ are closed and convex subsets. Then the Split Variational Inequality Problem is formulated as

$$\text{find a point } \zeta^\dagger \in \mathcal{E}_1 \text{ such that } \langle F(\zeta^\dagger), \zeta - \zeta^\dagger\rangle \geq 0, \forall \zeta \in \mathcal{E}_1,$$

and such that

$$\text{the point } \eta^\dagger \in \Psi(\zeta^\dagger) \in \mathcal{E}_2 \text{ solves } \langle \Lambda(\eta^\dagger), \eta - \eta^\dagger\rangle \geq 0, \forall \eta \in \mathcal{E}_2.$$

In 2011, Moudafi extended their split variational inequality problem and introduced split monotone variational inclusion (SMVI) problem for finding a point $\zeta^\dagger \in \Sigma_1$ in such a way that

$$\text{find } \zeta^\dagger \in \Sigma_1 \text{ such that } 0 \in F(\zeta^\dagger) + \Upsilon_1(\zeta^\dagger) \tag{1.1}$$

and such that

$$\mu^\dagger = \chi(\zeta^\dagger) \in \Sigma_2 \text{ solves } 0 \in \Lambda(\mu^\dagger) + \Upsilon_2(\mu^\dagger), \tag{1.2}$$

where $\Upsilon_1 : \Sigma_1 \to 2^{\Sigma_1}$, $\Upsilon_2 : \Sigma_2 \to 2^{\Sigma_2}$ are multivalued maximal monotone mappings and $F : \Sigma_1 \to \Sigma_1$ and $\Lambda : \Sigma_2 \to \Sigma_2$ are nonlinear mappings and $\chi : \Sigma_1 \to \Sigma_2$ a bounded and linear mapping. Further, (1.1) and (1.2) are monotone variational inclusion problems in two different Hilbert spaces. Their solutions are denoted by Sol(MVIP(1.1)) and Sol(MVIP(1.2)), respectively. The solution set of SMVI is denoted by Sol(SMVI(1.1)-(1.2)).

Recently, in 2023, Mehra *et.al* [5] introduced a mapping and they named it as ξ-demimetric mapping, the mapping is defined as follows:

Definition 1.1. A mapping $\Psi : \Sigma_1 \to \Sigma_1$ is said to be ξ-demimetric with respect to M-norm, where $\xi \in (-\infty, 1)$ if $F(\Psi) \neq \emptyset$ such that

$$\langle \mu - \mu^\dagger, (I - \Psi)\mu\rangle_M \geq \frac{1}{2}(1-\xi)\|(I-\Psi)\mu\|_M^2, \forall \mu \in \Sigma_1, \mu^\dagger \in F(\Psi).$$

In this paper, we present an algorithm and prove that the sequence generated by that algorithm converges strongly to a common solution of the split monotone variational inclusion problem and the set of fixed points of a finite family of ξ-demimetric mappings.

2. PRELIMINARIES

Suppose $\mathcal{E}$ be a nonempty closed convex subset of a real Hilbert space $(\Sigma, \langle ., . \rangle)$ and $\Psi : \mathcal{E} \to \mathcal{E}$ a mapping. A point $\zeta^\dagger \in \mathcal{E}$ is said to be a fixed point of Ψ if $\Psi(\zeta^\dagger) = \zeta^\dagger$. The set of all fixed points of Ψ will be denoted by $F(\Psi)$.

Lemma 2.1. *Suppose that Ψ be a nonexpansive mapping on a convex, closed subset of a Hilbert space Σ_1. If the mapping Ψ has a fixed point, then the mapping $I - \Psi$ is demiclosed.*

Lemma 2.2. [5] *Let $\Psi : \Sigma_1 \to \Sigma_1$ is ξ-demimetric mapping with respect to M-norm, where $\xi \in (-\infty, 1)$ and $F(\Psi) \neq \emptyset$. Let $P = (1-\gamma)I + \gamma\Psi$, where $\gamma \in (-\infty, \infty)$ with $\gamma \in (0, 1-\xi]$, then $P : \Sigma_1 \to \Sigma_1$ is a quasi nonexpansive mapping.*

Lemma 2.3. [5] *The mapping Ψ_n defined by $\Psi_n = \frac{1}{N}\sum_{i=1}^{N}(1-q_n)I + q_n\Psi_i$ is quasi nonexpansive mapping.*

Definition 2.1. [7]. A Hilbert space Σ satisfies Opial property if, for every weakly convergent sequence (ζ_n) with weak limit $\zeta \in \Sigma$ it holds:

$$\liminf_{n\to\infty} \|\zeta_n - \zeta\| < \liminf_{n\to\infty} \|\zeta_n - \mu\|$$

for all $\mu \in \Sigma$ with $\zeta \neq \mu$.

Lemma 2.4. [2]. *Suppose Σ be a real Hilbert space then for any $\zeta, \mu \in \Sigma$ and $\alpha \in [0, 1]$ following holds*

$$\|\alpha\zeta + (1-\alpha)\mu\|^2 = \alpha\|\zeta\|^2 + (1-\alpha)\|\mu\|^2 - \alpha(1-\alpha)\|\zeta - \mu\|^2.$$

Definition 2.2. A multivalued mapping $\Upsilon_1 : \Sigma_1 \to 2^{\Sigma_1}$ is said to be a monotone mapping if for any $\zeta, \mu \in \Sigma_1$ and $\eta \in \Upsilon_1(\zeta), \vartheta \in \Upsilon_1(\mu)$ such that

$$\langle \zeta - \mu, \eta - \vartheta \rangle \geq 0.$$

Definition 2.3. A monotone mapping $\Upsilon_1 : \Sigma_1 \to 2^{\Sigma_1}$ is said to be maximal if the Graph(Υ_1) is not properly contained in the graph of any other monotone mapping.

We also know that a monotone mapping Υ_1 is said to be maximal if and only if for $(\zeta, \mu) \in \Sigma_1 \times \Sigma_1$, $\langle \zeta - \eta, \mu - \vartheta \rangle \geq 0 \; \forall \; (\eta, \vartheta) \in$ Graph (Υ_1) implies $\mu \in \Upsilon_1(\zeta)$.

Definition 2.4. Suppose $\Upsilon_1 : \Sigma_1 \to 2^{\Sigma_1}$ be a multivalued maximal monotone mapping then the resolvent mapping $J_\rho^{\Upsilon_1} : \Sigma_1 \to \Sigma_1$ associated with Σ_1 for some $\rho > 0$ is given as

$$J_\rho^{\Upsilon_1}(\zeta) = (I + \rho\Upsilon_1)^{-1}(\zeta), \forall \zeta \in \Sigma_1.$$

We also note that for any $\rho > 0$ the resolvent mapping $J_\rho^{\Upsilon_1}$ is a single valued, firmly nonexpansive and nonexpansive mapping.

Lemma 2.5. [11]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n$$

where $\{\delta_n\} \subseteq (0, 1)$, $\{b_n\}$ is sequence in $\mathbb{R}$ such that

(a) $\sum_{n=1}^{\infty} \delta_n = \infty$;

(b) $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $a_n \to 0$.

3. MAIN RESULT

Theorem 3.1. *Let Σ_1, Σ_2 be Hilbert spaces and $\chi : \Sigma_1 \to \Sigma_1$ a bounded linear mapping. Suppose $\Upsilon_1 : \Sigma_1 \to 2^{\Sigma_1}$ and $\Upsilon_2 : \Sigma_2 \to 2^{\Sigma_2}$ are two multivalued monotone mappings, $F : \Sigma_1 \to \Sigma_1$ and $\Lambda : \Sigma_2 \to \Sigma_2$ are θ_1, θ_2-inverse strongly monotone mappings and $\Xi : \Sigma_1 \to \Sigma_1$ a contraction mapping with constant $\alpha \in (0,1)$. Suppose $\Psi_i : \Sigma_1 \to \Sigma_1$ be a finite family of ξ-demimetric mappings with $\xi \in (-\infty, 1)$ such that $I - \Psi_i$ is demiclosed at origin for all $i = 1, 2, \dots N$ and $\Theta = \bigcap_{i=1}^{n} F(\Psi_i) \bigcap \mathrm{Sol}(SMVI(1.1) - (1.2)) \neq \emptyset$. For any given $\zeta_1 \in \Sigma_1$ we define sequence as follows*

$$\begin{cases} \vartheta_n = J_{\rho_n}^{\Upsilon_1}(\zeta_n - \rho_n F(\zeta_n)), \\ \eta_n = J_{\rho_n}^{\Upsilon_2}(I - \rho_n \Lambda)\chi(\vartheta_n), \\ \varpi_n = P_{\mathcal{E}_1}[\vartheta_n + \delta\chi^*(\eta_n - \chi(\vartheta_n))], \\ \zeta_{n+1} = \alpha_n \Xi(\zeta_n) + (1-\alpha_n)\Psi_n(\varpi_n). \end{cases} \tag{3.1}$$

where, $\Psi_n = \frac{1}{N}\sum_{i=1}^{N}(1-q_n)I + q_n\Psi_i$, $\delta \in \left(0, \frac{1}{\|\chi\|^2}\right)$, $\{\rho_n\}$ and $\{\alpha_n\}$ are the sequences in $(0,1)$ and satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(iii) $\lim_{n\to\infty} \rho_n = 0$, $\sum_{n=0}^{\infty} \rho_n = \infty$, *and* $\sum_{n=1}^{\infty} |\rho_n - \rho_{n-1}| < \infty$;

then the sequences $\{\zeta_n\}$ and $\{\vartheta_n\}$ converges strongly to an element $\zeta^\dagger \in \Theta$, where $\zeta^\dagger = P_\Theta \Xi(\zeta^\dagger)$.

Proof. Suppose $\zeta^\dagger \in \Theta$, then $\zeta^\dagger \in \mathrm{Sol}(SMVI(1.1) - (1.2))$, and hence $\zeta^\dagger = J_{\rho_n}^{\Upsilon_1}(\zeta^\dagger - \rho_n F(\zeta^\dagger))$ and $\chi(\zeta^\dagger) = J_{\rho_n}^{\Upsilon_2}(I - \rho_n\Lambda)\chi(\zeta^\dagger)$. Now

$$\begin{aligned} \|\vartheta_n - \zeta^\dagger\|^2 &= \left\| J_{\rho_n}^{\Upsilon_1}(\zeta_n - \rho_n F(\zeta_n)) - J_{\rho_n}^{\Upsilon_1}(\zeta^\dagger - \rho_n F(\zeta^\dagger)) \right\|^2 \\ &\le \|(\zeta_n - \zeta^\dagger) - \rho_n(F(\zeta_n) - F(\zeta^\dagger))\|^2 \\ &= \|\zeta_n - \zeta^\dagger\|^2 + \rho_n^2\|F(\zeta_n) - F(\zeta^\dagger)\|^2 + 2\rho_n\langle \zeta_n - \zeta^\dagger, F(\zeta_n) - F(\zeta^\dagger)\rangle \\ &\le \|\zeta_n - \zeta^\dagger\|^2 - \rho_n(2\theta_1 - \rho_n)\|F(\zeta_n) - F(\zeta^\dagger)\|^2 \\ &\le \|\zeta_n - \zeta^\dagger\|^2. \end{aligned} \tag{3.2}$$

Now

$$\begin{aligned} \|\eta_n - \chi(\zeta^\dagger)\|^2 &= \left\| J_{\rho_n}^{\Upsilon_2}(I - \rho_n\Lambda)\chi(\vartheta_n) - J_{\rho_n}^{\Upsilon_2}(I - \rho_n\Lambda)\chi(\zeta^\dagger) \right\|^2 \\ &\le \|(\chi(\vartheta_n) - \chi(\zeta^\dagger)) - \rho_n(\Lambda\chi(\vartheta_n) - \Lambda\chi(\zeta^\dagger))\|^2 \\ &= \|\chi(\vartheta_n) - \chi(\zeta^\dagger)\|^2 + \rho_n^2\|\Lambda\chi(\vartheta_n) - \Lambda\chi(\zeta^\dagger)\|^2 \\ &+ 2\rho_n\langle \chi(\vartheta_n) - \chi(\zeta^\dagger), \Lambda\chi(\vartheta_n) - \Lambda\chi(\zeta^\dagger)\rangle \\ &\le \|\chi(\vartheta_n) - \chi(\zeta^\dagger)\| - \rho_n(2\theta_2 - \rho_n)\|\Lambda\chi(\vartheta_n) - \Lambda\chi(\zeta^\dagger)\|^2 \\ &\le \|\chi(\vartheta_n) - \chi(\zeta^\dagger)\|^2. \end{aligned} \tag{3.3}$$

Now, we have

$$
\begin{aligned}
\|\varpi_n - \zeta^\dagger\|^2 &= \|P_{\mathcal{E}_1}[\vartheta_n + \delta\chi^*(\eta_n - \chi\vartheta_n)] - \zeta^\dagger\|^2 \\
&\leq \|\vartheta_n + \delta\chi^*(\eta_n - \chi(\vartheta_n)) - \zeta^\dagger\|^2 \\
&= \|\vartheta_n - \zeta^\dagger\|^2 + \|\delta\chi^*(\eta_n - \chi(\vartheta_n))\|^2 + 2\delta\langle\vartheta_n - \zeta^\dagger, \chi^*(\eta_n - \chi(\vartheta_n))\rangle \\
&\leq \|\vartheta_n - \zeta^\dagger\|^2 + \delta^2\|\chi^*\|^2\|\eta_n - \chi(\vartheta_n)\|^2 \\
&+ 2\delta\langle\chi(\vartheta_n - \zeta^\dagger) + (\eta_n - \chi(\vartheta_n)) - (\eta_n - \chi(\vartheta_n)), (\eta_n - \chi(\vartheta_n))\rangle \\
&= \|\vartheta_n - \zeta^\dagger\|^2 + \delta^2\|\chi^*\|^2\|\eta_n - \chi(\vartheta_n)\|^2 + 2\delta\left[\frac{1}{2}\|\eta_n - \chi(\zeta^\dagger)\|^2 + \frac{1}{2}\|\eta_n - \chi(\vartheta_n)\|^2\right. \\
&\left.-\|\eta_n - \chi(\zeta^\dagger)\|^2 - \|\eta_n - \chi(\vartheta_n)\|^2\right] \\
&= \|\vartheta_n - \zeta^\dagger\|^2 - \delta(1 - \delta\|\chi^*\|^2)\|\eta_n - \chi(\vartheta_n)\|^2 \\
&\leq \|\vartheta_n - \zeta^\dagger\|^2 \leq \|\zeta_n - \zeta^\dagger\|^2. \qquad (3.4)
\end{aligned}
$$

And

$$
\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\| &= \|\alpha_n\Xi(\zeta_n) + (1 - \alpha_n)\Psi_n(\varpi_n) - \zeta^\dagger\| \\
&\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\| + (1 - \alpha_n)\|\Psi_n(\varpi_n) - \zeta^\dagger\| \\
&\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\| + (1 - \alpha_n)\|\varpi_n - \zeta^\dagger\| \\
&\leq \alpha_n[\|\Xi(\zeta_n) - \Xi(\zeta^\dagger)\| + \|\Xi(\zeta^\dagger) - \zeta^\dagger\|] + (1 - \alpha_n)\|\varpi_n - \zeta^\dagger\| \\
&\leq \alpha_n\alpha\|\zeta_n - \zeta^\dagger\| + \alpha_n\|\Xi(\zeta^\dagger) - \zeta^\dagger\| + (1 - \alpha_n)\|\zeta_n - \zeta^\dagger\| \\
&\leq [1 - \alpha_n(1 - \alpha)]\|\zeta_n - \zeta^\dagger\| + \alpha_n\|\Xi(\zeta^\dagger) - \zeta^\dagger\| \\
&\leq \max\left\{\|\zeta_n - \zeta^\dagger\|, \frac{\|\Xi(\zeta^\dagger) - \zeta^\dagger\|}{1 - \alpha}\right\} \\
&\cdots \\
&\leq \max\left\{\|\zeta_0 - \zeta^\dagger\|, \frac{\|\Xi(\zeta^\dagger) - \zeta^\dagger\|}{1 - \alpha}\right\}. \qquad (3.5)
\end{aligned}
$$

Hence the sequence $\{\zeta_n\}$ is bounded so the sequences $\{\vartheta_n\}$, $\{\eta_n\}$, $\{\varpi_n\}$, $\{\Xi(\zeta_n)\}$ and $\{\Psi(\varpi_n)\}$ are also bounded. Now we prove that the sequence $\{\zeta_n\}$ is asymptotically regular, that is

$$\lim_{n\to\infty} \|\zeta_{n+1} - \zeta_n\| = 0.$$

Now we have,

$$
\begin{aligned}
\|\vartheta_n - \vartheta_{n-1}\| &= \left\|J_{\rho_n}^{\Upsilon_1}(\zeta_n - \rho_n F(\zeta_n)) - J_{\rho_{n-1}}^{\Upsilon_1}(\zeta_{n-1} - \rho_{n-1}F(\zeta_{n-1}))\right\| \\
&\leq \left\|J_{\rho_n}^{\Upsilon_1}(\zeta_n - \rho_n F(\zeta_n)) - J_{\rho_n}^{\Upsilon_1}(\zeta_{n-1} - \rho_n F(\zeta_{n-1}))\right\| \\
&+ \left\|J_{\rho_n}^{\Upsilon_1}(\zeta_{n-1} - \rho_n F(\zeta_{n-1}) - J_{\rho_{n-1}}^{\Upsilon_1}(\zeta_{n-1} - \rho_{n-1}F(\zeta_{n-1}))\right\| \\
&\leq \|(\zeta_n - \zeta_{n-1}) - \rho_n(F(\zeta_n) - F(\zeta_{n-1})) + (\rho_n - \rho_{n-1})F(\zeta_{n-1})\| \qquad (3.6) \\
&\leq \|\zeta_n - \zeta_{n-1}\| + |\rho_n - \rho_{n-1}|\|F(\zeta_{n-1})\|. \qquad (3.7)
\end{aligned}
$$

Similarly, we can also easily get

$$\|\eta_n - \eta_{n-1}\| \leq \|\Lambda\chi(\vartheta_n) - \Lambda\chi(\vartheta_{n-1})\| + |\rho_n - \rho_{n-1}|\|\Lambda\chi(\vartheta_{n-1})\|. \qquad (3.8)$$

Now we compute,

$$\begin{aligned}
\|\varpi_n-\varpi_{n-1}\|^2 &\le \|\vartheta_n+\delta\chi^*(\eta_n-\chi(\vartheta_n))-\vartheta_{n-1}-\delta\chi^*(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2\\
&\le \|\vartheta_n-\vartheta_{n-1}\|^2+\|\delta\chi^*((\eta_n-\chi(\vartheta_n))-(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2\\
&+2\delta\langle\vartheta_n-\vartheta_{n-1},\chi^*((\eta_n-\chi(\vartheta_n))-(\eta_{n-1}-\chi(\vartheta_{n-1}))\rangle\\
&\le \|\vartheta_n-\vartheta_{n-1}\|^2+\delta\|\chi^*\|^2+\|((\eta_n-\chi(\vartheta_n))-(\eta_{n-1}-\chi(\vartheta_{n-1})))\|^2\\
&+2\delta\langle\chi(\vartheta_n)-\chi(\vartheta_{n-1})+\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1})),\\
&\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1}))\rangle\\
&= \|\vartheta_n-\vartheta_{n-1}\|^2+\delta^2\|\chi^*\|^2+\|\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2\\
&+2\delta\left[\frac{1}{2}\|\eta_n-\eta_{n-1}\|^2+\frac{1}{2}\|\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2-\frac{1}{2}\|\chi(\vartheta_n)-\chi(\vartheta_{n-1})\|^2\right]\\
&-2\delta\|\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2\\
&= \|\vartheta_n-\vartheta_{n-1}\|^2+\delta(1-\delta\|\chi^*\|^2)\|\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2\\
&+\delta\left[\|\eta_n-\eta_{n-1}\|^2-\|\chi(\vartheta_n)-\chi(\vartheta_{n-1})\|^2\right]\\
&= \|\vartheta_n-\vartheta_{n-1}\|^2+\delta(1-\delta\|\chi^*\|^2)\|\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2\\
&+\delta|\rho_n-\rho_{n-1}|(\|\eta_n-\eta_{n-1}\|^2-\|\chi(\vartheta_n)-\chi(\vartheta_{n-1})\|^2)\|\Lambda\chi(\vartheta_{n-1})\|\\
&\le \|\zeta_n-\zeta_{n-1}\|^2+|\rho_n-\rho_{n-1}|\left[\|F(\zeta_{n-1})\|\|\zeta_n-\zeta_{n-1}\|+|\rho_n-\rho_{n-1}|\|F(\zeta_{n-1})\|\right]\\
&-\delta(\delta\|\chi^*\|^2-1)\|\eta_n-\chi(\vartheta_n)-(\eta_{n-1}-\chi(\vartheta_{n-1}))\|^2\\
&+\delta|\rho_n-\rho_{n-1}|(\|\eta_n-\eta_{n-1}\|^2-\|\chi(\vartheta_n)-\chi(\vartheta_{n-1})\|^2)\|\Lambda\chi(\vartheta_{n-1})\|\\
&\le \|\zeta_n-\zeta_{n-1}\|^2+|\rho_n-\rho_{n-1}|L
\end{aligned}$$

$$\le \left|\|\zeta_n-\zeta_{n-1}\|^2+|\rho_n-\rho_{n-1}|L\right|. \tag{3.9}$$

Where L is a constant in such a way that

$$\begin{aligned}
&\left[\|\zeta_n-\zeta_{n-1}\|+|\rho_n-\rho_{n-1}|\right]\|F(\zeta_{n-1})\|+\delta|\rho_n-\rho_{n-1}|\left[\|\eta_n-\eta_{n-1}\|^2\right.\\
&\qquad\left.-\|\chi(\vartheta_n)-\chi(\vartheta_{n-1})\|^2\right]\|\Lambda\chi(\vartheta_{n-1})\|\le L.
\end{aligned}\tag{3.10}$$

From (3.9) we can get

$$\|\varpi_n-\varpi_{n-1}\|\le\|\zeta_n-\zeta_{n-1}\|+\sqrt{|\rho_n-\rho_{n-1}|L}. \tag{3.11}$$

Next, we have

$$\begin{aligned}
\|\zeta_{n+1}-\zeta_n\| &= \|\alpha_n\Xi(\zeta_n)+(1-\alpha_n)\Psi_n(\varpi_n)-[\alpha_{n-1}\Xi(\zeta_{n-1})+(1-\alpha_{n-1})\Psi_{n-1}(\varpi_{n-1})]\|\\
&= \|\alpha_n\Xi(\zeta_n)-\alpha_n\Xi(\zeta_{n-1})+\alpha_n\Xi(\zeta_{n-1})-\alpha_{n-1}\Xi(\zeta_{n-1})+(1-\alpha_n)\Psi_n(\varpi_n)\\
&-(1-\alpha_n)\Psi_n(\varpi_{n-1})+(1-\alpha_n)\Psi_n(\varpi_{n-1})-(1-\alpha_{n-1})\Psi_{n-1}(\varpi_{n-1})\|\\
&\le \alpha_n\alpha\|\zeta_n-\zeta_{n-1}\|+(1-\alpha_n)\|\Psi_n(\varpi_n)-\Psi_n(\varpi_{n-1})\|+2|\alpha_n-\alpha_{n-1}|L_1\\
&\le \alpha_n\alpha\|\zeta_n-\zeta_{n-1}\|+(1-\alpha_n)\|\varpi_n-\varpi_{n-1}\|+2|\alpha_n-\alpha_{n-1}|L_1.
\end{aligned}\tag{3.12}$$

Where $L_1=\sup\{\|\Xi(\zeta_n)\|+\|\Psi_n(\varpi_n)\|:n\in\mathbb{N}\}$. Using (3.11) in (3.12), we get,

$$\|\zeta_{n+1}-\zeta_n\|\le(1-\alpha_n(1-\alpha))\|\zeta_n-\zeta_{n-1}\|+2|\alpha_n-\alpha_{n-1}|L_1+\sqrt{|\rho_n-\rho_{n-1}|L}.$$

By taking $\delta_n = \alpha_n(1-\alpha)$ and $b_n = 2|\alpha_n - \alpha_{n-1}|L_1 + \sqrt{|\rho_n - \rho_{n-1}|L}$ in the above equation and applying Lemma 2.5, we get

$$\lim_{n\to\infty} \|\zeta_{n+1} - \zeta_n\| = 0.$$

We have

$$\begin{aligned} \zeta_{n+1} - \zeta_n &= \alpha_n \Xi(\zeta_n) + (1-\alpha_n)\Psi_n(\varpi_n) - \zeta_n \\ &= \alpha_n(\Xi(\zeta_n) - \zeta_n) + (1-\alpha_n)(\Psi_n(\varpi_n) - \zeta_n). \end{aligned} \tag{3.13}$$

We also have

$$(1-\alpha_n)\|\Psi_n(\varpi_n) - \zeta_n\| \leq \|\zeta_{n+1} - \zeta_n\| + \alpha_n\|\Xi(\zeta_n) - \zeta_n\|.$$

Since, $\alpha_n \to 0$ and $\|\zeta_{n+1} - \zeta_n\| \to 0$ as $n \to \infty$ applying limit in the above equation we get

$$\lim_{n\to\infty} \|\Psi_n(\varpi_n) - \zeta_n\| = 0.$$

Now we show that $\lim\limits_{n\to\infty} \|\zeta_n - \vartheta_n\| = 0.$

$$\begin{aligned} \|\zeta_{n+1} - \zeta^\dagger\|^2 &= \|\alpha_n\Xi(\zeta_n) + (1-\alpha_n)\Psi_n(\varpi_n) - \zeta^\dagger\|^2 \\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (1-\alpha_n)\|\Psi_n(\varpi_n) - \zeta^\dagger\|^2 \\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (1-\alpha_n)\|\vartheta_n - \zeta^\dagger\|^2 \\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (1-\alpha_n)\left[\|\zeta_n - \zeta^\dagger\|^2 + \rho_n(\rho_n - 2\theta_1)\|F(\zeta_n) - F(\zeta^\dagger)\|^2\right] \\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + \|\zeta_n - \zeta^\dagger\|^2 + \rho_n(\rho_n - 2\theta_1)\|F(\zeta_n) - F(\zeta^\dagger)\|^2, \end{aligned}$$

it gives

$$\begin{aligned} \rho_n(\rho_n - 2\theta_1)\|F(\zeta_n) - F(\zeta^\dagger)\|^2 &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + \|\zeta_n - \zeta^\dagger\|^2 - \|\zeta_{n+1} - \zeta^\dagger\|^2 \\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (\|\zeta_n - \zeta^\dagger\| + \|\zeta_{n+1} - \zeta^\dagger\|)\|\zeta_n - \zeta_{n+1}\|. \end{aligned}$$

Since $\lim\limits_{n\to\infty} \|\zeta_{n+1} - \zeta_n\| = 0$ and $\lim\limits_{n\to\infty} \alpha_n = 0$, we get

$$\lim_{n\to\infty} \|F(\zeta_n) - F(\zeta^\dagger)\| = 0.$$

We also have

$$\begin{aligned} \|\vartheta_n - \zeta^\dagger\|^2 &= \left\|J^{\Upsilon_1}_{\rho_n}(\zeta_n - \rho_n F(\zeta_n)) - J^{\Upsilon_1}_{\rho_n}(\zeta^\dagger - \rho_n F(\zeta^\dagger))\right\|^2 \\ &\leq \langle \vartheta_n - \zeta^\dagger, (\zeta_n - \rho_n F(\zeta_n)) - (\zeta^\dagger - \rho_n F(\zeta^\dagger))\rangle \\ &\leq \frac{1}{2}\left\{\|\vartheta_n - \zeta^\dagger\|^2 + \|(\zeta_n - \rho_n F(\zeta_n)) - (\zeta^\dagger - \rho_n F(\zeta^\dagger))\|^2\right. \\ &\left. - \|(\vartheta_n - \zeta_n) + \rho_n(F(\zeta_n) - F(\zeta^\dagger))\|^2\right\} \\ &\leq \frac{1}{2}\left\{\|\vartheta_n - \zeta^\dagger\|^2 + \|\zeta_n - \zeta^\dagger\|^2 - \|\vartheta_n - \zeta_n + \rho_n(F(\zeta_n) - F(\zeta^\dagger))\|^2\right\}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\vartheta_n - \zeta^\dagger\|^2 &\leq \|\zeta_n - \zeta^\dagger\|^2 - \|\vartheta_n - \zeta_n\|^2 - \rho_n^2\|F(\zeta_n) - F(\zeta^\dagger)\|^2 + 2\rho_n\langle \vartheta_n - \zeta_n, F(\zeta_n) - F(\zeta^\dagger)\rangle \\ &\leq \|\zeta_n - \zeta^\dagger\|^2 - \|\vartheta_n - \zeta_n\|^2 + 2\rho_n\|\vartheta_n - \zeta_n\|\|F(\zeta_n) - F(\zeta^\dagger)\|. \end{aligned}$$

It gives

$$\begin{aligned}\|\zeta_{n+1}-\zeta^{\dagger}\|^2 &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(1-\alpha_n)\|\Psi_n(\varpi_n)-\zeta^{\dagger}\|^2\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(1-\alpha_n)\|\vartheta_n-\zeta^{\dagger}\|^2\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(1-\alpha_n)[\|\zeta_n-\zeta^{\dagger}\|^2-\|\vartheta_n-\zeta_n\|^2\\ &+2\rho_n\|\vartheta_n-\zeta_n\|\|F(\zeta_n)-F(\zeta^{\dagger})\|]\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+\|\zeta_n-\zeta^{\dagger}\|^2-\|\vartheta_n-\zeta_n\|^2+2\rho_n\|\vartheta_n-\zeta_n\|\|F(\zeta_n)-F(\zeta^{\dagger})\|.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|\vartheta_n-\zeta_n\|^2 &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+\|\zeta_n-\zeta^{\dagger}\|^2-\|\zeta_{n+1}-\zeta^{\dagger}\|^2+2\rho_n\|\vartheta_n-\zeta_n\|\|F(\zeta_n)-F(\zeta^{\dagger})\|\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(\|\zeta_n-\zeta^{\dagger}\|+\|\zeta_{n+1}-\zeta^{\dagger}\|)\|\zeta_n-\zeta_{n+1}\|\\ &+2\rho_n\|\vartheta_n-\zeta_n\|\|F(\zeta_n)-F(\zeta^{\dagger})\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|\zeta_{n+1}-\zeta_n\|=0$, $\lim_{n\to\infty}\alpha_n=0$, and $\lim_{n\to\infty}\|F(\zeta_n)-F(\zeta^{\dagger})\|=0$, we get

$$\lim_{n\to\infty}\|\vartheta_n-\zeta_n\|=0. \tag{3.14}$$

Similarly, we can also get

$$\begin{aligned}\|\zeta_{n+1}-\zeta^{\dagger}\|^2 &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(1-\alpha_n)\|\Psi_n(\varpi_n)-\zeta^{\dagger}\|^2\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(1-\alpha_n)\|\varpi_n-\zeta^{\dagger}\|^2\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(1-\alpha_n)[\|\vartheta_n-\zeta^{\dagger}\|^2-\delta(1-\delta\|\chi^*\|^2)\|\eta_n-\chi(\vartheta_n)\|^2]\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+\|\zeta_n-\zeta^{\dagger}\|^2-\delta(1-\delta\|\chi^*\|^2)\|\eta_n-\chi(\vartheta_n)\|^2.\end{aligned}$$

And we get

$$\begin{aligned}\delta(1-\delta\|\chi^*\|^2)\|\eta_n-\chi(\vartheta_n)\|^2 &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+\|\zeta_n-\zeta^{\dagger}\|^2-\|\zeta_{n+1}-\zeta^{\dagger}\|^2\\ &\leq \alpha_n\|\Xi(\zeta_n)-\zeta^{\dagger}\|^2+(\|\zeta_n-\zeta^{\dagger}\|+\|\zeta_{n+1}-\zeta^{\dagger}\|)\|\zeta_n-\zeta_{n+1}\|.\end{aligned}$$

Since $\delta(1-\delta\|\chi^*\|^2)>0$, $\|\zeta_{n+1}-\zeta_n\|\to 0$ and $\alpha_n\to 0$ as $n\to\infty$, we get

$$\lim_{n\to\infty}\|\eta_n-\chi(\vartheta_n)\|=0. \tag{3.15}$$

It follows from (3.3) and (3.15) that

$$\begin{aligned}&\rho_n(2\theta_2-\rho_n)\|\Lambda\chi(\vartheta_n)-\Lambda\chi(\zeta^{\dagger})\|^2\\ &\leq \|\chi(\vartheta_n)-\chi(\zeta^{\dagger})\|^2-\|\eta_n-\chi(\zeta^{\dagger})\|^2\\ &=(\|\chi(\vartheta_n)-\chi(\zeta^{\dagger})\|+\|\eta_n-\chi(\zeta^{\dagger})\|)(\|\chi(\vartheta_n)-\chi(\zeta^{\dagger})\|-\|\eta_n-\chi(\zeta^{\dagger})\|)\\ &\leq (\|\chi(\vartheta_n)-\chi(\zeta^{\dagger})\|+\|\eta_n-\chi(\zeta^{\dagger})\|)\|\eta_n-\chi(\vartheta_n)\|.\end{aligned}$$

Since $\rho_n(2\theta_2-\rho_n)>0$, $\|\eta_n-\chi(\vartheta_n)\|\to 0$ as $n\to\infty$, we get

$$\lim_{n\to\infty}\|\Lambda\chi(\vartheta_n)-\Lambda\chi(\zeta^{\dagger})\|=0. \tag{3.16}$$

Next, we have

$$\begin{aligned}\|\varpi_n - \zeta^\dagger\|^2 &= \|P_{\mathcal{E}_1}[\vartheta_n + \delta\chi^*(\eta_n - \chi(\vartheta_n))] - \zeta^\dagger\|^2\\ &\leq \langle \vartheta_n + \delta\chi^*(\eta_n - \chi(\vartheta_n)) - \zeta^\dagger, \varpi_n - \zeta^\dagger\rangle\\ &= \frac{1}{2}[\|(\vartheta_n - \zeta^\dagger) + \delta\chi^*(\eta_n - \chi(\vartheta_n))\|^2 + \|\varpi_n - \zeta^\dagger\|^2\\ &- \|\vartheta_n + \delta\chi^*(\eta_n - \chi(\vartheta_n)) - \zeta^\dagger - \varpi_n + \zeta^\dagger\|^2]\\ &= \frac{1}{2}[\|\vartheta_n - \zeta^\dagger\|^2 + \|\varpi_n - \zeta^\dagger\|^2 + \|\delta\chi^*(\eta_n - \chi(\vartheta_n))\|^2\\ &+ 2\delta\langle \chi(\vartheta_n) - \chi(\zeta^\dagger), \eta_n - \chi(\vartheta_n)\rangle - \|(\vartheta_n - \varpi_n)\| + \delta\chi^*(\eta_n - \chi(\vartheta_n))]\\ &\leq \frac{1}{2}[\|\vartheta_n - \zeta^\dagger\|^2 + \|\varpi_n - \zeta^\dagger\|^2 + \|\delta\chi^*(\eta_n - \chi(\vartheta_n))\|^2 + 2\delta\|\chi(\vartheta_n) - \chi(\zeta^\dagger)\|\|\eta_n - \chi(\vartheta_n)\|\\ &- \|\vartheta_n - \varpi_n\|^2 - \|\delta\chi^*(\eta_n - \chi(\vartheta_n))\|^2 + \|\varpi_n - \zeta^\dagger\|^2 + 2\delta\langle \vartheta_n - \varpi_n, \chi^*(\eta_n - \chi(\vartheta_n))\rangle],\end{aligned}$$

it gives

$$\begin{aligned}\|\varpi_n - \zeta^\dagger\|^2 &\leq \|\vartheta_n - \zeta^\dagger\|^2 - \|\vartheta_n - \varpi_n\|^2 + 2\delta\|\chi(\vartheta_n) - \chi(\zeta^\dagger)\|\|\eta_n - \chi(\vartheta_n)\|\\ &+ 2\delta\|\vartheta_n - \varpi_n\|\|\chi^*\|\|\eta_n - \chi(\vartheta_n)\|\\ &\leq \|\vartheta_n - \zeta^\dagger\|^2 - \|\vartheta_n - \varpi_n\|^2\\ &+ 2\delta\|\eta_n - \chi(\vartheta_n)\|(\|\chi(\vartheta_n) - \chi(\zeta^\dagger)\| + \|\chi^*\|\|\vartheta_n - \varpi_n\|). \end{aligned} \tag{3.17}$$

So, we will have

$$\begin{aligned}\|\zeta_{n+1} - \zeta^\dagger\|^2 &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (1 - \alpha_n)\|\Psi_n(\varpi_n) - \zeta^\dagger\|^2\\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (1 - \alpha_n)\|\varpi_n - \zeta^\dagger\|^2\\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (1 - \alpha_n)[\|\vartheta_n - \zeta^\dagger\|^2 - \|\vartheta_n - \varpi_n\|^2\\ &+ 2\delta\|\eta_n - \chi(\vartheta_n)\|(\|\chi(\vartheta_n) - \chi(\zeta^\dagger)\| + \|\chi^*\|\|\vartheta_n - \varpi_n\|)]\\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + \|\zeta_n - \zeta^\dagger\|^2 - \|\vartheta_n - \varpi_n\|^2\\ &+ 2\delta\|\eta_n - \chi(\vartheta_n)\|(\|\chi(\vartheta_n) - \chi(\zeta^\dagger)\| + \|\chi^*\|\|\vartheta_n - \varpi_n\|),\end{aligned}$$

and it gives

$$\begin{aligned}\|\vartheta_n - \varpi_n\|^2 &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + \|\zeta_n - \zeta^\dagger\|^2 - \|\zeta_{n+1} - \zeta^\dagger\|^2\\ &+ 2\delta[\|\eta_n - \chi(\vartheta_n)\|(\|\chi(\vartheta_n) - \chi(\zeta^\dagger)\| + \|\chi^*\|\|\vartheta_n - \varpi_n\|)]\\ &\leq \alpha_n\|\Xi(\zeta_n) - \zeta^\dagger\|^2 + (\|\zeta_n - \zeta^\dagger\| + \|\zeta_{n+1} - \zeta^\dagger\|)\|\zeta_n - \zeta_{n+1}\|\\ &+ 2\delta\|\eta_n - \chi(\vartheta_n)\|(\|\chi(\vartheta_n) - \chi(\zeta^\dagger)\| + \|\chi^*\|\|\vartheta_n - \varpi_n\|).\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \|\zeta_{n+1} - \zeta_n\| = 0$ and $\lim_{n\to\infty} \|\eta_n - \chi(\vartheta_n)\| = 0$ applying limit $n \to \infty$ in the above equation, we get

$$\lim_{n\to\infty} \|\vartheta_n - \varpi_n\| = 0. \tag{3.18}$$

Now, we will also have

$$\|\varpi_n - \zeta_n\| \leq \|\varpi_n - \vartheta_n\| + \|\vartheta_n - \zeta_n\| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

and

$$\|\Psi_n(\varpi_n) - \varpi_n\| \leq \|\Psi_n(\varpi_n) - \zeta_n\| + \|\zeta_n - \vartheta_n\| + \|\vartheta_n - \varpi_n\| \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Since the sequence $\{\varpi_n\}$ is bounded, so $\exists$ a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ such that $\varpi_{n_k} \rightharpoonup \hat{\zeta}$(let). Therefore, from (3.20) we can also say that $\exists$ a subsequence $\{\vartheta_{n_k}\}$ of $\{\vartheta_n\}$ such that $\vartheta_{n_k} \rightharpoonup \hat{\zeta}$. First we prove that $\hat{\zeta} \in F(\Psi_n)$. On contrary, let $\hat{\zeta} \notin F(\Psi_n)$. Since $\Psi_n(\hat{\zeta}) \neq \hat{\zeta}$, then using Opial property and applying (3.20), we get

$$\begin{aligned}\liminf_{k\to\infty} \|\varpi_{n_k} - \hat{\zeta}\| &< \liminf_{k\to\infty} \|\varpi_{n_k} - \Psi_n(\hat{\zeta})\| \\ &\leq \liminf_{k\to\infty} \{\|\varpi_{n_k} - \Psi_n(\varpi_{n_k})\| + \|\Psi_n(\varpi_{n_k}) - \Psi_n(\hat{\zeta})\|\} \\ &\leq \liminf_{k\to\infty} \|\varpi_{n_k} - \hat{\zeta}\|,\end{aligned}$$

which is a contradiction, and hence $\hat{\zeta} \in F(\Psi_n)$. On the other hand, we also have

$$\vartheta_{n_k} = J_{\rho_n}^{\Upsilon_1}(\zeta_{n_k} - \rho_{n_k} F(\zeta_{n_k})),$$

and it can be written as

$$\frac{(\zeta_{n_k} - \vartheta_{n_k}) - \rho_{n_k} F(\zeta_{n_k})}{\rho_{n_k}} \in \Upsilon_1(\vartheta_{n_k}). \tag{3.21}$$

Applying limit $k \to \infty$ in the above equation and using (3.14), and the fact that the graph of maximal monotone mapping is weakly-strongly closed, we get $0 \in \Upsilon_1(\hat{\zeta}) + F(\hat{\zeta})$ that is $\hat{\zeta} \in \text{Sol}(MVIP(1.1))$. Further, since sequences $\{\zeta_n\}$ and $\{\vartheta_n\}$ have the same asymptotical behaviour, $\{\chi(\zeta_n)\}$ weakly converges to $\chi(\hat{\zeta})$. Using (3.15) and the nonexpansiveness of the mapping $J_{\rho_n}^{\Upsilon_2}(I - \rho_n\Lambda)$ and Lemma 2.1 we get $0 \in \Upsilon_2(\chi(\hat{\zeta})) + \Lambda(\chi(\hat{\zeta}))$ that is $\chi(\hat{\zeta}) \in \text{Sol}(MVIP(1.2))$. Further, we claim that $\limsup_{n\to\infty}\langle \Xi(\zeta^\dagger) - \zeta^\dagger, \zeta_n - \zeta^\dagger\rangle \leq 0$, where $\zeta^\dagger \in P_\Theta \Xi(\zeta^\dagger)$. We have

$$\begin{aligned}\limsup_{n\to\infty}\langle \Xi(\zeta^\dagger) - \zeta^\dagger, \zeta_n - \zeta^\dagger\rangle &= \limsup_{n\to\infty}\langle \Xi(\zeta^\dagger) - \zeta^\dagger, \Psi_n(\varpi_n) - \zeta^\dagger\rangle \\ &\leq \limsup_{n\to\infty}\langle F(\zeta^\dagger) - \zeta^\dagger, \Psi_n(\varpi_n) - \zeta^\dagger\rangle \\ &= \langle \Xi(\zeta^\dagger) - \zeta^\dagger, \hat{\zeta} - \zeta^\dagger\rangle \\ &\leq 0. \end{aligned} \tag{3.22}$$

Last, we show that the sequence $\zeta_n \to \zeta^\dagger$. We have

$$\begin{aligned}\|\zeta_{n+1} - \zeta^\dagger\|^2 &= \|\alpha_n \Xi(\zeta_n) + (1 - \alpha_n)\Psi_n(\varpi_n) - \zeta^\dagger\|^2 \\ &= \|\alpha_n(\Xi(\zeta_n) - \zeta^\dagger) + (1 - \alpha_n)(\Psi_n(\varpi_n) - \zeta^\dagger)\|^2 \\ &= \|\alpha_n(\Xi(\zeta_n) - \Xi(\zeta^\dagger)) + \alpha_n(F(\zeta^\dagger) - \zeta^\dagger) + (1 - \alpha_n)\Psi_n(\varpi_n) - \zeta^\dagger\|^2 \\ &\leq \|\alpha_n(\Xi(\zeta_n) - \Xi(\zeta^\dagger)) + (1 - \alpha_n)\Psi_n(\varpi_n) - \zeta^\dagger\|^2 + 2\alpha_n\langle \Xi(\zeta_n) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\ &\leq |\alpha_n|\|\Xi(\zeta_n) - \Xi(\zeta^\dagger)\|^2 + (1 - \alpha_n)\|\varpi_n - \zeta^\dagger\|^2 + 2\alpha_n\langle \Xi(\zeta_n) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\ &\leq \alpha_n\alpha^2\|\zeta_n - \zeta^\dagger\|^2 + (1 - \alpha_n)\|\zeta_n - \zeta^\dagger\|^2 + 2\alpha_n\langle \Xi(\zeta_n) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\ &\leq (1 - (1 - \alpha^2)\alpha_n)\|\zeta_n - \zeta^\dagger\|^2 + 2\alpha_n\langle \Xi(\zeta_n) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle.\end{aligned}$$

Applying lemma 2.5 and (3.22) in the above equation we get $\zeta_n \to \zeta^\dagger$ as $n \to \infty$. From $\|\vartheta_n - \zeta_n\| \to 0$, $\vartheta_n \rightharpoonup \hat{\zeta} \in \Theta$ and $\zeta_n \to \zeta^\dagger$ as $n \to \infty$. We get $\zeta^\dagger = \hat{\zeta}$, it completes the proof.
∎

CONCLUSION

This paper presents a novel viscosity approximation algorithm designed to address the problem of finding a common element in the fixed point set of a finite family of κ-demimetric mappings and the solution set of split monotone variational inclusion problems in Hilbert spaces. By integrating the properties of κ-demimetric mappings with a contraction mapping, as established in Theorem 3.1, our algorithm achieves strong convergence under well-defined conditions.

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