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ON A NOVEL CLASS OF ADVANCED INTEGRALS OF MACROBERT TYPE INCORPORATING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The main objective of this study is to evaluate two master formulas expressed as integrals over the interval from zero to $\pi/2$. These integrals involve exponential functions and trigonometric functions sine and cosine, along with ${}_3F_2$ hypergeometric functions parameterized by general variables i and j, where $i = j = 0, \pm 1, \pm 2, \ldots$. From these master formulas, we derive twenty-six MacRobert-type integrals by evaluating them at specific values of i and j. The results are expressed using Gamma functions. The findings given here are derived from summation formulas for the series ${}_4F_3$ obtained recently by Choi and Rathie along with a general result attributed to MacRobert.

Key words and phrases: Generalized Hypergeometric function; Summation theorem; MacRobert-integral.

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1. INTRODUCTION

Let $\mathbb{C}, \mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive and non-positive integers respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric series ${}_pF_q\,(p, q \in \mathbb{N}_0)$ defined by [1, 2, 7, 8, 13]:

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q\left(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z\right),$$

where $\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-; j = 1, \dots, q$.

Here $(a)_n$ is the Pochhammer symbol defined (for $a \in \mathbb{C}$) by [1]

$$(a)_n = \begin{cases} 1 & ;(n=0) \\ a(a+1)\dots(a+n-1) & ;(n \in \mathbb{N}) \end{cases} = \frac{\Gamma(a+n)}{\Gamma(a)} \quad \left(a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right),$$

and $\Gamma(z)$ is the familiar Gamma function defined as:

$$(1.2)\qquad \Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx,\ \mathrm{Re}(z) > 0.$$

Using the ratio test [2, 3] we can verify that the series is convergent for all $p \leq q$, converges in $|z| < 1$ for $p = q + 1$ and converges everywhere for $p < q + 1$ and while it is divergent for all z, $(z \neq 0)$ if $p > q + 1$.

While substituting $p = 1, q = 1$, in (1.1) we get:

$${}_1F_1\left[\begin{matrix}a \\ c\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}$$

which converges everywhere and popularly known in the literature as Kummer's function or confluent hypergeometric function. Both Gauss hypergeometric and Kummers hypergeometric functions have found applications in many physical problems[10, 11, 12].

Summation theorems play a crucial role in evaluating hypergeometric series. The traditional summation theorems, such as Gauss, Gauss's second, Bailey, and Kummer for the series ${}_2F_1$ and Watson, Dixon, and Whipple for the series ${}_3F_2$ allow scientists and mathematicians to formulate hypergeometric series in different ways.

The classical Watson's summation theorem for a ${}_3F_2$ hypergeometric function [1] has the following form:

$$(1.3)\qquad {}_3F_2\left[\begin{matrix}a,\ b,\ c, \\ \frac{1}{2}(a+b+1),\ 2c\end{matrix}; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} = S$$

provided $\mathrm{Re}(2c - a - b) > -1$.

In 2010, Kim et al.[4] provided two extensions of classical Watson's summation theorem (1.3), one of which is given by the following result:

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+1),\ 2c+1,\ d \end{matrix};1\right] \\
&=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\
&\quad+\left(\frac{2c}{d}-1\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}
\end{aligned}\tag{1.4}$$

provided $\mathrm{Re}(2c-a-b)>-1$, $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

On substituting $d=2c$, (1.4) reduces to (1.3). Kim et al.[4] obtained these results with the help of generalization of Watson's summation theorem obtained earlier by Lavoie et al. [5]. In 2016, Choi and Rathie [6] provided thirteen summation formulas including (1.4) for the series ${}_4F_3$ with the help of results obtained by Lavoie et al. [5], which are essential for our current investigation and are presented below:

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+1),\ 2c,\ d \end{matrix};1\right] \\
&=\left[1+\frac{ab}{d(2c-a-b-1)}\right]\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\
&\quad+\left(\frac{2}{d}\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b\right)} \\
&=S_1
\end{aligned}\tag{1.5}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b)>1$.

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+1),\ 2c+1,\ d \end{matrix};1\right] \\
&=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\
&\quad+\left(\frac{2c}{d}-1\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)} \\
&=S_2
\end{aligned}\tag{1.6}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b)>-1$.

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+1),\ 2c+2,\ d \end{matrix};1\right] \\
&=\left(\frac{(c-a+1)(c-b+1)+c(c+1)}{2(c+1)}-\frac{abc}{2d(c+1)}\right) \\
&\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{3}{2}\right)} \\
&\quad+\frac{2(c-d)}{d(c+1)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)} \\
&=S_3
\end{aligned}\tag{1.7}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b) > -1$.

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b),\ 2c,\ d \end{matrix};1\right] \\ &= \left(1+\frac{a(2c-a+b)}{d(2c-a-b-2)}\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)} \\ &+ \left(1+\frac{b(2c+a-b)}{d(2c-a-b-2)}\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\ &= S_4 \end{aligned} \tag{1.8}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b) > 2$.

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b),\ 2c+1,\ d \end{matrix};1\right] \\ &= \frac{1}{2c-a}\left(2c-a-b+\frac{2bc}{d}\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\ &+ \frac{1}{2c-b}\left(2c-a-b+\frac{2ac}{d}\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)} \\ &= S_5 \end{aligned} \tag{1.9}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b) > 0$.

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+2),\ 2c,\ d \end{matrix};1\right] \\ &= \frac{2(d-b)}{d(a-b)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\ &- \frac{2(d-a)}{d(a-b)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)} \\ &= S_6 \end{aligned} \tag{1.10}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b) > 0$.

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+2),\ 2c+1,\ d \end{matrix};1\right] \\ &= \frac{2\left(2c-a+b-\frac{2bc}{d}\right)}{(2c-a)(a-b)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\ &- \frac{2\left(2c+a-b-\frac{2ac}{d}\right)}{(2c-b)(a-b)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)} \\ &= S_7 \end{aligned} \tag{1.11}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\operatorname{Re}(2c-a-b) > 0$.

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+2),\ 2c+2,\ d \end{matrix};1\right] \\
&= D_{1,2}^{(1)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+\frac{3}{2}\right)} \\
&\quad - D_{1,2}^{(2)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}b+1\right)} \\
&= S_8
\end{aligned} \tag{1.12}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\operatorname{Re}(2c-a-b) > 0$, where

$$D_{1,2}^{(1)} = \frac{2c(c+1)-(a-b)(c-b+1)-\frac{bc(2c+a-b+2)}{d}}{(c+1)(a-b)}$$

and

$$D_{1,2}^{(2)} = \frac{2c(c+1)+(a-b)(c-a+1)-\frac{ac(2c-a+b+2)}{d}}{(c+1)(a-b)}$$

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+1),\ 2c-1,\ d \end{matrix};1\right] \\
&= D_{0,-1}^{(1)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} \\
&\quad + D_{0,-1}^{(2)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b\right)} \\
&= S_9
\end{aligned} \tag{1.13}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\operatorname{Re}(2c-a-b) > 3$, where

$$D_{0,-1}^{(1)} = \frac{1}{2}(2c-a-b-3)+\frac{ab}{d}$$

and

$$D_{0,-1}^{(2)} = \frac{1}{2}(2c-a-b-3)+\frac{(c-a-1)(c-b-1)+c(c-1)}{d}$$

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+2),\ 2c-1,\ d \end{matrix};1\right] \\
&= D_{1,-1}^{(1)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-1\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} \\
&\quad + D_{1,-1}^{(2)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+1\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)} \\
&= S_{10}
\end{aligned} \tag{1.14}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\operatorname{Re}(2c-a-b) > 2$, where

$$D_{1,-1}^{(1)} = \frac{2c-a-b-2-\frac{2b(c-a-1)}{d}}{a-b}$$

and

$$D_{1,-1}^{(2)} = -\frac{2c-a-b-2-\frac{2a(c-b-1)}{d}}{a-b}$$

$$
(1.15)\quad \begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b-1),\ 2c+1,\ d \end{matrix};1\right] \\
&= D^{(1)}_{-2,1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\
&\quad + D^{(2)}_{-2,1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)} \\
&= S_{11}
\end{aligned}
$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b) > 1$, where

$$D^{(1)}_{-2,1} = \frac{1}{2}(2c-a-b-1) + \frac{2abc}{d(a+b-1)}$$

and

$$D^{(2)}_{-2,1} = \frac{1}{4}(4c-a-b+1)(2c-a-b-1) + \frac{2c}{d}\{(a+1)(2c-a+1)+(b+1)(2c-b+1)-2c-1\}$$

$$
(1.16)\quad \begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b),\ 2c-1,\ d \end{matrix};1\right] \\
&= D^{(1)}_{-1,-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-2\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} \\
&\quad + D^{(2)}_{-1,-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-2\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)} \\
&= S_{12}
\end{aligned}
$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b) > 4$, where

$$D^{(1)}_{-1,-1} = \frac{1}{4}(2c-a+b-2)(2c-a-b-4) + \frac{b}{2d}\{2c(c-1)+(a-b)(c-a-1)\}$$

and

$$D^{(2)}_{-1,-1} = \frac{1}{4}(2c+a-b-2)(2c-a-b-4) + \frac{1}{2d}\{2c(c-1)-(a-b)(c-b-1)\}$$

$$
(1.17)\quad \begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b),\ 2c+2,\ d \end{matrix};1\right] \\
&= \frac{\left(c-b+1+\frac{bc}{d}\right)}{c+1}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+1\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+\frac{3}{2}\right)} \\
&\quad + \frac{\left(c-a+1+\frac{ac}{d}\right)}{c+1}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+1\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}b+1\right)} \\
&= S_{13}
\end{aligned}
$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\mathrm{Re}(2c-a-b) > -2$.

The following integral is attributed to MacRobert [9], and will be used in proving our main results:

$$(1.18)\quad \int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}d\theta = e^{\frac{\omega\pi\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

provided $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ and $\omega = \sqrt{-1}$.

In the next section, we aim to produce twenty-six integrals in the form of two main formulae involving generalised hypergeometric functions, by deploying the summation theorems (1.5) through (1.17), along with the (1.18).

2. MACROBERT-TYPE INTEGRALS CONSISTING OF GENERALISED HYPERGEOMETRIC FUNCTIONS

To determine our primary results, first we will establish two main formulae involving generalized hypergeometric functions, asserted in the following theorems.

Theorem 2.1. *For* $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(c+j) > 0$ *and* $\mathrm{Re}(2c-a-b+i+2j) > 1$, *for* $i = j = 0, \pm 1, \pm 2, \cdots$ *the following result holds.*

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} e^{\omega(2c+j)\theta}(sin\theta)^{c+j-1}(cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b+i+1),\ d \end{matrix}\ ; e^{\omega\theta}cos\theta\right] d\theta \\
&= e^{\omega(c+j)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)}{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j,\ d \end{matrix}\ ; 1\right].
\end{aligned} \tag{2.1}$$

Theorem 2.2. *For* $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(c+j) > 0$ *and* $\mathrm{Re}(2c-a-b+i+2j) > 1$, *for* $i = j = 0, \pm 1, \pm 2, \cdots$ *the following result holds.*

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} e^{\omega(2c+j)\theta}(sin\theta)^{c-1}(cos\theta)^{c+j-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b+i+1),\ d \end{matrix}\ ; e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right] d\theta \\
&= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)}{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j,\ d \end{matrix}\ ; 1\right].
\end{aligned} \tag{2.2}$$

Proof. The process used to get the result (2.1) is rather straightforward.

In order to establish (2.1), we proceed as follows:

Using the letter I to represent the left side of (2.1), we get

$$I = \int_0^{\frac{\pi}{2}} e^{\omega(2c+j)\theta}(sin\theta)^{c+j-1}(cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b+i+1),\ d \end{matrix}\ ; e^{\omega\theta}cos\theta\right] d\theta.$$

Expressing ${}_3F_2$ as a series, we obtain

$$I = \int_0^{\frac{\pi}{2}} e^{\omega(2c+j)\theta}(sin\theta)^{c+j-1}(cos\theta)^{c-1}\sum_{n=0}^{\infty}\frac{(a)_n\ (b)_n\ (d+1)_n\ (e^{\omega\theta}cos\theta)^n}{(\frac{1}{2}(a+b+i+1))_n\ (d)_n n!}\, d\theta.$$

Altering the order of integration and summation, and evaluating the integral using the result (1.18), we arrive at the following equation:

$$I = \sum_{n=0}^{\infty}\frac{(a)_n\ (b)_n\ (d+1)_n\ (c)_n}{(\frac{1}{2}(a+b+i+1))_n\ (d)_n\ (2c+j)_n n!}\ e^{\omega(c+j)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)}.$$

By summing up the series, we finally get

$$I = e^{\omega(c+j)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)}{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j,\ d \end{matrix}\ ; 1\right],$$

which is the right hand side of (2.1).

This completes the proof of Theorem (2.1). ∎

In exactly the same manner, we can establish theorem (2.2).

3. VARIOUS RESULTS IN COMPACT FORMS

In this section, we shall deduce twenty-six MacRobert type integrals by taking suitable values for i and j in (2.1) and (2.2), using corresponding results from (1.5) to (1.17).

By substituting various values for i and j into Theorem (2.1), we obtain the following thirteen results, labeled as (3.1) through (3.13).

(a) For $i = j = 0$, we have

$$\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b+1),\ d \end{matrix};e^{\omega\theta}cos\theta\right]d\theta = e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}S_1, \tag{3.1}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > 1$.
Here, S_1 is same as given in (1.5).

(b) For $i = 0$ and $j = 1$, we have

$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b+1),\ d \end{matrix};e^{\omega\theta}cos\theta\right]d\theta = e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_2, \tag{3.2}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > -1$.
Here, S_2 is same as given in (1.6).

(c) For $i = 0$ and $j = 2$, we have

$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b+1),\ d \end{matrix};e^{\omega\theta}cos\theta\right]d\theta = e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}S_3, \tag{3.3}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > -3$.
Here, S_3 is same as given in (1.7).

(d) For $i = -1$ and $j = 0$, we have

$$\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b),\ d \end{matrix};e^{\omega\theta}cos\theta\right]d\theta = e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}S_4, \tag{3.4}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > 2$.
Here, S_4 is same as given in (1.8).

(e) For $i = -1$ and $j = 1$, we have

$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b),\ d \end{matrix};e^{\omega\theta}cos\theta\right]d\theta = e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_5, \tag{3.5}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > 0$.
Here, S_5 is same as given in (1.9).

(f) For $i=1$ and $j=0$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\&=e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}S_6,\end{aligned}\tag{3.6}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>0$.
Here, S_6 is same as given in (1.10).

(g) For $i=1$ and $j=1$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\&=e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_7,\end{aligned}\tag{3.7}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>-2$.
Here, S_7 is same as given in (1.11).

(h) For $i=1$ and $j=2$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\&=e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}S_8,\end{aligned}\tag{3.8}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>-4$.
Here, S_8 is same as given in (1.12).

(i) For $i=0$ and $j=-1$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\&=e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}S_9,\end{aligned}\tag{3.9}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>3$.
Here, S_9 is same as given in (1.13).

(j) For $i=1$ and $j=-1$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\&=e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}S_{10},\end{aligned}\tag{3.10}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>2$.
Here, S_{10} is same as given in (1.14).

(k) For $i=-2$ and $j=1$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b-1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\&=e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_{11},\end{aligned}\tag{3.11}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > 1$.
Here, S_{11} is same as given in (1.15).

(l) For $i = -1$ and $j = -1$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta \\ &= e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}S_{12},\end{aligned} \tag{3.12}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > 4$.
Here, S_{12} is same as given in (1.16).

(m) For $i = -1$ and $j = 2$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta \\ &= e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}S_{13},\end{aligned} \tag{3.13}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > -2$.
Here, S_{13} is same as given in (1.17).

Similarly, substituting various values for i and j into Theorem (2.2), we obtain the following thirteen results, labeled as (3.14) through (3.26).

(n) For $i = j = 0$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+1),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta \\ &= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}S_1,\end{aligned} \tag{3.14}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > 1$.
Here, S_1 is same as given in (1.5).

(o) For $i = 0$ and $j = 1$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2c+1)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+1),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta \\ &= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_2,\end{aligned} \tag{3.15}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > -1$.
Here, S_2 is same as given in (1.6).

(p) For $i = 0$ and $j = 2$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2c+2)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c+1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+1),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta \\ &= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}S_3,\end{aligned} \tag{3.16}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > -3$.
Here, S_3 is same as given in (1.7).

(q) For $i=-1$ and $j=0$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta\\ &=e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}S_4,\end{aligned} \tag{3.17}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>2$.
Here, S_4 is same as given in (1.8).

(r) For $i=-1$ and $j=1$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta\\ &=e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_5,\end{aligned} \tag{3.18}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>0$.
Here, S_5 is same as given in (1.9).

(s) For $i=1$ and $j=0$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+2),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta\\ &=e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}S_6,\end{aligned} \tag{3.19}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>0$.
Here, S_6 is same as given in (1.10).

(t) For $i=1$ and $j=1$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+2),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta\\ &=e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_7,\end{aligned} \tag{3.20}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>-2$.
Here, S_7 is same as given in (1.11).

(u) For $i=1$ and $j=2$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c+1}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+2),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta\\ &=e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}S_8,\end{aligned} \tag{3.21}$$

provided $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(2c-a-b)>-4$.
Here, S_8 is same as given in (1.12).

(v) For $i=0$ and $j=-1$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-2}{}_3F_2\left[\begin{matrix}a,\ b,\ d+1\\ \frac{1}{2}(a+b+1),\ d\end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta\\ &=e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}S_9,\end{aligned} \tag{3.22}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\text{Re}(c) > 0$ and $\text{Re}(2c - a - b) > 3$.
Here, S_9 is same as given in (1.13).

(w) For $i = 1$ and $j = -1$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-2}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b+2),\ d \end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta \\ = e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}S_{10},\end{aligned} \tag{3.23}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\text{Re}(c) > 0$ and $\text{Re}(2c - a - b) > 2$.
Here, S_{10} is same as given in (1.14).

(x) For $i = -2$ and $j = 1$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b-1),\ d \end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta \\ = e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}S_{11},\end{aligned} \tag{3.24}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\text{Re}(c) > 0$ and $\text{Re}(2c - a - b) > 1$.
Here, S_{11} is same as given in (1.15).

(y) For $i = -1$ and $j = -1$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-2}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b),\ d \end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta \\ = e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}S_{12},\end{aligned} \tag{3.25}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\text{Re}(c) > 0$ and $\text{Re}(2c - a - b) > 4$.
Here, S_{12} is same as given in (1.16).

(z) For $i = -1$ and $j = 2$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c+1}{}_3F_2\left[\begin{matrix} a,\ b,\ d+1 \\ \frac{1}{2}(a+b),\ d \end{matrix};e^{\omega(\theta-\frac{\pi}{2})}sin\theta\right]d\theta \\ = e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}S_{13},\end{aligned} \tag{3.26}$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\text{Re}(c) > 0$ and $\text{Re}(2c - a - b) > -2$.
Here, S_{13} is same as given in (1.17).

4. SPECIAL CASES

In this section, we present interesting special cases of the results (3.1) to (3.13) by substituting specific values for the parameters a and b.

First, we let $b = -2n$ and replace a by $a + 2n$, or we let $b = -2n - 1$ and replace a by $a + 2n + 1$, where $n \in \mathbb{N}_0$. We now observe that in both cases, one of the two terms on the right-hand side of the resulting integral formulas (3.1) to (3.13) will vanish. Subsequently, we derive the following twenty-four new integral formulas, valid for all $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$. In constructing the right-hand side of these formulas, we employ the summation theorems obtained by Choi and Rathie[6].

Throughout this section, we assume that

$$n \in \mathbb{N}_0 \quad \text{and} \quad d \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

Under this assumption, we now present the following results:

$$\begin{aligned}(4.1)\qquad &\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\left(1-\frac{2n(a+2n)}{d(2c-a-1)}\right)\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}.\end{aligned}$$

$$\begin{aligned}(4.2)\qquad &\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\left(-\frac{1}{d}\right)\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}.\end{aligned}$$

$$\begin{aligned}(4.3)\qquad &\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}.\end{aligned}$$

$$\begin{aligned}(4.4)\qquad &\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}\frac{(d-2c)}{d(2c+1)}\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{3}{2}\right)_n}.\end{aligned}$$

$$\begin{aligned}(4.5)\qquad &\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}\frac{\alpha}{(c+1)(2c-a+1)}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c-\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{3}{2}\right)_n}.\end{aligned}$$

where

$$\alpha := (c-a-2n+1)(c+2n+1)+c(c+1)+\frac{2nc(a+2n)}{d}.$$

$$\begin{aligned}(4.6)\qquad &\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}\frac{d-c}{d(c+1)}\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{3}{2}\right)_n}.\end{aligned}$$

$$\begin{aligned}(4.7)\qquad &\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\left(1-\frac{2n(2c+a+4n)}{d(2c-a-2)}\right)\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{1}{2}\right)_n}.\end{aligned}$$

(4.8)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\left(\frac{-1}{a+2n}\right)\left(1+\frac{(a+2n+1)(2c-a-4n-2)}{d(2c-a-2)}\right)\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{1}{2}\right)_n}.$$

(4.9)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}\frac{\left(2c-a-\frac{4nc}{d}\right)}{(2c-a-2n)}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{1}{2}\right)_n}.$$

(4.10)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}\frac{a-2c+2n+1+\frac{4nc}{d}}{(2c-a-4n-1)(a+2n)}\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{1}{2}\right)_n}.$$

(4.11)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\frac{a(d+2n)}{d(a+4n)}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{1}{2}\right)_n}.$$

(4.12)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega c\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\frac{(d-a-2n-1)}{d(a+4n+2)}\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a+1\right)_n\left(c+\frac{1}{2}\right)_n}.$$

(4.13)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}\frac{a\left(2c-a-4n+\frac{4nc}{d}\right)}{(2c-a-2n)(a+4n)}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{1}{2}\right)_n}.$$

(4.14)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c+1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}\frac{\left(2c+a+4n+2-\frac{2c}{d}(a+2n+1)\right)}{(2c+2n+1)(a+4n+2)}\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a+1\right)_n\left(c+\frac{1}{2}\right)_n}.$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}\frac{\beta}{(2c-a)(c+1)(a+4n)}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{3}{2}\right)_n}.\end{aligned}\tag{4.15}$$

where

$$\beta := a\left(2c(c+1)-(a+4n)(c+2n+1)+\frac{2nc}{d}(2c+a+4n+2)\right).$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}\frac{\gamma}{(a-2c)(c+1)(a+4n+2)}\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a+1\right)_n\left(c+\frac{3}{2}\right)_n}.\end{aligned}\tag{4.16}$$

where

$$\gamma := \frac{c}{d}(a+2n+1)(2c-4n-a)-2c(c+1)-(a+4n+2)(c-a-2n).$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}\left(1-\frac{4n(a+2n)}{d(2c-a-3)}\right)\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c-\frac{1}{2}\right)_n}.\end{aligned}\tag{4.17}$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+1),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}\frac{\delta}{d(2c-1)(2c-a-3)}\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}.\end{aligned}\tag{4.18}$$

where

$$\delta := d(a-2c+3)-2\{c(c-1)+(c+2n)(c-a-2n-2)\}.$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}\frac{a}{a+4n}\left(1+\frac{4n(c-a-2n-1)}{d(2c-a-2)}\right)\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n\left(c-\frac{1}{2}\right)_n}.\end{aligned}\tag{4.19}$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}(a+2),\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta\\ &= e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}\frac{2c-a-2-\frac{2}{d}(c+2n)(a+2n+1)}{(2c-1)(a+4n+2)}\\ &\quad\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}.\end{aligned}\tag{4.20}$$

(4.21)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c-1)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(2c-1)}\frac{\epsilon\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a\right)_n\left(c-\frac{1}{2}\right)_n}.$$

where
$$\epsilon := 1-\frac{2n\{d(2c-a+4)+4c(c-1)+2(a+4n)(c-a-2n-1)\}}{d(2c-a-4)(2c-a-4n-2)}.$$

(4.22)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c-1)\theta}(\sin\theta)^{c-2}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c-1)\frac{\pi}{2}}\frac{-\Gamma(c)\Gamma(c-1)\epsilon\ \left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+2\right)_n}{\Gamma(2c-1)ad(2c-1+2n)(2c-a-4)\left(\frac{1}{2}a+1\right)_n\left(c-\frac{1}{2}\right)_n}.$$

where
$$\epsilon := d(2c+a+4n)(2c-a-4)+2(a+2n+1)\{2c(c-1)-(a+4n+2)(c+2n)\}.$$

(4.23)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}\frac{\left(c+2n+1-\frac{2nc}{d}\right)}{(c+1)}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a\right)_n\left(c+\frac{3}{2}\right)_n}.$$

(4.24)
$$\int_0^{\frac{\pi}{2}} e^{\omega(2c+2)\theta}(\sin\theta)^{c+1}(\cos\theta)^{c-1}{}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ d+1\\ \frac{1}{2}a,\ d\end{matrix};e^{\omega\theta}cos\theta\right]d\theta$$
$$= e^{\omega(c+2)\frac{\pi}{2}}\frac{\Gamma(c)\Gamma(c+2)}{\Gamma(2c+2)}\left(-\frac{\left(c-a-2n+\frac{c}{d}(a+2n+1)\right)}{a(c+1)}\right)$$
$$\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a+1\right)_n\left(c+\frac{3}{2}\right)_n}.$$

Similarly special cases for (3.14) to (3.26) can be obtained.

5. CONCLUSION

Our research has produced a set of twenty-six MacRobert type integrals involving generalized hypergeometric functions of series ${}_3F_2$ interms of gamma functions. We look forward to explore these results in more detail in our upcoming work.Further research and study in this area have the potential to lead to advancement across diverse disciplines, as well as deepen our understanding of mathematical analysis and its practical implications.

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