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BANACH-SAKS PROPERTY AND THE DEGREE OF NONDENSIFIABILITY

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ABSTRACT. We present new upper bounds based on the so-called degree of nondensifiability (DND), for some quantification (see the references and definitions in the paper) of the Banach–Saks property. To be more precise, we prove that the mentioned quantification of a bounded subset of a Banach space can be bounded above by the DND of the convex hull of such a subset, multiplied by a constant. As a consequence of our main result, we derive an upper bound for the Banach-Saks property of bounded linear operators between Banach spaces. Through several examples, we show that such bounds are the best possible.

Key words and phrases: Banach-Saks property; Degree of nondensifiability; α-dense curves.

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1. INTRODUCTION

In 1930 Banach and Saks proved in [4] (see also [5]) that, with the actual terminology, every bounded sequence in $L^p(0,1)$, with $1 < p < \infty$, has a convergent Cesàro subsequence. That is to say, each bounded sequence $(x_n)_{n\geq 1} \subset L^p(0,1)$ contains a subsequence, put $(x_{n_k})_{k\geq 1}$, such that the sums $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converge in $L^p(0,1)$. Since the separation between the elements of a sequence in a bounded subset of a Banach space is closely related to the compactness of such a subset, the Banach and Saks result mentioned above is interesting and relevant. Therefore, it is not a surprising fact that there is a vast literature related to this problem (or, more generally, to the study of the separation between the elements of a bounded sequence), see, for instance, [2, 5, 7, 16, 17, 18] and references therein.

We give the following formal definition of a Banach-Saks set and the Banach-Saks property, see, for instance, [5, 17].

Definition 1.1. A bounded subset B of a Banach space X is said to be a Banach-Saks set if each sequence in B has a Cesàro convergent subsequence. A Banach space X is said to have the Banach–Saks property if its closed unit ball is a Banach-Saks set.

In our context, by a *quantitative* version of a theorem, which relates some notions, roughly speaking, we mean that the implications between such notions are replaced by inequalities between certain quantities. Thus, several quantitative versions to the Banach-Saks sets as well as the Banach-Saks property have been defined and studied in [5, 17, 18] and references therein. In Section 3, for a given bounded and non-empty subset B of a Banach space X such *quantification* of the Banach-Saks sets and the Banach-Saks property will be denoted, respectively, as $\mathrm{bs}(B)$ and $\mathrm{bs}(U_X)$, U_X being the close unit ball of X. In other words, for a given bounded subset B of a Banach space X, the number $\mathrm{bs}(B)$ can be considered as the distance (in the specified sense) from B to the class of Banach-Saks subsets of X, while the number $\mathrm{bs}(U_X)$ measures (in the specified sense) the lack of the Banach-Saks property of X.

On the other hand, in the present paper we use the so-called degree of nondensifiability (DND), explained in detail in Section 2, to provide an upper bound for the numbers $\mathrm{bs}(B)$ and $\mathrm{bs}(U_X)$. The DND has been already used to prove a quantitative version of some classical theorems from Functional Analysis in [9, 10, 11]. To be more precise, in Theorem 3.2 we provide an upper bound for the number $\mathrm{bs}(B)$ based on the DND of the convex hull of B. Such an upper bound is improved when B is convex.

Also, as a consequence of our main result, we provide an upper bound (also based on the DND) for other quantities related to the Banach-Saks property of a bounded subset as well as of a given bounded linear operator between two Banach spaces. Such an upper bound for bounded linear operators is proved to be the best possible.

2. THE DEGREE OF NONDENSIFIABILITY

Before recalling the concept of the degree of nondensifiability of a non-empty and bounded subset of a Banach space, it is convenient to recall the following concepts introduced in [20]. In what follows, (M,d) is a metric space and $\mathcal{B}(M)$ the class of the non-empty and bounded subsets of M.

Definition 2.1. Let $B \in \mathcal{B}(M)$ and $\alpha \geq 0$. A continuous mapping $\gamma : [0,1] \longrightarrow (M,d)$ is said to be an α-dense curve in B if the following conditions hold:

(i) $\gamma([0,1]) \subset B$.
(ii) For each $x \in B$ there is $y \in \gamma([0,1])$ such that $d(x,y) \leq \alpha$.

If for each $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Note that given $B \in \mathcal{B}(M)$, fixing any $x_0 \in B$, the mapping $\gamma(t) := x_0$ for all $t \in [0,1]$ is, trivially, an α-dense curve in B for any α greater than or equal to the diameter of B. Also, the α-dense generalize the so-called *space-filling curves*, see [24]. For a detailed exposition of the α-dense curves and the densifiable sets, see [6, 20, 19, 21, 22].

Example 2.1. *Let any integer $N > 1$, $-\infty < a_i \le b_i < +\infty$ be real numbers and $C := \prod_{i=1}^N [a_i, b_i]$. For a given integer $m > 1$, define $\gamma : [0,1] \longrightarrow \mathbb{R}^N$ as*

$$\gamma(t) := \big(a_1 + (b_1 - a_1)t, a_2 + \frac{b_2 - a_2}{2}(1 - \cos(\pi m t)), \ldots, a_N + \frac{b_N - a_N}{2}(1 - \cos(\pi m^{N-1} t))\big),$$

for each $t \in [0,1]$.Then, γ is a $M\frac{\sqrt{N-1}}{m}$-dense curve in C, where $M = \max\{b_i - a_i : i = 1, \ldots, N\}$ (see [6]*).*

Now, we can give the following definition, see [14, 21].

Definition 2.2. For a given $B \in \mathcal{B}(M)$, the degree of nondensifiability (DND) of B is defined as

$$\phi(B) := \inf\left\{\alpha \ge 0 : \Gamma_{B,\alpha} \ne \emptyset\right\},$$

where $\Gamma_{B,\alpha}$ stands for the class of α-dense curves in B.

Let us note that, from the above considerations, given any $B \in \mathcal{B}(M)$, the DND of B is well defined, because $0 \le \phi(B) \le \operatorname{Diam}(B)$ (the diameter of B).

Example 2.2. *If U_X denotes the closed unit ball of a Banach space X then*

$$\phi(U_X) = \begin{cases} 1, & \text{if } X \text{ is infinite dimensional} \\ 0, & \text{otherwise} \end{cases}$$

By the well-known Hahn-Mazurkiewicz theorem (see, for instance, [24, 25]) a set $B \in \mathcal{B}(M)$ is the continuous image of $[0,1]$ if, and only if, B is a Peano Continuum (i.e., compact, connected, and locally connected). So, the DND measures, in the specified sense, the distance from B to the class of its Peano Continua.

Some basic properties of the DND, proved in [14], are listed in the following result.

Proposition 2.1. *The DND ϕ satisfies the following properties:*

(M-1) Regularity on the class of non-empty, bounded, and arc-connected sets of M: $\phi(B) = 0$ if, and only if, B is precompact, for each $B \in \mathcal{B}(M)$ arc-connected.
(M-2) Invariant under closure: $\phi(B) = \phi(\bar{B})$, for each $B \in \mathcal{B}(M)$.

Additionally, if $M := X$ is a Banach space, then the following conditions are also satisfied:

(B-1) Semi-homogeneity: $\phi(cB) = |c|\phi(B)$, for each $c \in \mathbb{R}$ and $B \in \mathcal{B}(X)$.
(B-2) Invariant under translations: $\phi(x_0 + B) = \phi(B)$, for each $x_0 \in X$ and $B \in \mathcal{B}(X)$.
(B-3) For each $B_1, B_2 \in \mathcal{B}(X)$,

$$\phi\big(\operatorname{Conv}(B_1 \cup B_2)\big) \le \max\left\{\phi\big(\operatorname{Conv}(B_1)\big), \phi\big(\operatorname{Conv}(B_2)\big)\right\} \le \max\left\{\phi(B_1), \phi(B_2)\right\}.$$

Some of the above properties are also satisfied by the measures of noncompactness but, as we have pointed out in Section 1, the DND is not a measure of noncompactness. However, as proved in [8, 14], the DND is related to some measures of noncompactness. For instance, by recalling that the Hausdorff measure of noncompactness of $B \in \mathcal{B}(X)$ (see, for instance, [3]), X being a Banach space, is defined as

$$\chi(B) := \inf\left\{\varepsilon > 0 : B \subset \{x_1, \ldots, x_n\} + \varepsilon U_X, \text{ with } x_1, \ldots, x_n \in X\right\},$$

where U_X stands for the closed unit ball of X, in [14, Theorem 2.5] we proved the following result:

Proposition 2.2. *For each $B \in \mathcal{B}(X)$ arc-wise connected, the inequalities*

$$\chi(B) \leq \phi(B) \leq 2\chi(B)$$

hold and are the best possible.

3. THE INEQUALITIES

In this section, $(X, \|\cdot\|)$ will denotes a real Banach space, infinite dimensional unless otherwise specified, and U_X its closed unit ball. As in the previous section, $\mathcal{B}(X)$ denotes the class of non-empty and bounded subsets of X. Also, for a given subset N of $\mathbb{N}$ (the natural numbers), $|N|$ is the cardinality of N. The following concepts are crucial for our goals.

Definition 3.1. Let $B \in \mathcal{B}(X)$. The arithmetic separation of a sequence $(x_n) \subset B$ is defined to be

$$\operatorname{asep}(x_n) := \inf\Big\{\frac{1}{m}\Big\|\Big(\sum_{n\in N_1} x_n - \sum_{n\in N_2} x_n\Big)\Big\| : m \in \mathbb{N}, N_1, N_2 \subset \mathbb{N}, |N_1| = |N_2| = m,$$

$$\max N_1 < \min N_2\Big\}.$$

Also, we will denote $\operatorname{bs}(B) := \sup\{\operatorname{asep}(x_n) : (x_n) \subset B\}$.

The above concepts were introduced and studied by Kryczka, the arithmetic separation of a sequence in [18] and the number $\operatorname{bs}(B)$ in [17]. Let us note that, roughly speaking, $\operatorname{bs}(B)$ measures, in the specified sense, how close is B to the class of Banach-Saks subsets of X. In particular, $\operatorname{bs}(U_X)$ measures the *deviation* of X from the Banach-Saks property. These considerations are clearer in light of the statement (1) of the following result.

Proposition 3.1. *Given $B_1, B_2 \in \mathcal{B}(X)$, the Banach-Saks measure satisfies the following properties:*

(1) $\operatorname{bs}(B_1) = 0$ *if, and only if,* B_1 *is a Banach-Saks set. In particular,* $\operatorname{bs}(U_X) = 0$ *if, and only if,* X *has the Banach-Saks property.*
(2) $\operatorname{bs}(rB_1) = |r|\operatorname{bs}(B_1)$ *for all* $r \in \mathbb{R}$.
(3) If $B_1 \subset B_2$ *then* $\operatorname{bs}(B_1) \leq \operatorname{bs}(B_2)$.
(4) $\operatorname{bs}(B_1) = \operatorname{bs}(\bar{B}_1)$.
(5) $\operatorname{bs}(B_1 \cup B_2) = \max\big\{\operatorname{bs}(B_1), \operatorname{bs}(B_2)\big\}$.
(6) If B_1 *and* B_2 *are convex, then* $\operatorname{bs}(B_1 + B_2) \leq \operatorname{bs}(B_1) + \operatorname{bs}(B_2)$.

Proof. We only prove (4), the other properties were proved in Corollary 8 and Proposition 9 of [17]. By (3), $\operatorname{bs}(B_1) \leq \operatorname{bs}(\bar{B}_1)$. To get the opposite inequality, let us note that if $(\bar{x}_n)$ is a sequence in $\bar{B}_1$, given any $\varepsilon > 0$ there is a sequence (x_n) is a sequence in B_1 such that $\|\bar{x}_n - x_n\| \leq \frac{\varepsilon}{2}$ for each $n \in \mathbb{N}$. So, given $m \in \mathbb{N}$, and $N_1, N_2 \subset \mathbb{N}$ with $|N_1| = |N_2| = m$ and $\max N_1 < \min N_2$ we have

$$\frac{1}{m}\Big\|\Big(\sum_{n\in N_1} \bar{x}_n - \sum_{n\in N_2} \bar{x}_n\Big)\Big\| \leq \varepsilon + \frac{1}{m}\Big\|\Big(\sum_{n\in N_1} x_n - \sum_{n\in N_2} x_n\Big)\Big\|.$$

Therefore, $\operatorname{asep}(\bar{x}_n) \leq \operatorname{asep}(x_n) + \varepsilon$, and from the arbitrariness of $\varepsilon > 0$, $\operatorname{asep}(\bar{x}_n) \leq \operatorname{asep}(x_n)$. Consequently, $\operatorname{bs}(\bar{B}_1) \leq \operatorname{bs}(B_1)$.

∎

At this point, it is convenient to give the following example.

Example 3.1. *Let c_0 and ℓ_1 be, respectively, the Banach spaces of the (real) null and summable absolute value sequences, both endowed with their usual norms. Then, in* [17] *it was proved the following equalities:*

$$\mathrm{bs}(U_{c_0}) = 1, \quad \mathrm{bs}(U_{\ell_1}) = 2.$$

Now, we can state and prove the following result.

Theorem 3.2. *For a given $B \in \mathcal{B}(X)$, we have the inequality*

$$\mathrm{bs}(B) \leq 2\phi\big(\mathrm{Conv}(B)\big). \tag{3.1}$$

Moreover, if B is convex then

$$\mathrm{bs}(B) \leq \mathrm{bs}(U_X)\phi(B). \tag{3.2}$$

In both cases, the above inequalities are the best possible.

Proof. Let any sequence $(x_n) \subset B$. For given $m \in \mathbb{N}$ and $N_1, N_2 \subset \mathbb{N}$, with $|N_1| = |N_2| = m$ and $\max N_1 < \min N_2$, let us define

$$\Delta(m, N_1, N_2) := \frac{1}{m}\big\|\big(\sum_{n\in N_1} x_n - \sum_{n\in N_2} x_n\big)\big\|.$$

Now, let any $\varepsilon > 0$ and γ a $(\phi(\mathrm{Conv}(B)) + \frac{\varepsilon}{4})$-dense curve in $\mathrm{Conv}(B)$. For each $n \in \mathbb{N}$ let $(y_n) \subset \gamma([0,1])$ a sequence such that

$$\|x_n - y_n\| \leq \phi\big(\mathrm{Conv}(B)\big) + \frac{\varepsilon}{4}, \quad \text{for all } n \in \mathbb{N}.$$

Then, we have

$$\begin{aligned}\Delta(m, N_1, N_2) \leq \frac{1}{m}\big\|\big(\sum_{n\in N_1} x_n - \sum_{n\in N_1} y_n\big)\big\| + \frac{1}{m}\big\|\big(\sum_{n\in N_1} y_n - \sum_{n\in N_2} y_n\big)\big\| + \\ \frac{1}{m}\big\|\big(\sum_{n\in N_2} y_n - \sum_{n\in N_2} x_n\big)\big\| \leq 2\phi\big(\mathrm{Conv}(B)\big) + \frac{\varepsilon}{2} + \frac{1}{m}\big\|\big(\sum_{n\in N_1} y_n - \sum_{n\in N_2} y_n\big)\big\|.\end{aligned} \tag{3.3}$$

As $\gamma([0,1])$ is compact, (y_n) has a convergent subsequence, for simplicity denoted in same way, and in particular, such subsequence is a Cauchy sequence. So, there are $N(\varepsilon) \in \mathbb{N}$ and $n_2(\varepsilon) > n_1(\varepsilon) \geq N(\varepsilon)$ such that

$$\big\|y_{n_1(\varepsilon)} - y_{n_2(\varepsilon)}\big\| \leq \frac{\varepsilon}{2}. \tag{3.4}$$

Therefore, form (3.3) and (3.4) we conclude that

$$\mathrm{asep}(x_n) \leq \Delta\big(1, \{n_1(\varepsilon)\}, \{n_2(\varepsilon)\}\big) \leq 2\phi\big(\mathrm{Conv}(B)\big) + \varepsilon,$$

and from the arbitrariness $\varepsilon > 0$, $\mathrm{asep}(x_n) \leq 2\phi(\mathrm{Conv}(B))$. So, taking supremum over all the sequences (x_n) in B, the inequality (3.1) follows.

The class of Banach spaces where inequality (3.1) is strict is large. Indeed, let X be an infinite dimensional uniformly convex Banach space and $B := (x_n)$ a bounded but not precompact sequence. Then, as Kakutani proved in [15], B has the Banach-Saks property, i.e. $\mathrm{bs}(B) = 0$. We recall that a set of a locally convex linear space is precompact if and only if its convex hull is, see, for instance, [23, p. 50]. So, by virtue of property (M-1) of Proposition 2.1, $\phi(\mathrm{Conv}(B)) > 0$ because of B is not precompact. Then, $\mathrm{bs}(B) < \phi(\mathrm{Conv}(B))$. In Example 3.2 we will show that inequality (3.1) can be an equality.

Next, assume B is convex. By Propositions 2.1 and 3.1 we can assume, without loss of generality, that B is also closed. Let any $\varepsilon > 0$. If γ is a $(\phi(B)+\varepsilon)$-dense curve in B, as B is closed and convex, we have

$$K := \overline{\mathrm{Conv}}\big(\gamma([0,1])\big) \subset B.$$

As K is convex and compact (see, for instance, [1, Theorem 5.35]), by the Hahn-Mazurkiewicz theorem, there exists a continuous mapping $\omega : [0,1] \longrightarrow X$ such that $\omega([0,1]) = K$. Clearly, ω is a $(\phi(B)+\varepsilon)$-dense curve in B. Therefore, we have

$$B \subset K + (\phi(B)+\varepsilon)U_X.$$

Thus, by Proposition 3.1, we find

$$\mathrm{bs}(B) \leq \mathrm{bs}(K) + (\phi(B)+\varepsilon)\mathrm{bs}(U_X) = (\phi(B)+\varepsilon)\mathrm{bs}(U_X).$$

From the arbitrariness of $\varepsilon > 0$, the inequality (3.2) holds.

On the other hand, in view of Examples 3.1 and 2.2, we have:

$$\mathrm{bs}(U_{c_0}) = 1 < 2 = 2\phi(U_{c_0}), \quad \mathrm{bs}(U_{\ell_1}) = 2 = 2\phi(U_{\ell_1}).$$

Consequently, inequality (3.2) is the best possible.

∎

Looking at the proof of Theorem 3.2, it is clear that the inequality $\mathrm{bs}(B) \leq 2\phi(B)$ also holds. However, according to property (B-3) of Proposition 2.1, the inequality of Theorem 3.2 is finer. Indeed, let us consider $B := \{x_1, x_2\} \subset X$ with $x_1, x_2 \in X$ and $x_1 \neq x_2$, being X an arbitrary Banach space. Then, $\mathrm{bs}(B) = 2\phi(\mathrm{Conv}(B)) = 0$ but $\phi(B) = \|x_1 - x_2\| > 0$.

As we have pointed out in Section 1, the Banach-Saks and the weak Banach-Saks properties are closely related with the compactness and weak compactness of such set. Also, from the well known Mazur and Krein-Šmulian theorems (see, for instance, [1, Theorems 3.65 and 5.35], we know that a compact or a weakly compact set have a precompact or relatively weakly compact convex hull. However, it is worth to say that, in general and as it was shown in [19], the convex hull of a Banach-Saks set not need to be a Banach-Saks set. In other words, there are Banach spaces such that the class $\mathcal{N}(X) := \{B \in \mathcal{B}(X) : \mathrm{bs}(B) = 0 < \mathrm{bs}(\mathrm{Conv}(B))$ is non-empty.

In the next examples we show that the equality $\mathrm{bs}(B) \leq 2\phi\big(\mathrm{Conv}(B)\big)$ is the best possible for non-convex and bounded subsets.

Example 3.2. *Let, for each $n \in \mathbb{N}$, e_n the n-th basic vector of ℓ_1, and consider the set $B := (e_n)$. Then, it is immediate to check that $\mathrm{bs}(B) = 2$.*

Now, let $\gamma_1(t) := e_1$ for all $t \in [0,1]$. As $\|x - \gamma_1(t)\| \leq 1$ for each $t \in [0,1]$ and $x \in \mathrm{Conv}(B)$, γ_1 is a 1-dense curve in $\mathrm{Conv}(B)$. Thus, $\phi(\mathrm{Conv}(B)) \leq 1$. We will prove in the below lines the opposite inequality.

Let γ be an α-dense curve in $\mathrm{Conv}(B)$, for some $\alpha > 0$. Then according to [3, Theorem II.4.1], *fixed any $\varepsilon > 0$ there is an integer $N_\varepsilon \geq 1$ such that*

$$\sum_{n \geq N_\varepsilon} |y_n| \leq \varepsilon, \quad \text{for all } y := (y_n) \in \gamma([0,1]).$$

So, for each $y := (y_n) \in \gamma([0,1])$, we have

$$\|e_{N_\varepsilon} - y\| \geq 1 - y_{N_\varepsilon} \geq 1 - \varepsilon,$$

and consequently $\alpha \geq 1 - \varepsilon$. Noticing the arbitrariness of $\varepsilon > 0$, we infer that $\alpha \geq 1$ and therefore

$$\phi(\mathrm{Conv}(B)) \geq 1.$$

So, we have

$$\mathrm{bs}(B) = 2 = 2\phi\big(\mathrm{Conv}(B)\big).$$

Let us note that, as $\mathrm{bs}(U_X) \leq 2$, for bounded and convex subsets of X inequality (3.2) is, in general, finer than (3.1). We illustrate this fact in the following example.

Example 3.3. *For $1 < p < \infty$ let J_p be the James' space of all real null sequences $x = (x_n)$ with finite*

$$\|x\|_p := \sup\Big\{\sum_{i=1}^{n-1} |x_{k_{i+1}} - x_{k_i}| : 0 < k_1 < \ldots < k_n, n \geq 2\Big\}.$$

Then, according to [17], $\mathrm{bs}(U_{J_p}) = 2^{1/p}$. *Therefore, for each convex and not precompact $B \in \mathcal{B}(J_p)$, by Theorem 3.2 we have*

$$\mathrm{bs}(B) \leq 2^{1/p}\phi(B) < 2\phi(B).$$

In order to state some inequalities between the DND and other quantities related with the Banach-Saks property, we need to recall the following numbers (see, for instance, [17]).

Definition 3.2. Let $B \in \mathcal{B}(X)$. The alternated arithmetic separation of B is defined as

$$\mathrm{aasep}(B) := \inf\Big\{\frac{1}{N}\sum_{n\in N} \epsilon_n x_n : N \subset \mathbb{N} \text{ is finite }, \epsilon_n := \pm 1\Big\}.$$

and $\mathrm{abs}(B) := \sup\{\mathrm{aasep}(x_n) : (x_n) \subset B\}$.

If X, Y are Banach spaces and $T : X \longrightarrow Y$ a bounded linear operator, we define the number $\widetilde{\mathrm{bs}}(T) := \mathrm{bs}(T(U_X))$. If $\widetilde{\mathrm{bs}}(T) = 0$, T is said to be a Banach-Saks operator.

The number $\mathrm{abs}(B)$ is a quantification of the so-called alternate signs Banach–Saks property, which is defined in the same way that the Banach-Saks property but replacing the Cesàro sums by the sums $\frac{1}{m}\sum_{k=1}^{m}(-1)^k x_{n_k}$. The number $\widetilde{\mathrm{bs}}(T)$ is a natural way to quantify the Banach-Saks property of a linear operator $T : X \longrightarrow Y$.

In the following results, as consequence of Theorem 3.2 and by using the DND, we provide an upper for the concepts of Definition 3.2.

Corollary 3.3. *Let $B \in \mathcal{B}(X)$. Then,* $\mathrm{abs}(B) \leq \phi(\mathrm{Conv}(B))$. *In particular, if B is convex then*

$$\mathrm{abs}(B) \leq \frac{1}{2}\mathrm{bs}(B)\phi(B).$$

Proof. According to [17, Corollary 8] $\mathrm{abs}(B) \leq \frac{1}{2}\mathrm{bs}(B)$, and the result follows from this equality and Theorem 3.2. ∎

Corollary 3.4. *Let X, Y two Banach spaces and $T : X \longrightarrow Y$ a bounded linear operator. Then,*

$$\widetilde{\mathrm{bs}}(T) \leq \phi\big(T(U_X)\big)\mathrm{bs}(U_Y). \tag{3.5}$$

Moreover, the above inequalities are the best possible.

Proof. As $T(U_X)$ is a bounded and convex subset of Y, by Theorem 3.2 inequality (3.5) holds. In the below examples we will show that such inequality is the best possible.
∎

Example 3.4. *Let $\hat{c}_0$ be the space c_0 endowed with the norm*

$$|||x||| := \|x\|_0 + \sum_{n\geq 1} \frac{|x_n|}{2^n} \quad \text{for all } x := (x_n) \in c_0,$$

where $\|\cdot\|_0$ stands for the usual norm of c_0, and $T : c_0 \longrightarrow \hat{c}_0$ the identity map. One can check that

$$\|T\| = 2, \quad \mathrm{bs}(U_{\hat{c}_0}) = 1, \quad \widetilde{\mathrm{bs}}(T) = 2. \tag{3.6}$$

Now, let any $\varepsilon > 0$ and $N := N(\varepsilon) > 1$ such that $\sum_{n\geq N+1} 2^{-n} \leq \varepsilon/2$. Let $g := (g_1, \dots, g_N) : [0,1] \longrightarrow \mathbb{R}^N$ be a $\frac{\varepsilon}{2}$-dense curve in $[-1,1]^N$; see Example 2.1. Define $\gamma : [0,1] \longrightarrow \hat{c}_0$ as

$$\gamma(t) := \big(g_1(t), \dots, g_N(t), 0, \dots, 0, \dots\big), \quad \text{for all } t \in [0,1].$$

It is clear that γ is continuous and $\gamma([0,1]) \subset T(U_{c_0})$. Also, given $y := (y_n) \in T(U_{c_0})$, for some let $t \in [0,1]$ be such that $\|g(t) - (y_1, \dots, y_N)\|_e \leq \varepsilon/2$, where $\| \cdot \|_e\|$ stands for the Euclidean norm. The existence of such a $t \in [0,1]$ follows from the fact that g is a ε-dense curve in $[-1,1]^N$ and $|y_n| \leq 1$ for each $n \geq 1$. Then, we have

$$|||y - \gamma(t)||| \leq \max\big\{\frac{\varepsilon}{2}, \sup\{|y_n| : n \geq N+1\}\big\} + \frac{\varepsilon}{2} + \sum_{n\geq N+1} \frac{|y_n|}{2^n} \leq 1 + \varepsilon.$$

From the arbitrariness of $\varepsilon > 0$ we infer that $\phi(T(c_0)) \leq 1$ and therefore, noticing Example 3.1 we conclude that

$$\max\big\{\|T\|\mathrm{bs}(U_{c_0}), \phi\big(T(U_{\ell_1})\big)\mathrm{bs}(U_{\hat{c}_0})\big\} = 2 = \widetilde{\mathrm{bs}}(T).$$

That is to say, inequality (3.5) of Corollary 3.4 become into an equality in this example.

Example 3.5. *Let us define the operator $T : \ell_1 \longrightarrow \ell_1$ as $T(x) := \frac{1}{2}x$ for all $x \in \ell_1$.*

Then, noticing Example 3.1, $\widetilde{\mathrm{bs}}(T) = 1$ and $\mathrm{bs}(U_{\ell_1}) = 2$. By Example 2.2 and Proposition 2.1, $\phi(T(U_{\ell_1})) = 1/2$. Therefore,

$$\widetilde{\mathrm{bs}}(T) = 1 = \phi\big(T(U_{\ell_1})\big)\mathrm{bs}(U_{\ell_1}).$$

So, here the inequality (3.5) become into an equality.

On the other hand it is important to stress that, in general, there is not $k > 0$ such that $k\phi(\mathrm{Conv}(B)) \leq \mathrm{bs}(B)$. Indeed, from the considerations of Section 1, for a given $1 < p < +\infty$ we have $\mathrm{bs}(U_{L^p(0,1)}) = 0$ but, from Example 2.2, $\phi(U_{L^p(0,1)}) = 1$. However, in the space ℓ_1 such inequality holds, as we prove in the following result.

Proposition 3.5. *Let $B \in \mathcal{B}(\ell_1)$. Then, $\phi(\mathrm{Conv}(B)) \leq \mathrm{bs}(B)$ and this inequality is the best possible. In particular, if B is convex then*

$$\phi(B) \leq \mathrm{bs}(B) \leq 2\phi(B). \tag{3.7}$$

Proof. Let χ denote the Hausdorff measure of noncompactness, given in Section 2. In [16] it was proved that $\mathrm{bs}(B) = 2\chi(B)$, for each $B \in \mathcal{B}(\ell_1)$. Therefore, by Proposition 2.2, we find $\phi(\mathrm{Conv}(B)) \leq \mathrm{bs}(B)$.

If B is precompact (or, equivalently, as ℓ_1 has the Schur property, relatively weakly compact), by (M-1) of Proposition 2.1 and noticing that $\mathrm{bs}(B) = 0$, we have $\phi(\mathrm{Conv}(B)) = 0 = \mathrm{bs}(B)$. Also, from Examples 2.2 and 3.1, we have $\phi(U_{\ell_1}) = 1 < 2 = \mathrm{bs}(U_{\ell_1})$.

The right-hand inequality in (3.7) follows directly from Theorem 3.2.

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4. FINAL REMARKS

In the present paper, by using the DND, we proved in Theorem 3.2 and Corollary 3.3 some upper bounds for the *quantification* of a non-empty and bounded subset B of a Banach space, $\mathrm{bs}(B)$, as well as of a given bounded linear operator T between Banach spaces, $\widetilde{\mathrm{bs}}(T)$. As we have illustrated with several examples, such bounds are the best possible. Also, in Corollary 3.4, we have provided an upper bound for the so-called alternated arithmetic separation of B, $\mathrm{abs}(B)$.

As we have shown, in general, the DND cannot be used to state a lower bound for the above-mentioned numbers related to the Banach-Saks properties. However, for the Banach space ℓ_1, it is possible to derive a lower bound for the number $\mathrm{bs}(B)$, where B is a non-empty and bounded subset of ℓ_1. In [5], lower bounds for the number $\mathrm{abs}(B)$ were stated using the so-called weak measures of noncompactness. Therefore, it seems that to provide a lower bound for the number $\mathrm{bs}(B)$ based on the DND, we need to define, in some sense, a DND for the weak topology of a Banach space.

In [18], the relationships between the Banach-Saks property and real interpolation of operators were studied. So, in future works, it could be interesting to analyze (if any) the relationships between the DND and the real interpolation of operators.

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