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SOME NEW INEQUALITIES FOR HYPO-q-NORMS ON A CARTESIAN PRODUCT OF NORMED LINEAR SPACES

SILVESTRU SEVER DRAGOMIR

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APPLIED MATHEMATICS RESEARCH GROUP, ISILC, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.
sever.dragomir@vu.edu.au
URL: https://rgmia.org/dragomir

SCHOOL OF COMPUTER SCIENCE & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA.

ABSTRACT. Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. If by $\mathbb{S}_{n,p}$ with $p \in [1, \infty]$ we denote the spheres generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we consider the following *hypo-q-norms* on E^n, $\|\mathbf{x}\|_{h,n,q} := \sup_{\boldsymbol{\lambda} \in \mathbb{S}_{n,p}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|$, with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$. For $p = 2$, we also consider the *hypo-Euclidean norm* on E^n, i.e.,$\|\mathbf{x}\|_{h,n,e} := \sup_{\boldsymbol{\lambda} \in \mathbb{S}_{n,2}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|$.

In this paper we have obtained among others the following inequalities

$$\left\| \sum_{i=1}^n \alpha_i x_i \right\|^2 \leq \begin{cases} \max_{1 \leq k \leq n} \left\{ |\alpha_k|^2 \right\} \|\mathbf{x}\|_{h,e}^2 \\ \\ \left(\sum_{k=1}^n |\alpha_k|^{2\alpha} \right)^{1/\alpha} \|\mathbf{x}\|_{h,n,\beta}^2, \\ \text{where } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \\ \sum_{k=1}^n |\alpha_k|^2 \|\mathbf{x}\|_{n,\infty}^2 \end{cases}$$

$$+ \begin{cases} \max_{1 \leq k \leq n} \left\{ |\alpha_k|^2 \right\} \sum_{1 \leq i \neq j \leq n}^n \|x_i\| \|x_j\| \\ \\ (n-1)^{1/\gamma} \left(\sum_{k=1}^n |\alpha_k|^{2\gamma} \right)^{1/\gamma} \\ \times \left(\sum_{1 \leq i \neq j \leq n}^n \|x_i\|^\delta \|x_j\|^\delta \right)^{1/\delta} \\ \text{where } \gamma, \delta > 1 \text{ with } \frac{1}{\gamma} + \frac{1}{\delta} = 1 \\ \\ (n-1) \sum_{k=1}^n |\alpha_k|^2 \max_{1 \leq i \neq j \leq n} \left\{ \|x_i\| \|x_j\| \right\}. \end{cases}$$

The case for $n = 2$ and the connection with the following new norms $\|\cdot\|_t, \|\cdot\|_s : E^2 \to [0, \infty)$, $\|(x, y)\|_t := \sup_{\theta \in \mathbb{R}} \|\cos\theta x + i \sin\theta y\|$ and $\|(x, y)\|_s := \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta} x + e^{-i\theta} y}{2} \right\|$ are also investigated. When the norm $\|\cdot\|$ is generated by an inner product, further bounds are given as well.

Key words and phrases: Normed spaces; Cartesian products of normed spaces; Inequalities.

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1. INTRODUCTION

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit sphere

$$\mathbb{S}(\|\cdot\|_n) := \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n | \, \|\lambda\|_n = 1\}.$$

As an example of such norms we should mention the usual p-*norms*

$$\|\boldsymbol{\lambda}\|_{n,p} := \begin{cases} \max\{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty; \\ \\ \left(\sum_{k=1}^n |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases} \tag{1.1}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\boldsymbol{\lambda}\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2\right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following p-*norms*:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max\{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty; \\ \\ \left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \end{cases} \tag{1.2}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

Following [4], for a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in \mathbb{S}(\|\cdot\|_n)} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.3}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

It is easy to see, by the properties of the norm $\|\cdot\|$, that:

(i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;
(ii) $\|\mathbf{x} + \mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}$, $\mathbf{y} \in E^n$;
(iii) $\|\alpha \mathbf{x}\|_{h,n} = |\alpha| \, \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

If by $\mathbb{S}_{n,p}$ with $p \in [1, \infty]$ we denote the spheres generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-q-norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\boldsymbol{\lambda} \in \mathbb{S}_{n,p}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.4}$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean sphere in $\mathbb{K}^n$, which we denote by $\mathbb{S}_n$, $\mathbb{S}_n = \left\{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \left| \sum_{i=1}^n |\lambda_i|^2 = 1\right.\right\}$ that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|\mathbf{x}\|_{h,n,e} := \sup_{\boldsymbol{\lambda} \in \mathbb{S}_{n,2}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \tag{1.5}$$

Moreover, if $E = H$, H is a inner product space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_{n,e} := \sup_{\boldsymbol{\lambda} \in \mathbb{S}_{n,2}} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{1.6}$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the n-tuples of vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^{n} \langle x_j, y_j \rangle, \qquad \mathbf{x},\ \mathbf{y} \in H^n, \tag{1.7}$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|\mathbf{x}\|_{n,2} := \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{\frac{1}{2}}, \qquad \mathbf{x} \in H^n. \tag{1.8}$$

The following result established in [4] connects the usual Euclidean norm $\|\cdot\|$ with the hypo-Euclidean norm $\|\cdot\|_{n,e}$.

Theorem 1.1 (Dragomir, 2007, [4]). *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_{n,2} \leq \|\mathbf{x}\|_{n,e} \leq \|\mathbf{x}\|_{n,2}, \tag{1.9}$$

i.e., $\|\cdot\|_{n,2}$ and $\|\cdot\|_{n,e}$ are equivalent norms on H^n.

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 1.2 (Dragomir, 2007, [4]). *For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we have*

$$\|\mathbf{x}\|_{n,e} = \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}}. \tag{1.10}$$

For various new results concerning norm and numerical radius inequalities for bounded linear operators on Hilbert spaces, see the recent papers [1]-[5], [10]-[14] and the references therein.

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. We denote by E^* its dual space endowed with the norm $\|\cdot\|$ defined by

$$\|f\| := \sup_{\|x\|=1} |f(x)| < \infty, \text{ where } f \in E^*.$$

We have the following representation result for the *hypo-q-norms* on E^n, see [6]:

Theorem 1.3. *Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. For any $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we have the representation*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|f\|=1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^q \right)^{1/q} \right\}, \tag{1.11}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{x}\|_{h,n,1} = \sup_{\|f\|=1} \left\{ \sum_{j=1}^{n} |f(x_j)| \right\} \tag{1.12}$$

and

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j\in\{1,\ldots,n\}} \{\|x_j\|\} . \tag{1.13}$$

In particular,

$$\|\mathbf{x}\|_{h,n,e} = \sup_{\|f\|=1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^2 \right)^{1/2} \right\} . \tag{1.14}$$

In [6] we also proved that, for $r \geq q \geq 1$, the following double inequality holds

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r}$$

for any $\mathbf{x} \in E^n$.

In the same paper we also obtained the following reverse inequalities

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^2 ,$$

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \|\mathbf{x}\|_{h,n,1} \|\mathbf{x}\|_{n,\infty}$$

and

$$0 \leq \|\mathbf{x}\|_{h,e} - \frac{1}{\sqrt{n}} \|\mathbf{x}\|_{h,n,1} \leq \frac{1}{4} \sqrt{n} \|\mathbf{x}\|_{n,\infty} .$$

For an n-tuple of complex numbers $\mathbf{a} = (a_1, \ldots, a_n)$ with $n \geq 2$ consider the $(n-1)$-tuple built by the aid of forward differences $\Delta\mathbf{a} = (\Delta a_1, ..., \Delta a_{n-1})$ where $\Delta a_k := a_{k+1} - a_k$ where $k \in \{1, ..., n-1\}$. Similarly, if $\mathbf{x} = (x_1, \ldots, x_n) \in E^n$ is an n-tuple of vectors we also can consider in a similar way the $(n-1)$-tuple $\Delta\mathbf{x} = (\Delta x_1, ..., \Delta x_{n-1})$.

In [6] we also obtained the following results:

$$\|\mathbf{x}\|_{h,n,q+r}^{q+r} \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^{q} \|\mathbf{x}\|_{h,n,r}^{r} + \begin{cases} \frac{1}{12} qr (n^2-1) n \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta\mathbf{x}\|_{n-1,\infty}^2 , \\ \\ \frac{1}{6} (n^2-1) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta\mathbf{x}\|_{h,n-1,\alpha} \|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2} (n-1) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta\mathbf{x}\|_{h,n-1,1}^2 \end{cases}$$

for $q, r \geq 1$.

Motivated by the above results, in this paper we have obtained among others the following inequalities

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|^2 \leq \begin{cases} \max_{1\leq k\leq n}\left\{|\alpha_k|^2\right\}\|\mathbf{x}\|_{h,e}^2 \\ \\ \left(\sum_{k=1}^{n}|\alpha_k|^{2\alpha}\right)^{1/\alpha}\|\mathbf{x}\|_{h,n,\beta}^2, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^{n}|\alpha_k|^2\|\mathbf{x}\|_{n,\infty}^2 \end{cases}$$
$$+\begin{cases} \max_{1\leq k\leq n}\left\{|\alpha_k|^2\right\}\sum_{1\leq i\neq j\leq n}^{n}\|x_i\|\|x_j\| \\ \\ (n-1)^{1/\gamma}\left(\sum_{k=1}^{n}|\alpha_k|^{2\gamma}\right)^{1/\gamma} \\ \times\left(\sum_{1\leq i\neq j\leq n}^{n}\|x_i\|^{\delta}\|x_j\|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ (n-1)\sum_{k=1}^{n}|\alpha_k|^2\max_{1\leq i\neq j\leq n}\left\{\|x_i\|\|x_j\|\right\}. \end{cases}$$

The case for $n=2$ and the connection with the following new norms $\|\cdot\|_t, \|\cdot\|_s : E^2 \to [0,\infty)$, $\|(x,y)\|_t := \sup_{\theta\in\mathbb{R}}\|\cos\theta x + i\sin\theta y\|$ and $\|(x,y)\|_s := \sup_{\theta\in\mathbb{R}}\left\|\frac{e^{i\theta}x+e^{-i\theta}y}{2}\right\|$ are also investigated. When the norm $\|\cdot\|$ is generated by an inner product, further bounds are given as well.

2. MAIN RESULTS

We have the following inequalities:

Lemma 2.1. *Let $x_1,...,x_n\in E$, $\alpha_1,...,\alpha_n\in\mathbb{C}$ and $f\in E^*$ with $f\neq 0$. Then*

$$\left|\sum_{i=1}^{n}\alpha_i f(x_i)\right|^2 \leq \begin{cases} \max_{1\leq k\leq n}\left\{|\alpha_k|^2\right\}\sum_{k=1}^{n}|f(x_k)|^2 \\ \\ \left(\sum_{k=1}^{n}|\alpha_k|^{2\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n}|f(x_k)|^{2\beta}\right)^{1/\beta}, \\ \text{where } \alpha,\ \beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^{n}|\alpha_k|^2\max_{1\leq k\leq n}|f(x_k)|^2 \end{cases} \tag{2.1}$$
$$+\begin{cases} \max_{1\leq i\neq j\leq n}\left\{|\alpha_i||\alpha_j|\right\}\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)||f(x_j)| \\ \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|^{\gamma}\right)^2-\sum_{k=1}^{n}|\alpha_k|^{2\gamma}\right]^{1/\gamma} \\ \times\left(\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\ \delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|\right)^2-\sum_{k=1}^{n}|\alpha_k|^2\right] \\ \times\max_{1\leq i\neq j\leq n}\left\{|f(x_i)||f(x_j)|\right\} \end{cases}$$

$$\leq \begin{cases} \max_{1\leq k\leq n}\left\{|\alpha_k|^2\right\}\sum_{k=1}^n |f(x_k)|^2 \\ \\ \left(\sum_{k=1}^n |\alpha_k|^{2\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^n |f(x_k)|^{2\beta}\right)^{1/\beta}, \\ \textit{where } \alpha,\beta>1 \textit{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^n |\alpha_k|^2 \max_{1\leq k\leq n}|f(x_k)|^2 \end{cases}$$

$$+\begin{cases} \max_{1\leq k\leq n}\left\{|\alpha_k|^2\right\}\sum_{1\leq i\neq j\leq n}^n |f(x_i)||f(x_j)| \\ \\ (n-1)^{1/\gamma}\left(\sum_{k=1}^n |\alpha_k|^{2\gamma}\right)^{1/\gamma} \\ \times\left(\sum_{1\leq i\neq j\leq n}^n |f(x_i)|^\delta |f(x_j)|^\delta\right)^{1/\delta} \\ \textit{where } \gamma,\delta>1 \textit{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ (n-1)\sum_{k=1}^n |\alpha_k|^2 \max_{1\leq i\neq j\leq n}\{|f(x_i)||f(x_j)|\}. \end{cases}$$

Proof. Let $f\in E^*$ with $f\neq 0$. Then

$$\left|\sum_{i=1}^n \alpha_i f(x_i)\right|^2 = \left|f\left(\sum_{i=1}^n \alpha_i x_i\right)\right|^2 = f\left(\sum_{i=1}^n \alpha_i x_i\right)\overline{f\left(\sum_{j=1}^n \alpha_j x_j\right)} \tag{2.2}$$
$$= \sum_{i=1}^n \alpha_i f(x_i)\sum_{j=1}^n \overline{\alpha_j}\,\overline{f(x_j)} = \left|\sum_{i=1}^n\sum_{j=1}^n \alpha_i\overline{\alpha_j} f(x_i)\overline{f(x_j)}\right|$$

$$= \left|\sum_{k=1}^n |\alpha_k|^2|f(x_k)|^2 + \sum_{1\leq i\neq j\leq n}^n \alpha_i\overline{\alpha_j}f(x_i)\overline{f(x_j)}\right|$$
$$\leq \left|\sum_{k=1}^n |\alpha_k|^2|f(x_k)|^2\right| + \left|\sum_{1\leq i\neq j\leq n}^n \alpha_i\overline{\alpha_j}f(x_i)\overline{f(x_j)}\right|$$
$$= \sum_{k=1}^n |\alpha_k|^2|f(x_k)|^2 + \left|\sum_{1\leq i\neq j\leq n}^n \alpha_i\overline{\alpha_j}f(x_i)\overline{f(x_j)}\right|$$
$$\leq \sum_{k=1}^n |\alpha_k|^2|f(x_k)|^2 + \sum_{1\leq i\neq j\leq n}^n |\alpha_i||\alpha_j||f(x_i)||f(x_j)|.$$

Using Hölder's inequality we have

$$\sum_{k=1}^n |\alpha_k|^2|f(x_k)|^2 \leq \begin{cases} \max_{1\leq k\leq n}\left\{|\alpha_k|^2\right\}\sum_{k=1}^n |f(x_k)|^2 \\ \\ \left(\sum_{k=1}^n |\alpha_k|^{2\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^n |f(x_k)|^{2\beta}\right)^{1/\beta}, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^n |\alpha_k|^2 \max_{1\leq k\leq n}|f(x_k)|^2. \end{cases} \tag{2.3}$$

By Hölder's inequality for double sums we also have

$$\sum_{1\le i\neq j\le n}^{n}|\alpha_i||\alpha_j||f(x_i)||f(x_j)| \tag{2.4}$$

$$\le \begin{cases} \max_{1\le i\neq j\le n}\{|\alpha_i||\alpha_j|\}\sum_{1\le i\neq j\le n}^{n}|f(x_i)||f(x_j)| \\ \left(\sum_{1\le i\neq j\le n}^{n}|\alpha_i|^{\gamma}|\alpha_j|^{\gamma}\right)^{1/\gamma}\left(\sum_{1\le i\neq j\le n}^{n}|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \sum_{1\le i\neq j\le n}^{n}|\alpha_i||\alpha_j|\max_{1\le i\neq j\le n}\{|f(x_i)||f(x_j)|\} \end{cases}$$

$$= \begin{cases} \max_{1\le i\neq j\le n}\{|\alpha_i||\alpha_j|\}\sum_{1\le i\neq j\le n}^{n}|f(x_i)||f(x_j)| \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|^{\gamma}\right)^2-\sum_{k=1}^{n}|\alpha_k|^{2\gamma}\right]^{1/\gamma}\left(\sum_{1\le i\neq j\le n}^{n}|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|\right)^2-\sum_{k=1}^{n}|\alpha_k|^{2}\right]\max_{1\le i\neq j\le n}\{|f(x_i)||f(x_j)|\}. \end{cases}$$

By utilizing (2.2)-(2.4) we deduce the desired result (2.1).

By utilizing the following Cauchy-Bunyakovsky-Schwarz type inequality

$$\left(\sum_{i=1}^{n}a_i\right)^2\le n\sum_{i=1}^{n}a_i^2,\ \text{ for } a_i\in\mathbb{R},\ 1\le i\le n,$$

we have

$$\left(\sum_{k=1}^{n}|\alpha_k|^{\gamma}\right)^2-\sum_{k=1}^{n}|\alpha_k|^{2\gamma}\le(n-1)\sum_{k=1}^{n}|\alpha_k|^{2\gamma}$$

and

$$\left(\sum_{k=1}^{n}|\alpha_k|\right)^2-\sum_{k=1}^{n}|\alpha_k|^{2}\le(n-1)\sum_{k=1}^{n}|\alpha_k|^{2},$$

which proves the last part of (2.1). ∎

Theorem 2.2. *Let* $x_1, ..., x_n \in E$, $\alpha_1, ..., \alpha_n \in \mathbb{C}$. *Then*

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|^2 \le \begin{cases} \max_{1\le k\le n}\left\{|\alpha_k|^2\right\}\|\mathbf{x}\|_{h,n,e}^2 \\ \\ \left(\sum_{k=1}^{n}|\alpha_k|^{2\alpha}\right)^{1/\alpha}\|\mathbf{x}\|_{h,n,\beta}^2, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^{n}|\alpha_k|^2\|\mathbf{x}\|_{n,\infty}^2 \end{cases} \tag{2.5}$$

$$+\begin{cases} \max_{1\le i\ne j\le n}\{|\alpha_i||\alpha_j|\}\sum_{1\le i\ne j\le n}^{n}\|x_i\|\|x_j\| \\ \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|^{\gamma}\right)^2-\sum_{k=1}^{n}|\alpha_k|^{2\gamma}\right]^{1/\gamma} \\ \times\left(\sum_{1\le i\ne j\le n}^{n}\|x_i\|^{\delta}\|x_j\|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|\right)^2-\sum_{k=1}^{n}|\alpha_k|^2\right] \\ \times\max_{1\le i\ne j\le n}\{\|x_i\|\|x_j\|\}. \end{cases}$$

$$\le \begin{cases} \max_{1\le k\le n}\left\{|\alpha_k|^2\right\}\|\mathbf{x}\|_{h,e}^2 \\ \\ \left(\sum_{k=1}^{n}|\alpha_k|^{2\alpha}\right)^{1/\alpha}\|\mathbf{x}\|_{h,n,\beta}^2, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^{n}|\alpha_k|^2\|\mathbf{x}\|_{n,\infty}^2 \end{cases}$$

$$+\begin{cases} \max_{1\le k\le n}\left\{|\alpha_k|^2\right\}\sum_{1\le i\ne j\le n}^{n}\|x_i\|\|x_j\| \\ \\ (n-1)^{1/\gamma}\left(\sum_{k=1}^{n}|\alpha_k|^{2\gamma}\right)^{1/\gamma} \\ \times\left(\sum_{1\le i\ne j\le n}^{n}\|x_i\|^{\delta}\|x_j\|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ (n-1)\sum_{k=1}^{n}|\alpha_k|^2\max_{1\le i\ne j\le n}\{\|x_i\|\|x_j\|\}. \end{cases}$$

Proof. If we take the supremum in (2.1), then we get

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|^2 = \sup_{\|f\|=1}\left|f\left(\sum_{i=1}^{n}\alpha_i x_i\right)\right|^2 \tag{2.6}$$

$$\leq \begin{cases} \max_{1\leq k\leq n}\left\{|\alpha_k|^2\right\}\sup_{\|f\|=1}\sum_{k=1}^{n}|f(x_k)|^2 \\ \\ \left(\sum_{k=1}^{n}|\alpha_k|^{2\alpha}\right)^{1/\alpha}\sup_{\|f\|=1}\left(\sum_{k=1}^{n}|f(x_k)|^{2\beta}\right)^{1/\beta}, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^{n}|\alpha_k|^2\sup_{\|f\|=1}\max_{1\leq k\leq n}|f(x_k)|^2. \end{cases}$$

$$+\begin{cases} \max_{1\leq i\neq j\leq n}\{|\alpha_i||\alpha_j|\} \\ \times\sup_{\|f\|=1}\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)||f(x_j)| \\ \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|^{\gamma}\right)^2-\sum_{k=1}^{n}|\alpha_k|^{2\gamma}\right]^{1/\gamma} \\ \times\sup_{\|f\|=1}\left(\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ \left[\left(\sum_{k=1}^{n}|\alpha_k|\right)^2-\sum_{k=1}^{n}|\alpha_k|^2\right] \\ \times\sup_{\|f\|=1}\max_{1\leq i\neq j\leq n}\{|f(x_i)||f(x_j)|\}. \end{cases}$$

Observe that

$$\sup_{\|f\|=1}\sum_{k=1}^{n}|f(x_k)|^2=\|\mathbf{x}\|_{h,e}^2,\quad \sup_{\|f\|=1}\left(\sum_{k=1}^{n}|f(x_k)|^{2\beta}\right)^{1/\beta}=\|\mathbf{x}\|_{h,n,\beta}^2$$

and

$$\sup_{\|f\|=1}\max_{1\leq k\leq n}|f(x_k)|^2=\max_{1\leq k\leq n}\sup_{\|f\|=1}|f(x_k)|^2=\max_{1\leq k\leq n}\|x_k\|^2=\|\mathbf{x}\|_{n,\infty}^2.$$

Also

$$\begin{aligned} \sup_{\|f\|=1}\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)||f(x_j)| &\leq \sum_{1\leq i\neq j\leq n}^{n}\sup_{\|f\|=1}\left(|f(x_i)||f(x_j)|\right) \\ &\leq \sum_{1\leq i\neq j\leq n}^{n}\sup_{\|f\|=1}|f(x_i)|\sup_{\|f\|=1}|f(x_j)| \\ &= \sum_{1\leq i\neq j\leq n}^{n}\|x_i\|\|x_j\|, \end{aligned}$$

$$\begin{aligned} \sup_{\|f\|=1}\left(\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)^{1/\delta} &\leq \left(\sup_{\|f\|=1}\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)^{1/\delta} \\ &\leq \left(\sum_{1\leq i\neq j\leq n}^{n}\sup_{\|f\|=1}\left(|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)\right)^{1/\delta} \\ &\leq \left(\sum_{1\leq i\neq j\leq n}^{n}\sup_{\|f\|=1}|f(x_i)|^{\delta}\sup_{\|f\|=1}|f(x_j)|^{\delta}\right)^{1/\delta} \\ &= \left(\sum_{1\leq i\neq j\leq n}^{n}\|x_i\|^{\delta}\|x_j\|^{\delta}\right)^{1/\delta} \end{aligned}$$

and

$$\sup_{\|f\|=1} \max_{1\leq i\neq j\leq n} \{|f(x_i)| |f(x_j)|\} = \max_{1\leq i\neq j\leq n} \sup_{\|f\|=1} \{|f(x_i)| |f(x_j)|\}$$
$$\leq \max_{1\leq i\neq j\leq n} \left\{ \sup_{\|f\|=1} |f(x_i)| \sup_{\|f\|=1} |f(x_j)| \right\}$$
$$= \max_{1\leq i\neq j\leq n} \{\|x_i\| \|x_j\|\}.$$

By utilizing (2.6) we derive the first part of (2.5). The second part is obvious. ∎

Corollary 2.3. *Let* $x_1, ..., x_n \in E$, $\alpha_1, ..., \alpha_n \in \mathbb{C}$. *Then*

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \leq \left(\sum_{k=1}^{n} |\alpha_k|^4 \right)^{1/2} \|\mathbf{x}\|_{h,n,e}^2 + \left[\left(\sum_{k=1}^{n} |\alpha_k|^2 \right)^2 - \sum_{k=1}^{n} |\alpha_k|^4 \right]^{1/\gamma} \left(\sum_{1\leq i\neq j\leq n} \|x_i\|^2 \|x_j\|^2 \right)^{1/2}$$
$$\leq \left(\sum_{k=1}^{n} |\alpha_k|^4 \right)^{1/2} \|\mathbf{x}\|_{h,n,e}^2 + (n-1)^{1/\gamma} \left(\sum_{k=1}^{n} |\alpha_k|^4 \right)^{1/2} \left(\sum_{1\leq i\neq j\leq n}^{n} \|x_i\|^2 \|x_j\|^2 \right)^{1/2}. \tag{2.7}$$

The inequality follows by (2.5) for $\alpha = \beta = \gamma = \delta = 2$.
We also have:

Theorem 2.4. *For any* $\mathbf{x} \in E^n$ *and* $p > 1$ *we have the following nine possible inequalities*

$$\|\mathbf{x}\|_{h,n,p}^{2p} \leq \begin{cases} \|\mathbf{x}\|_{n,\infty}^{2(p-1)} \|\mathbf{x}\|_{h,n,e}^2 \\ \\ \|\mathbf{x}\|_{h,n,2\alpha(p-1)}^{2(p-1)} \|\mathbf{x}\|_{h,n,\beta}^2, \\ \text{where } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \\ \|\mathbf{x}\|_{h,n,2(p-1)}^{2(p-1)} \|\mathbf{x}\|_{n,\infty}^2 \end{cases} + \begin{cases} \|\mathbf{x}\|_{n,\infty}^{2(p-1)} \sum_{1\leq i\neq j\leq n}^{n} \|x_i\| \|x_j\| \\ \\ (n-1)^{1/\gamma} \|\mathbf{x}\|_{h,n,2\gamma(p-1)}^{2(p-1)} \left(\sum_{1\leq i\neq j\leq n}^{n} \|x_i\|^\delta \|x_j\|^\delta \right)^{1/\delta}, \\ \text{where } \gamma, \delta > 1 \text{ with } \frac{1}{\gamma} + \frac{1}{\delta} = 1 \\ \\ (n-1) \|\mathbf{x}\|_{h,n,2(p-1)}^{2(p-1)} \max_{1\leq i\neq j\leq n} \{\|x_i\| \|x_j\|\}. \end{cases} \tag{2.8}$$

For $p=1$ *we also have:*

(2.9)
$$\|\mathbf{x}\|_{h,n,1}^2 \leq \begin{cases} \|\mathbf{x}\|_{h,n,e}^2 \\ \\ n^{1/\alpha}\|\mathbf{x}\|_{h,n,\beta}^2, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ n\|\mathbf{x}\|_{n,\infty}^2 \end{cases} + \begin{cases} \sum_{1\leq i\neq j\leq n}^n \|x_i\|\|x_j\| \\ \\ (n-1)^{1/\gamma} n^{1/\gamma}\left(\sum_{1\leq i\neq j\leq n}^n \|x_i\|^\delta \|x_j\|^\delta\right)^{1/\delta}, \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ (n-1)\, n \max_{1\leq i\neq j\leq n}\{\|x_i\|\|x_j\|\}. \end{cases}$$

Proof. Let $1 \leq i \leq n$ and $f \in E^*$ with $f \neq 0$. From (2.1) for $\alpha_i = \overline{f(x_i)}\,|f(x_i)|^{p-2}$ and $\alpha_i = 0$ if $f(x_i)=0$ we get

$$\left(\sum_{i=1}^n |f(x_i)|^p\right)^2 \leq \begin{cases} \max_{1\leq k\leq n}\left\{|f(x_k)|^{2(p-1)}\right\}\sum_{k=1}^n |f(x_k)|^2 \\ \\ \left(\sum_{k=1}^n |f(x_k)|^{2\alpha(p-1)}\right)^{1/\alpha}\left(\sum_{k=1}^n |f(x_k)|^{2\beta}\right)^{1/\beta}, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \sum_{k=1}^n |f(x_k)|^{2(p-1)} \max_{1\leq k\leq n}|f(x_k)|^2 \end{cases}$$

$$+\begin{cases} \max_{1\leq k\leq n}\left\{|f(x_k)|^{2(p-1)}\right\}\sum_{1\leq i\neq j\leq n}^n |f(x_i)|\,|f(x_j)| \\ \\ (n-1)^{1/\gamma}\left(\sum_{k=1}^n |f(x_k)|^{2\gamma(p-1)}\right)^{1/\gamma} \\ \times\left(\sum_{1\leq i\neq j\leq n}^n |f(x_i)|^\delta |f(x_j)|^\delta\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ (n-1)\sum_{k=1}^n |f(x_k)|^{2(p-1)} \max_{1\leq i\neq j\leq n}\{|f(x_i)|\,|f(x_j)|\}. \end{cases}$$

If we take the supremum over $f \in E^*$ with $\|f\| = 1$, then we get

$$\sup_{\|f\|=1} \left(\sum_{i=1}^{n} |f(x_i)|^p \right)^2 \tag{2.10}$$

$$\leq \begin{cases} \sup_{\|f\|=1} \max_{1\leq k\leq n} \left\{ |f(x_k)|^{2(p-1)} \right\} \sup_{\|f\|=1} \sum_{k=1}^{n} |f(x_k)|^2 \\ \\ \sup_{\|f\|=1} \left(\sum_{k=1}^{n} |f(x_k)|^{2\alpha(p-1)} \right)^{1/\alpha} \\ \times \sup_{\|f\|=1} \left(\sum_{k=1}^{n} |f(x_k)|^{2\beta} \right)^{1/\beta}, \\ \text{where } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \\ \sup_{\|f\|=1} \sum_{k=1}^{n} |f(x_k)|^{2(p-1)} \sup_{\|f\|=1} \max_{1\leq k\leq n} |f(x_k)|^2 \end{cases}$$

$$+ \begin{cases} \sup_{\|f\|=1} \max_{1\leq k\leq n} \left\{ |f(x_k)|^{2(p-1)} \right\} \\ \times \sup_{\|f\|=1} \sum_{1\leq i\neq j\leq n}^{n} |f(x_i)| |f(x_j)| \\ \\ (n-1)^{1/\gamma} \sup_{\|f\|=1} \left(\sum_{k=1}^{n} |f(x_k)|^{2\gamma(p-1)} \right)^{1/\gamma} \\ \times \sup_{\|f\|=1} \left(\sum_{1\leq i\neq j\leq n}^{n} |f(x_i)|^{\delta} |f(x_j)|^{\delta} \right)^{1/\delta} \\ \text{where } \gamma, \delta > 1 \text{ with } \frac{1}{\gamma} + \frac{1}{\delta} = 1 \\ \\ (n-1) \sup_{\|f\|=1} \sum_{k=1}^{n} |f(x_k)|^{2(p-1)} \\ \times \sup_{\|f\|=1} \max_{1\leq i\neq j\leq n} \{ |f(x_i)| |f(x_j)| \}. \end{cases}$$

Observe that

$$\sup_{\|f\|=1} \left(\sum_{i=1}^{n} |f(x_i)|^p \right)^2 = \left[\sup_{\|f\|=1} \left(\sum_{i=1}^{n} |f(x_i)|^p \right)^{1/p} \right]^{2p} = \|\mathbf{x}\|_{h,n,p}^{2p},$$

$$\sup_{\|f\|=1} \max_{1\leq k\leq n} \left\{ |f(x_k)|^{2(p-1)} \right\} = \max_{1\leq k\leq n} \left\{ \sup_{\|f\|=1} |f(x_k)|^{2(p-1)} \right\} = \|\mathbf{x}\|_{n,\infty}^{2(p-1)},$$

$$\sup_{\|f\|=1} \sum_{k=1}^{n} |f(x_k)|^2 = \|\mathbf{x}\|_{h,n,e}^2,$$

$$\sup_{\|f\|=1} \left(\sum_{k=1}^{n} |f(x_k)|^{2\alpha(p-1)} \right)^{1/\alpha} = \left[\sup_{\|f\|=1} \left(\sum_{k=1}^{n} |f(x_k)|^{2\alpha(p-1)} \right)^{1/[2\alpha(p-1)]} \right]^{2(p-1)}$$
$$= \|\mathbf{x}\|_{h,n,2\alpha(p-1)}^{2(p-1)}$$

and the similar representations for the other terms in the first branch of (2.10), and since

$$\sup_{\|f\|=1} \sum_{1\leq i\neq j\leq n}^{n} |f(x_i)| |f(x_j)| \leq \sum_{1\leq i\neq j\leq n}^{n} \sup_{\|f\|=1} |f(x_i)| \sup_{\|f\|=1} |f(x_j)|$$
$$= \sum_{1\leq i\neq j\leq n}^{n} \sum_{1\leq i\neq j\leq n}^{n} \|x_i\| \|x_j\|$$

and the similar inequalities from the other terms in the second branch of (2.10), we derive the desired result (2.8).

For $p = 1$ in (2.10) we get

$$\sup_{\|f\|=1}\left(\sum_{i=1}^{n}|f(x_i)|\right)^2 \leq \begin{cases} \sup_{\|f\|=1}\sum_{k=1}^{n}|f(x_k)|^2 \\ \\ n^{1/\alpha}\sup_{\|f\|=1}\left(\sum_{k=1}^{n}|f(x_k)|^{2\beta}\right)^{1/\beta}, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ n\sup_{\|f\|=1}\max_{1\leq k\leq n}|f(x_k)|^2 \end{cases} + \begin{cases} \sup_{\|f\|=1}\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)||f(x_j)| \\ \\ (n-1)^{1/\gamma}n^{1/\gamma}\sup_{\|f\|=1}\left(\sum_{1\leq i\neq j\leq n}^{n}|f(x_i)|^{\delta}|f(x_j)|^{\delta}\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ (n-1)\,n\sup_{\|f\|=1}\max_{1\leq i\neq j\leq n}\{|f(x_i)||f(x_j)|\}, \end{cases}$$

which gives (2.9). ∎

Corollary 2.5. *For any* $\mathbf{x} \in E^n$ *we have*

$$\|\mathbf{x}\|_{h,n,e}^4 \leq \begin{cases} \|\mathbf{x}\|_{n,\infty}^2\|\mathbf{x}\|_{h,n,e}^2 \\ \\ \|\mathbf{x}\|_{h,n,2\alpha}^2\|\mathbf{x}\|_{h,n,\beta}^2, \\ \textit{where } \alpha,\beta>1 \textit{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \|\mathbf{x}\|_{h,n,2}^2\|\mathbf{x}\|_{n,\infty}^2 \end{cases} + \begin{cases} \|\mathbf{x}\|_{n,\infty}^2\sum_{1\leq i\neq j\leq n}^{n}\|x_i\|\|x_j\| \\ \\ (n-1)^{1/\gamma}\|\mathbf{x}\|_{h,n,2\gamma}^2\left(\sum_{1\leq i\neq j\leq n}^{n}\|x_i\|^{\delta}\|x_j\|^{\delta}\right)^{1/\delta}, \\ \textit{where } \gamma,\delta>1 \textit{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ (n-1)\|\mathbf{x}\|_{h,n,2}^2\max_{1\leq i\neq j\leq n}\{\|x_i\|\|x_j\|\}. \end{cases} \tag{2.11}$$

The inequality follows by (2.8) for $p = 2$.

3. THE CASE OF TWO VECTORS

Let $(x,y)\in E^2$ and α, β complex numbers. By taking $n=2$ in the inequalities (2.1) above we can state that

$$|\alpha f(x)+\beta f(y)|^2 \le \begin{cases} \max\left\{|\alpha|^2,|\beta|^2\right\}\left(|f(x)|^2+|f(y)|^2\right) \\ \\ \left(|\alpha|^{2\varphi}+|\beta|^{2\varphi}\right)^{1/\varphi}\left(|f(x)|^{2\psi}+|f(y)|^{2\psi}\right)^{1/\psi}, \\ \text{where } \varphi,\psi>1 \text{ with } \frac{1}{\varphi}+\frac{1}{\psi}=1 \\ \\ \left(|\alpha|^2+|\beta|^2\right)\max\left\{|f(x)|^2,|f(y)|^2\right\} \end{cases}$$
$$+\begin{cases} 2|\alpha||\beta||f(x)||f(y)| \\ \\ \left[(|\alpha|^\gamma+|\beta|^\gamma)^2-|\alpha|^{2\gamma}-|\beta|^{2\gamma}\right]^{1/\gamma} \\ \times\left(2|f(x)|^\delta|f(y)|^\delta\right)^{1/\delta} \\ \text{where } \gamma,\delta>1 \text{ with } \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \\ \left[(|\alpha|+|\beta|)^2-|\alpha|^2-|\beta|^2\right]|f(x)||f(y)|, \end{cases}$$

which gives the simpler three inequalities

$$|\alpha f(x)+\beta f(y)|^2 \le 2|\alpha||\beta||f(x)||f(y)| + \begin{cases} \max\left\{|\alpha|^2,|\beta|^2\right\}\left(|f(x)|^2+|f(y)|^2\right) \\ \\ \left(|\alpha|^{2\alpha}+|\beta|^{2\alpha}\right)^{1/\alpha}\left(|f(x)|^{2\beta}+|f(y)|^{2\beta}\right)^{1/\beta}, \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \left(|\alpha|^2+|\beta|^2\right)\max\left\{|f(x)|^2,|f(y)|^2\right\}, \end{cases} \quad . \tag{3.1}$$

for all $f\in E^*$.

From (2.5) we get

$$\|\alpha x+\beta y\|^2 \le 2|\alpha||\beta|\|x\|\|y\| + \begin{cases} \max\left\{|\alpha|^2,|\beta|^2\right\}\|(x,y)\|_{h,2,e}^2 \\ \\ \left(|\alpha|^{2\varphi}+|\beta|^{2\varphi}\right)^{1/\varphi}\|(x,y)\|_{h,2,\psi}^2, \\ \text{where } \varphi,\psi>1 \text{ with } \frac{1}{\varphi}+\frac{1}{\psi}=1; \\ \\ \left(|\alpha|^2+|\beta|^2\right)\|(x,y)\|_{2,\infty}^2 \end{cases} , \tag{3.2}$$

for all $(x,y)\in E^2$ and α, β complex numbers, where

$$\|(x,y)\|_{h,2,e} = \sup_{|\lambda|^2+|\mu|^2=1}\|\lambda x+\mu y\|,$$

$$\|(x,y)\|_{h,2,q} = \sup_{|\lambda|^q+|\mu|^q=1}\|\lambda x+\mu y\|$$

and

$$\|(x,y)\|_{2,\infty} = \max\{\|x\|,\|y\|\}.$$

For $\varphi = \psi = 2$ we get

$$\|\alpha x + \beta y\|^2 \le 2|\alpha||\beta|\|x\|\|y\| + \left(|\alpha|^4 + |\beta|^4\right)^{1/2} \|(x,y)\|_{h,2,e}^2,$$

for all $(x, y) \in E^2$ and α, β complex numbers.

If we take in (3.1)

$$\alpha = \overline{f(x)}\,|f(x)|^{p-2} \text{ and } \beta = \overline{f(y)}\,|f(y)|^{p-2}$$

we obtain for $p \ge 1$ that

$$(3.3)\quad \begin{aligned} &(|f(x)|^p + |f(y)|^p)^2 \\ &\le 2|f(x)|^p |f(y)|^p \\ &+ \begin{cases} \max\left\{|f(x)|^{2(p-1)}, |f(y)|^{2(p-1)}\right\}\left(|f(x)|^2 + |f(y)|^2\right) \\ \\ \left(|f(x)|^{2\varphi(p-1)} + |f(y)|^{2\varphi(p-1)}\right)^{1/\varphi}\left(|f(x)|^{2\psi} + |f(y)|^{2\psi}\right)^{1/\psi}, \\ \text{where } \varphi,\ \psi > 1 \text{ with } \frac{1}{\varphi} + \frac{1}{\psi} = 1 \\ \\ \left(|f(x)|^{2(p-1)} + |f(y)|^{2(p-1)}\right)\max\left\{|f(x)|^2, |f(y)|^2\right\}, \end{cases} \end{aligned}$$

for all $(x, y) \in E^2$ and $f \in E^*$.

If we take the supremum over $f \in E^*$, $\|f\| = 1$, then we get

$$(3.4)\quad \|(x,y)\|_{h,2,p}^{2p} \le 2\|x\|^p\|y\|^p + \begin{cases} \max\left\{\|x\|^{2(p-1)}, \|y\|^{2(p-1)}\right\}\|(x,y)\|_{h,2,e}^2 \\ \\ \|(x,y)\|_{h,2,2\varphi(p-1)}^{2(p-1)}\|(x,y)\|_{h,2,2\psi}^2, \\ \text{where } \varphi,\ \psi > 1 \text{ with } \frac{1}{\varphi} + \frac{1}{\psi} = 1 \\ \\ \max\left\{\|x\|^2, \|y\|^2\right\}\|(x,y)\|_{h,2,2(p-1)}^{2(p-1)} \end{cases}.$$

for all $(x, y) \in E^2$ and $p > 1$.

If we take $p = 2$ in (3.4), then we get

$$(3.5)\quad \|(x,y)\|_{h,2,e}^4 \le 2\|x\|^2\|y\|^2 + \begin{cases} \max\left\{\|x\|^2, \|y\|^2\right\}\|(x,y)\|_{h,2,e}^2 \\ \\ \|(x,y)\|_{h,2,2\varphi}^2\|(x,y)\|_{h,2,2\psi}^2, \\ \text{where } \varphi,\ \psi > 1 \text{ with } \frac{1}{\varphi} + \frac{1}{\psi} = 1 \\ \\ \max\left\{\|x\|^2, \|y\|^2\right\}\|(x,y)\|_{h,2,2}^2. \end{cases}.$$

For $p = 1$ we have

$$(3.6)\quad \|(x,y)\|_{h,2,1}^2 \le 2\|x\|\|y\| + \begin{cases} \|(x,y)\|_{h,2,e}^2 \\ \\ 2^{1/\varphi}\|(x,y)\|_{h,2,2\psi}^2, \\ \text{where } \varphi, \psi > 1 \text{ with } \frac{1}{\varphi} + \frac{1}{\psi} = 1 \\ \\ 2\max\left\{\|x\|^2, \|y\|^2\right\}. \end{cases}.$$

We introduce the following functionals $\|\cdot\|_t, \|\cdot\|_s : E^2 \to [0,\infty)$

$$\|(x,y)\|_t = \sup_{\theta\in\mathbb{R}} \|\cos\theta x + i\sin\theta y\|$$

and

$$\|(x,y)\|_s = \sup_{\theta\in\mathbb{R}} \left\| \frac{e^{i\theta}x + e^{-i\theta}y}{2} \right\|.$$

We observe that $\|\cdot\|_t$ is *a norm* on E^2 and

$$\|(x,y)\|_{2,\infty} \le \|(x,y)\|_t \le \|(x,y)\|_{h,2,e}$$

for all $(x,y) \in E^2$.

We also have that

$$\frac{\sqrt{2}}{2} \max\{\|x+iy\|, \|x-iy\|\} \le \|(x,y)\|_t$$

for all $(x,y) \in E^2$.

Observe that

$$\frac{e^{i\theta}x + e^{-i\theta}y}{2} = \frac{(\cos\theta + i\sin\theta)x + (\cos\theta - i\sin\theta)y}{2} = \cos\theta\left(\frac{x+y}{2}\right) + i\sin\theta\left(\frac{x-y}{2}\right),$$

which gives that

$$\|(x,y)\|_s = \left\| \left(\frac{x+y}{2}, \frac{x-y}{2}\right) \right\|_t. \tag{3.7}$$

The functional $\|\cdot\|_s$ is also a norm and

$$\left\| \left(\frac{x+y}{2}, \frac{x-y}{2}\right) \right\|_{2,\infty} \le \|(x,y)\|_s \le \left\| \left(\frac{x+y}{2}, \frac{x-y}{2}\right) \right\|_{h,2,e}.$$

Proposition 3.1. *For any $(x,y) \in E^2$, we have*

$$\|(x,y)\|_t^2 \le \|x\|\|y\| + \begin{cases} \|(x,y)\|_{h,2,\beta}^2, \text{ where } \beta > 1, \\ \\ \|(x,y)\|_{2,\infty}^2 \end{cases}, \tag{3.8}$$

and

$$\|(x,y)\|_s^2 \le \left\|\frac{x+y}{2}\right\| \left\|\frac{x-y}{2}\right\| + \begin{cases} \left\|\left(\frac{x+y}{2}, \frac{x-y}{2}\right)\right\|_{h,2,\beta}^2, \\ \text{where } \beta > 1 \\ \\ \left\|\left(\frac{x+y}{2}, \frac{x-y}{2}\right)\right\|_{2,\infty}^2. \end{cases} \tag{3.9}$$

Proof. From (3.2) we get for $\alpha = \cos\theta$, $\beta = i\sin\theta$ that

$$\|\cos\theta x + i\sin\theta y\|^2 \le 2|\sin\theta||\cos\theta|\|x\|\|y\| + \begin{cases} \max\left\{|\sin\theta|^2, |\cos\theta|^2\right\}\|(x,y)\|_{h,2,e}^2 \\ \\ \left(|\sin\theta|^{2\alpha} + |\cos\theta|^{2\alpha}\right)^{1/\alpha}\|(x,y)\|_{h,2,\beta}^2, \\ \text{where } \alpha,\beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \\ \left(|\sin\theta|^2 + |\cos\theta|^2\right)\|(x,y)\|_{2,\infty}^2 \end{cases}, \tag{3.10}$$

for all $\theta \in \mathbb{R}$.

Since

$$2\left|\sin\theta\right|\left|\cos\theta\right|=\left|\sin\left(2\theta\right)\right|\leq 1$$

and

$$\left|\sin\theta\right|^{2\alpha}+\left|\cos\theta\right|^{2\alpha}\leq\left|\sin\theta\right|^{2}+\left|\cos\theta\right|^{2}=1,$$

because $\alpha>1$, then by (3.10) we deduce that

$$\left\|\cos\theta x+i\sin\theta y\right\|^{2}\leq\left\|x\right\|\left\|y\right\|+\begin{cases}\left\|(x,y)\right\|_{h,2,\beta}^{2}, \text{ where } \beta>1\\ \\ \left\|(x,y)\right\|_{2,\infty}^{2}\end{cases},$$

and if we take the supremum over $\theta\in\mathbb{R}$ we deduce the desired result (3.8).

The inequality (3.9) follows from (3.8) and the identity (3.7). ∎

Also, we have:

Proposition 3.2. *For any $(x,y)\in E^{2}$, we have*

$$\left\|(x,y)\right\|_{s}^{2}\leq\frac{1}{2}\left\|x\right\|\left\|y\right\|+\begin{cases}\frac{1}{4}\left\|(x,y)\right\|_{h,2,e}^{2}\\ \\ \frac{1}{2^{1+1/\beta}}\left\|(x,y)\right\|_{h,2,\beta}^{2}, \textit{ where } \beta>1\\ \\ \frac{1}{2}\left\|(x,y)\right\|_{2,\infty}^{2}\end{cases}\tag{3.11}$$

and

$$\left\|(x,y)\right\|_{t}^{2}\leq\frac{1}{2}\left\|x+y\right\|\left\|x-y\right\|+\begin{cases}\frac{1}{4}\left\|(x+y,x-y)\right\|_{h,2,e}^{2}\\ \\ \frac{1}{2^{1+1/\beta}}\left\|(x+y,x-y)\right\|_{h,2,\beta}^{2},\\ \textit{where } \beta>1\\ \\ \frac{1}{2}\left\|(x+y,x-y)\right\|_{2,\infty}^{2}.\end{cases}\tag{3.12}$$

Proof. If we take $\alpha=\frac{e^{i\theta}}{2}$ and $\beta=\frac{e^{-i\theta}}{2}$ in (3.2), then we get

$$\begin{aligned}\left\|\frac{e^{i\theta}}{2}x+\frac{e^{-i\theta}}{2}y\right\|^{2}&\leq\frac{1}{2}\left\|x\right\|\left\|y\right\|\\&+\begin{cases}\frac{1}{4}\left\|(x,y)\right\|_{h,2,e}^{2}\\ \\ \frac{1}{2^{2-1/\alpha}}\left\|(x,y)\right\|_{h,2,\beta}^{2},\\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1\\ \\ \frac{1}{2}\left\|(x,y)\right\|_{2,\infty}^{2},\end{cases}\end{aligned},$$

which proves the desired result (3.11).

If we take $x=u+v$ and $y=u-v$ in (3.7) then we have

$$\left\|(u,v)\right\|_{t}=\left\|(u+v,u-v)\right\|_{s}.$$

From (3.11) we then get

$$\|(u+v,u-v)\|_s^2 \le \frac{1}{2}\|u+v\|\|u-v\| + \begin{cases} \frac{1}{4}\|(u+v,u-v)\|_{h,2,e}^2 \\ \frac{1}{2^{1+1/\beta}}\|(u+v,u-v)\|_{h,2,\beta}^2, \text{ where } \beta>1 \\ \frac{1}{2}\|(u+v,u-v)\|_{2,\infty}^2, \end{cases}$$

which proves (3.12). ∎

4. THE CASE OF INNER PRODUCT SPACES

Let $(H,\langle .,.\rangle)$ be a complex Hilbert space. Then we have the following upper bounds for the norms $\|\cdot\|_t$, $\|\cdot\|_s$ defined on H^2.

Proposition 4.1. *For all $(x,y)\in H^2$ we have the inequalities*

$$\|(x,y)\|_t^2 \le |\operatorname{Im}\langle x,y\rangle| + \begin{cases} \|x\|^2+\|y\|^2 \\ \left(\|x\|^{2p}+\|y\|^{2p}\right)^{1/p}, \text{ for } p>1 \\ \max\left\{\|x\|^2,\|y\|^2\right\} \end{cases} \tag{4.1}$$

and the inequalities

$$\|(x,y)\|_s^2 \le \frac{1}{2}|\operatorname{Im}\langle x,y\rangle| + \begin{cases} \frac{1}{2}\left(\|x\|^2+\|y\|^2\right) \\ \left(\left\|\frac{x+y}{2}\right\|^{2p}+\left\|\frac{x-y}{2}\right\|^{2p}\right)^{1/p}, \\ \text{for } p>1 \\ \frac{1}{4}\left(\|x\|^2+\|y\|^2\right)+\frac{1}{2}|\operatorname{Re}\langle x,y\rangle|. \end{cases} \tag{4.2}$$

Proof. Observe that

$$\begin{aligned} \|(x,y)\|_t^2 &= \sup_{\theta\in\mathbb{R}}\|\cos\theta x+i\sin\theta y\|^2 \\ &= \sup_{\theta\in\mathbb{R}}\left[\|\cos\theta x\|^2+\|i\sin\theta y\|^2+2\operatorname{Re}\langle\cos\theta x,i\sin\theta y\rangle\right] \\ &= \sup_{\theta\in\mathbb{R}}\left[\cos^2\theta\|x\|^2+\sin^2\theta\|y\|^2-2\operatorname{Re}\left(\sin\theta\cos\theta i\langle x,y\rangle\right)\right] \\ &= \sup_{\theta\in\mathbb{R}}\left[\cos^2\theta\|x\|^2+\sin^2\theta\|y\|^2-2\sin\theta\cos\theta\operatorname{Re}\left(i\langle x,y\rangle\right)\right] \\ &= \sup_{\theta\in\mathbb{R}}\left[\cos^2\theta\|x\|^2+\sin^2\theta\|y\|^2-\sin(2\theta)\operatorname{Re}\left(i\left(\operatorname{Re}\langle x,y\rangle+i\operatorname{Im}\langle x,y\rangle\right)\right)\right] \\ &= \sup_{\theta\in\mathbb{R}}\left[\cos^2\theta\|x\|^2+\sin^2\theta\|y\|^2+\sin(2\theta)\operatorname{Im}\langle x,y\rangle\right] \end{aligned} \tag{4.3}$$

for all $(x,y)\in H^2$.

Observe that

$$\sin(2\theta)\operatorname{Im}\langle x,y\rangle \le |\sin(2\theta)\operatorname{Im}\langle x,y\rangle| \le |\operatorname{Im}\langle x,y\rangle| \tag{4.4}$$

for all $(x,y)\in H^2$ and $\theta\in\mathbb{R}$.

Also, by Hölder's elementary inequality we have

$$\cos^2\theta\,\|x\|^2+\sin^2\theta\,\|y\|^2\le\begin{cases}\max\left\{\cos^2\theta,\sin^2\theta\right\}\left(\|x\|^2+\|y\|^2\right)\\[2ex]\left(\cos^{2q}\theta+\sin^{2q}\theta\right)^{1/q}\left(\|x\|^{2p}+\|y\|^{2p}\right)^{1/p},\\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\[2ex]\left(\cos^2\theta+\sin^2\theta\right)\max\left\{\|x\|^2,\|y\|^2\right\}\end{cases}\tag{4.5}$$

$$\le\begin{cases}\|x\|^2+\|y\|^2\\[2ex]\left(\|x\|^{2p}+\|y\|^{2p}\right)^{1/p},\ \text{for } p>1\\[2ex]\max\left\{\|x\|^2,\|y\|^2\right\}\end{cases}$$

for all $(x,y)\in H^2$ and $\theta\in\mathbb{R}$, since $\max\left\{\cos^2\theta,\sin^2\theta\right\}\le 1$ and for $q>1$, $\cos^{2q}\theta+\sin^{2q}\theta\le\cos^2\theta+\sin^2\theta=1.$

By (4.3)-(4.5) we then deduce (4.1).

If we utilize (3.7) and (4.1), then we get

$$\|(x,y)\|_s^2=\left\|\left(\frac{x+y}{2},\frac{x-y}{2}\right)\right\|_t^2\tag{4.6}$$

$$\le\left|\operatorname{Im}\left\langle\frac{x+y}{2},\frac{x-y}{2}\right\rangle\right|+\begin{cases}\left\|\frac{x+y}{2}\right\|^2+\left\|\frac{x-y}{2}\right\|^2\\[2ex]\left(\left\|\frac{x+y}{2}\right\|^{2p}+\left\|\frac{x-y}{2}\right\|^{2p}\right)^{1/p},\\ \text{for } p>1\\[2ex]\max\left\{\left\|\frac{x+y}{2}\right\|^2,\left\|\frac{x-y}{2}\right\|^2\right\}.\end{cases}$$

Observe that

$$\begin{aligned}\left\langle\frac{x+y}{2},\frac{x-y}{2}\right\rangle&=\frac{1}{4}\left(\|x\|^2+\langle y,x\rangle-\langle x,y\rangle-\|y\|^2\right)\\&=\frac{1}{4}\left(\|x\|^2+\overline{\langle x,y\rangle}-\langle x,y\rangle-\|y\|^2\right)\\&=\frac{1}{4}\left(\|x\|^2-\|y\|^2+\overline{\langle x,y\rangle}-\langle x,y\rangle\right)\\&=\frac{1}{4}\left(\|x\|^2-\|y\|^2-i\operatorname{Im}\langle x,y\rangle-i\operatorname{Im}\langle x,y\rangle\right)\end{aligned}$$

which gives

$$\operatorname{Im}\left\langle\frac{x+y}{2},\frac{x-y}{2}\right\rangle=-\frac{1}{2}\operatorname{Im}\langle x,y\rangle,$$

$$\left\|\frac{x+y}{2}\right\|^2+\left\|\frac{x-y}{2}\right\|^2=\frac{1}{2}\left(\|x\|^2+\|y\|^2\right)$$

and

$$\max\left\{\left\|\frac{x+y}{2}\right\|^2,\left\|\frac{x-y}{2}\right\|^2\right\}$$
$$=\frac{1}{4}\max\left\{\|x\|^2+\|y\|^2+2\operatorname{Re}\langle x,y\rangle,\|x\|^2+\|y\|^2-2\operatorname{Re}\langle x,y\rangle\right\}$$
$$=\frac{1}{4}\left(\|x\|^2+\|y\|^2\right)+\frac{1}{2}\left|\operatorname{Re}\langle x,y\rangle\right|.$$

By (4.6) we then get (4.2). ∎

We also have

Proposition 4.2. *For all* $(x,y)\in H^2$ *we have the inequalities*

$$\|(x,y)\|_s^2\le\frac{1}{4}\left(\|x\|^2+\|y\|^2\right)+\frac{1}{2}\times\begin{cases}\left[\left|\operatorname{Re}\langle x,y\rangle\right|+\left|\operatorname{Im}\langle x,y\rangle\right|\right],\\ \\ \gamma_p\left[\left|\operatorname{Re}\langle x,y\rangle\right|^q+\left|\operatorname{Im}\langle x,y\rangle\right|^q\right]^{1/q}\\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \sqrt{2}\max\left\{\left|\operatorname{Re}\langle x,y\rangle\right|,\left|\operatorname{Im}\langle x,y\rangle\right|\right\},\end{cases}\tag{4.7}$$

where

$$\gamma_p:=\begin{cases}2^{\frac{2-p}{2p}} \text{ if } p\in(1,2)\\ \\ 1 \text{ if } p\in[2,\infty),\end{cases}$$

and

$$\|(x,y)\|_t^2\le\frac{1}{2}\left(\|x\|^2+\|y\|^2\right)\tag{4.8}$$
$$+\frac{1}{2}\times\begin{cases}\left[\left|\|x\|^2-\|y\|^2\right|+2\left|\operatorname{Im}\langle x,y\rangle\right|\right]\\ \\ \gamma_p\left[\left|\|x\|^2-\|y\|^2\right|^q+2^q\left|\operatorname{Im}\langle x,y\rangle\right|^q\right]^{1/q}\\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \sqrt{2}\max\left\{\left|\|x\|^2-\|y\|^2\right|,2\left|\operatorname{Im}\langle x,y\rangle\right|\right\}.\end{cases}$$

Proof. We have that

$$\|(x,y)\|_s^2=\sup_{\theta\in\mathbb{R}}\left\|\frac{e^{i\theta}x+e^{-i\theta}y}{2}\right\|^2\tag{4.9}$$
$$=\sup_{\theta\in\mathbb{R}}\left[\frac{\left\|e^{i\theta}x\right\|^2+\left\|e^{-i\theta}y\right\|^2+2\operatorname{Re}\left\langle e^{i\theta}x,e^{-i\theta}y\right\rangle}{4}\right]$$
$$=\frac{1}{4}\left(\|x\|^2+\|y\|^2\right)+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\operatorname{Re}\left(e^{2i\theta}\langle x,y\rangle\right)$$
$$=\frac{1}{4}\left(\|x\|^2+\|y\|^2\right)$$
$$+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\operatorname{Re}\left(\left[\cos(2\theta)+i\sin(2\theta)\right]\left[\operatorname{Re}\langle x,y\rangle+i\operatorname{Im}\langle x,y\rangle\right]\right)$$
$$=\frac{1}{4}\left(\|x\|^2+\|y\|^2\right)+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left[\cos(2\theta)\operatorname{Re}\langle x,y\rangle-\sin(2\theta)\operatorname{Im}\langle x,y\rangle\right]$$

for all $(x,y)\in H^2$.

Using Hölder's inequality, we have

$$\begin{aligned}
&\cos(2\theta)\operatorname{Re}\langle x,y\rangle-\sin(2\theta)\operatorname{Im}\langle x,y\rangle\\
&\le\left|\cos(2\theta)\operatorname{Re}\langle x,y\rangle-\sin(2\theta)\operatorname{Im}\langle x,y\rangle\right|\\
&\le\left|\cos(2\theta)\right|\left|\operatorname{Re}\langle x,y\rangle\right|+\left|\sin(2\theta)\right|\left|\operatorname{Im}\langle x,y\rangle\right|\\
&\le\begin{cases}\max\left\{\left|\cos(2\theta)\right|,\left|\sin(2\theta)\right|\right\}\left[\left|\operatorname{Re}\langle x,y\rangle\right|+\left|\operatorname{Im}\langle x,y\rangle\right|\right]\\[2ex]
\left(\left|\cos(2\theta)\right|^p+\left|\sin(2\theta)\right|^p\right)^{1/p}\left[\left|\operatorname{Re}\langle x,y\rangle\right|^q+\left|\operatorname{Im}\langle x,y\rangle\right|^q\right]^{1/q}\\
\text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2ex]
\left(\left|\cos(2\theta)\right|+\left|\sin(2\theta)\right|\right)\max\left\{\left|\operatorname{Re}\langle x,y\rangle\right|,\left|\operatorname{Im}\langle x,y\rangle\right|\right\}\end{cases}
\end{aligned}$$

for all $(x,y)\in H^2$ and $\theta\in\mathbb{R}$.

By taking the supremum over $\theta\in\mathbb{R}$ we get

$$\begin{aligned}
&\sup_{\theta\in\mathbb{R}}\left[\cos(2\theta)\operatorname{Re}\langle x,y\rangle-\sin(2\theta)\operatorname{Im}\langle x,y\rangle\right]\\
&\le\begin{cases}\sup_{\theta\in\mathbb{R}}\max\left\{\left|\cos(2\theta)\right|,\left|\sin(2\theta)\right|\right\}\left[\left|\operatorname{Re}\langle x,y\rangle\right|+\left|\operatorname{Im}\langle x,y\rangle\right|\right]\\[2ex]
\sup_{\theta\in\mathbb{R}}\left(\left|\cos(2\theta)\right|^p+\left|\sin(2\theta)\right|^p\right)^{1/p}\left[\left|\operatorname{Re}\langle x,y\rangle\right|^q+\left|\operatorname{Im}\langle x,y\rangle\right|^q\right]^{1/q}\\
\text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\[2ex]
\sup_{\theta\in\mathbb{R}}\left(\left|\cos(2\theta)\right|+\left|\sin(2\theta)\right|\right)\max\left\{\left|\operatorname{Re}\langle x,y\rangle\right|,\left|\operatorname{Im}\langle x,y\rangle\right|\right\}\end{cases}
\end{aligned}\tag{4.10}$$

We have the following bounds

$$\sup_{\theta\in\mathbb{R}}\max\left\{\left|\cos(2\theta)\right|,\left|\sin(2\theta)\right|\right\}=1$$

and for $p\ge 2$

$$0\le\left|\cos(2\theta)\right|^p+\left|\sin(2\theta)\right|^p\le\cos^2(2\theta)+\left|\sin^2(2\theta)\right|=1,$$

which gives that

$$\sup_{\theta\in\mathbb{R}}\left(\left|\cos(2\theta)\right|^p+\left|\sin(2\theta)\right|^p\right)=1.$$

Let $p\in[1,2)$ and $t\in[0,\pi/2]$ and consider $g_p(t)=\cos^p(t)+\sin^p(t)$. Then for $t\in(0,\pi/2)$

$$g_p'(t)=p\cos t\sin^{p-1}(t)-p\sin t\cos^{p-1}(t)=p\sin t\cos t\left[\sin^{p-2}(t)-\cos^{p-2}(t)\right],$$

which shows that $g_p'(t)=0$ only for $t=\pi/4$, g' is positive on $(0,\pi/4)$ and negative on $(\pi/4,\pi/2)$. Therefore

$$\sup_{t\in[0,\pi/2]}g_p(t)=g_p(\pi/4)=\cos^p(\pi/4)+\sin^p(\pi/4)=2^{\frac{2-p}{2}}>1.$$

This implies that

$$\sup_{\theta\in\mathbb{R}}\left(\left|\cos(2\theta)\right|^p+\left|\sin(2\theta)\right|^p\right)^{1/p}=2^{\frac{2-p}{2p}}\text{ for }p\in(1,2)$$

and

$$\sup_{\theta\in\mathbb{R}}\left(\left|\cos(2\theta)\right|+\left|\sin(2\theta)\right|\right)=\sqrt{2}.$$

By (4.10) we then get

$$\sup_{\theta\in\mathbb{R}}\left[\cos(2\theta)\operatorname{Re}\langle x,y\rangle-\sin(2\theta)\operatorname{Im}\langle x,y\rangle\right]\le\begin{cases}\left[\left|\operatorname{Re}\langle x,y\rangle\right|+\left|\operatorname{Im}\langle x,y\rangle\right|\right]\\ \\ \gamma_p\left[\left|\operatorname{Re}\langle x,y\rangle\right|^q+\left|\operatorname{Im}\langle x,y\rangle\right|^q\right]^{1/q}\\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \sqrt{2}\max\left\{\left|\operatorname{Re}\langle x,y\rangle\right|,\left|\operatorname{Im}\langle x,y\rangle\right|\right\}\end{cases}$$

and by (4.9) we derive (4.7).

By (4.7) we have

$$\begin{aligned}\|(x,y)\|_t^2&=\|(x+y,x-y)\|_s^2\\&\le\frac{1}{4}\left(\|x+y\|^2+\|x-y\|^2\right)\\&+\frac{1}{2}\times\begin{cases}\left[\left|\operatorname{Re}\langle x+y,x-y\rangle\right|+\left|\operatorname{Im}\langle x+y,x-y\rangle\right|\right]\\ \\ \gamma_p\left[\left|\operatorname{Re}\langle x+y,x-y\rangle\right|^q+\left|\operatorname{Im}\langle x+y,x-y\rangle\right|^q\right]^{1/q}\\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \sqrt{2}\max\left\{\left|\operatorname{Re}\langle x+y,x-y\rangle\right|,\left|\operatorname{Im}\langle x+y,x-y\rangle\right|\right\}.\end{cases}\end{aligned}\tag{4.11}$$

Observe that

$$\langle x+y,x-y\rangle=\|x\|^2-\|y\|^2-2i\operatorname{Im}\langle x,y\rangle,$$

which gives

$$\operatorname{Re}\langle x+y,x-y\rangle=\|x\|^2-\|y\|^2 \text{ and } \operatorname{Im}\langle x+y,x-y\rangle=-2\operatorname{Im}\langle x,y\rangle$$

and by (4.11) we deduce the desired result (4.8). ∎

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