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THE DRAZIN-STAR AND STAR-DRAZIN SOLUTIONS TO QUATERNION MATRIX EQUATIONS

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ABSTRACT. The notions of the Drazin-star and star-Drazin matrices are expanded to quaternion matrices in this paper. Their determinantal representations are developed in both cases in terms of noncommutative row-column determinants of quaternion matrices and for minors of appropriate complex matrices. We study all possible two-sided quaternion matrix equations with their one-sided partial cases whose uniquely determined solutions are based on the Drazin-star and star-Drazin matrices. Solutions of these equations are represented by Cramer's rules in both cases for quaternion and complex matrix equations. A numerical example is presented to illustrate our results.

Key words and phrases: Drazin-star matrix; Star-Drazin matrix; Quaternion matrix; Determinantal representation.

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1. INTRODUCTION

Usually, $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively, $\mathbb{H}^{m\times n}$ contains $m\times n$ matrices on the quaternion skew field $\mathbb{H}=\{v_0+v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}\mid \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,\ v_0,v_1,v_2,v_3\in\mathbb{R}\}$. Also, $\mathbb{H}^{m\times n}_r$ marks the subset of $\mathbb{H}^{m\times n}$ with matrices of rank r. For $v=v_0+v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}\in\mathbb{H}$, its conjugate is $\overline{v}=v_0-v_1\mathbf{i}-v_2\mathbf{j}-v_3\mathbf{k}$, and its norm is $\|v\|=\sqrt{v\overline{v}}=\sqrt{\overline{v}v}=\sqrt{v_0^2+v_1^2+v_2^2+v_3^2}$.

The rank, trace and conjugate transpose (Hermitian) of $\mathbf{A}\in\mathbb{H}^{m\times n}$, respectively, are denoted as $\operatorname{rank}(\mathbf{A})$, $\operatorname{tr}(\mathbf{A})$ and $\mathbf{A}^*$. Due to noncommutativity in the quaternion skew field, the following notions are used:

- $\mathcal{N}_l(\mathbf{A})=\{\mathbf{n}\in\mathbb{H}^{1\times m}:\ \mathbf{nA}=0\}$ is the left null space (or left kernel) of $\mathbf{A}$;
- $\mathcal{N}_r(\mathbf{A})=\{\mathbf{n}\in\mathbb{H}^{n\times 1}:\ \mathbf{An}=0\}$ is the right null space (or right kernel) of $\mathbf{A}$;
- $\mathcal{R}_l(\mathbf{A})=\{\mathbf{m}\in\mathbb{H}^{1\times n}:\ \mathbf{m}=\mathbf{nA},\ \mathbf{s}\in\mathbb{H}^{1\times m}\}$ is the left row space (or left range) of $\mathbf{A}$;
- $\mathcal{C}_r(\mathbf{A})=\{\mathbf{m}\in\mathbb{H}^{m\times 1}:\mathbf{m}=\mathbf{An},\ \mathbf{s}\in\mathbb{H}^{n\times 1}\}$ is the right column space (or right range) of $\mathbf{A}$.

The rank of $\mathbf{A}\in\mathbb{H}^{m\times n}$ is determined as $\operatorname{rank}(\mathbf{A})=\dim\mathcal{C}_r(\mathbf{A})=\dim\mathcal{R}_l(\mathbf{A}^*)$.

For $\mathbf{A}\in\mathbb{H}^{n\times m}$, its *Moore-Penrose (or MP) inverse* $\mathbf{A}^\dagger$ is a cunique solution $\mathbf{X}$ to the system

$$(1)\ \mathbf{A}=\mathbf{AXA},\quad (2)\ \mathbf{X}=\mathbf{XAX},\quad (3)\ \mathbf{AX}=(\mathbf{AX})^*,\quad (4)\ \mathbf{XA}=(\mathbf{XA})^*.$$

By $\mathbf{A}^{(\delta)}$ we denote any matrix that satisfies the equations determined by $\delta\subseteq\{1,2,3,4\}$ and it is called the δ-inverse of $\mathbf{A}$. In particular, $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are called the *inner inverse* and the *outer inverse*, respectively. One of the outer inverses is the *Drazin (D-)inverse* $\mathbf{A}^{\mathrm{D}}$ of $\mathbf{A}\in\mathbb{H}^{n\times n}$ that is defined as a unique solution to the system

$$(2)\ \mathbf{X}=\mathbf{XAX},\quad (5)\ \mathbf{XA}=\mathbf{AX},\quad (6)\ \mathbf{A}^k=\mathbf{XA}^{k+1},$$

where $k=\operatorname{Ind}(\mathbf{A})=\min\{k\in\mathbb{N}\cup\{0\}\mid \operatorname{rank}(\mathbf{A}^k)=\operatorname{rank}(\mathbf{A}^{k+1})\}$ denotes the index of $\mathbf{A}$. If $\operatorname{Ind}(\mathbf{A})\le 1$, then $\mathbf{A}^{\mathrm{D}}$ reduces to the *group inverse* $\mathbf{A}^{\#}$.

New generalized inverses can be generated by combining different generalized inverses or applying them in certain (range or kernel) matrix spaces. In particular, such generalized inverses involve the core inverse [3], the core-EP (CEP-)inverse [30], the MPD-inverse [23], the MPCEP-inverse [7], etc. Some extensions of these generalized inverses were given for tensors [34, 42], operators [26], and for elements of rings [9, 24, 47]. Their extensions with determinantal representations for quaternion matrices were introduced in [14, 18].

Recently, Mosić in [25] presented two new classes of square complex matrices that can be expanded to quaternion matrices.

Lemma 1.1. *Let* $\mathbf{A}\in\mathbb{H}^{n\times n}$ *and* $k=\operatorname{Ind}(\mathbf{A})$.

(a) *The system of equations*

$$\mathbf{X}(\mathbf{A}^\dagger)^*\mathbf{X}=\mathbf{X},\ \mathbf{A}^k\mathbf{X}=\mathbf{A}^k\mathbf{A}^*,\ \mathit{and}\ \mathbf{X}(\mathbf{A}^\dagger)^*=\mathbf{A}^{\mathrm{D}}\mathbf{A},$$

is consistent and its unique solution is $\mathbf{X}=\mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{A}^*$.

(b) *The system of equations*

$$\mathbf{X}(\mathbf{A}^\dagger)^*\mathbf{X}=\mathbf{X},\ \mathbf{XA}^k=\mathbf{A}^*\mathbf{A}^k,\ \mathit{and}\ (\mathbf{A}^\dagger)^*\mathbf{X}=\mathbf{AA}^{\mathrm{D}},$$

is consistent and its unique solution is $\mathbf{X}=\mathbf{A}^*\mathbf{AA}^{\mathrm{D}}$.

Definition 1.1. Let $\mathbf{A}\in\mathbb{H}^{n\times n}$ and $k=\operatorname{Ind}(\mathbf{A})$.

(a) The Drazin-star matrix of $\mathbf{A}$ (or the Drazin-star inverse of $(\mathbf{A}^\dagger)^*$) is defined as

$$\mathbf{A}^{\mathrm{D},*}=\mathbf{A}^{\mathrm{D}}\mathbf{AA}^*. \tag{1.1}$$

(b) The star-Drazin matrix of $\mathbf{A}$ (or the star-Drazin inverse of $(\mathbf{A}^{\dagger})^{*}$) is defined as

$$\mathbf{A}^{*,\mathrm{D}} = \mathbf{A}^{*}\mathbf{A}\mathbf{A}^{\mathrm{D}}. \tag{1.2}$$

Notice that if $\mathrm{Ind}(\mathbf{A}) = 0$ for $\mathbf{A} \in \mathbb{H}^{n\times n}$, then $\mathbf{A}^{\mathrm{D}} = \mathbf{A}^{-1}$ and in this case it follows $\mathbf{A}^{\mathrm{D},*} = \mathbf{A}^{*,\mathrm{D}} = \mathbf{A}^{*}$. For $\mathbf{A} \in \mathbb{H}^{n\times n}$ with $\mathrm{Ind}(\mathbf{A}) = 1$, the Drazin-star and star-Drazin matrices reduce to *the group-star matrix*, $\mathbf{A}^{\#,*} = \mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{*}$, and *the star-group matrix*, $\mathbf{A}^{*,\#} = \mathbf{A}^{*}\mathbf{A}\mathbf{A}^{\#}$, respectively.

Recently, the topic of the Drazin-star and star-Drazin matrices has found its development in [46] by the complex rectangular W-weighted Drazin-star matrix, in [28, 27] by operators on Hilbert spaces, and in [37] by Drazin-theta and theta-Drazin matrices.

Due to the important role of generalized inverses in many application fields, significant efforts have been made toward numerical algorithms for their efficient and accurate computation. Most existing methods for calculating complex generalized inverses are iterative algorithms for approximating generalized inverses [1, 35]. There are only several direct methods for finding generalized inverse. One of the direct methods is constructing its determinantal representation ($\mathfrak{D}$-representation shortly). The $\mathfrak{D}$-representation of the ordinary inverse as the matrix with cofactors in entries inducts the well-known Cramer rule for solving systems of equations. However, constructing of $\mathfrak{D}$-representations of generalized inverses is not as obvious and unambiguous, even for matrices with complex or real entries. In the search for more applicable explicit expressions, there are various widespread $\mathfrak{D}$-representations of generalized inverses of matrices over complex numbers [6, 22, 36, 39, 31, 40], integral domains [5, 4, 29, 44, 43], and the Riemannian space [38, 41]. The task of $\mathfrak{D}$-representing quaternion generalized inverses is more complicated than the complex case, due to the non-commutativity of quaternions. Difficulties arise in defining the determinant with noncommutative entries, known as a noncommutative determinant (see survey articles [2, 8, 45] for details). In this paper, we utilize the theory of row-column noncommutative determinants recently developed in [11, 17] to derive $\mathfrak{D}$-representations of the Drazin-star and star-Drazin matrices. We also rely on representations of quaternion Drazin inverses previously obtained using the limit-rank method. This method has also been used to derive new determinantal representations of various complex generalized inverses [10, 16]. As a result, we provide new determinantal representations of quaternion Drazin-star matrices and establish corresponding Cramer's rules for complex matrix equations. The primary focus of our research is the study of two-sided quaternion matrix equations (TQME) of the form $\mathbf{AXB} = \mathbf{C}$. This equation, a special case of the Sylvester equation, has wide-ranging applications in fields such as image and signal processing [33], photogrammetry [32], etc. It is known that the unique best approximate solution to this equation is $\mathbf{X} = \mathbf{A}^{\dagger}\mathbf{C}\mathbf{B}^{\dagger}$. In this paper, we will study all possible two-sided quaternion matrix equations with restrictions that are uniquely determined solutions based on the Drazin-star and star-Drazin matrices. Based on obtained determinantal representations, these equations are solved by Cramer's rules in both cases for quaternion and complex matrix equations. This paper is a continuation of a number of research studies [20, 18, 19, 21] dedicated to the study of two-sided quaternion matrix equations with restrictions uniquely determined by various generalized inverses and focused on solving these equations using Cramer's rules by row-column determinants.

By bold capital letters we denote quaternion matrices, while capital letters are used for complex matrices. As usual, $\mathbb{C}^{m\times n}$ denotes the set of $m \times n$ complex matrices. We use the next notation

$$\mathbb{H}^{(m)(k)} = \left\{ \mathbf{A} \in \mathbb{H}^{m\times m} \mid \ k = \mathrm{Ind}(\mathbf{A}) \right\},$$

and

$$\mathbf{A} \in \mathcal{O}_{\subset}(\mathbf{B}, \mathbf{C}) \Longleftrightarrow \mathcal{C}_r(\mathbf{A}) \subset \mathcal{C}_r(\mathbf{B}),\ \mathcal{R}_l(\mathbf{A}) \subset \mathcal{R}_l(\mathbf{C}),$$
$$(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|l)} \Longleftrightarrow \mathbf{A} \in \mathbb{H}^{(m)(k)}, \mathbf{B} \in \mathbb{H}^{(n)(l)}.$$

The main research streams of this paper are briefly described as follows.

1. Determinantal representations of the Drazin-star and star-Drazin matrices for quaternion and complex matrices are presented.
2. When $\mathbf{C} \in \mathbb{H}^{m\times n}$, $\mathbf{A} \in \mathbb{H}^{(m)(k)}$ and $\mathbf{B} \in \mathbb{H}^{(n)(q)}$, we prove solvability of the quaternion restricted matrix equation (or shortly Q-RME):

$$\mathbf{A}^k\mathbf{X}\mathbf{B}^q = \mathbf{A}^k\mathbf{A}^*\mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q) \tag{1.3}$$

   and show that equation (1.3) possesses a uniquely determined solution based on the Drazin-star matrix of $\mathbf{A}$ and the star-Drazin matrix of $\mathbf{B}$.
3. Particular kinds of equation (1.3) are studied when $\mathbf{A} = \mathbf{I}_m$ or $\mathbf{B} = \mathbf{I}_n$ or $\mathbf{A}$ and $\mathbf{B}$ are partial isometries.
4. When $\mathbf{C} \in \mathbb{H}^{m\times n}$, $\mathbf{A} \in \mathbb{H}^{(m)(k)}$ and $\mathbf{B} \in \mathbb{H}^{(n)(q)}$, we verify solvability of the Q-RME

$$(\mathbf{A}^\dagger)^*\mathbf{X}(\mathbf{B}^\dagger)^* = \mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D}}\mathbf{B}, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^*, \mathbf{B}^*) \tag{1.4}$$

   and express its unique solution using the star-Drazin matrix of $\mathbf{A}$ and the Drazin-star matrix of $\mathbf{B}$.
5. Special types of (1.4) are considered.
6. Several more Q-RMEs are solved based on Drazin-star matrices of $\mathbf{A}$ and $\mathbf{B}$ or star-Drazin matrices of $\mathbf{A}$ and $\mathbf{B}$.
7. Cramer's rules for obtained solutions to above Q-RMEs are given.
8. An illustrative example illustrates the obtained results.

The remainder of our article is directed as follows. $\mathfrak{D}$-representations of the quaternion Drazin-star and star-Drazin matrices are derived in Section 2. Section 3 investigates the solvability of QRMEs of the form (1.3) and (1.4) and their special cases. Cramer's rule for considered solutions is derived in Section 4. A numerical example is given in Section 5 to illustrate gained results. Concluding comments are stated in Section 6.

2. DETERMINANTAL REPRESENTATIONS OF THE QUATERNION DRAZIN-STAR AND STAR-DRAZIN MATRICES

By the theory of row-column determinants, for $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ there is a method to produce n row ($\mathfrak{R}$-)determinants and n column ($\mathfrak{C}$-)determinants by stating a certain order of factors in each term.

- *The ith $\mathfrak{R}$-determinant* of $\mathbf{A}$, for an arbitrary row index $i \in I_n = \{1, \dots, n\}$, is given by

$$\mathrm{rdet}_i\mathbf{A} := \sum_{\sigma\in S_n} (-1)^{n-r} \left(a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i}\right) \dots \left(a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}}\right),$$

  whereat S_n denotes the symmetric group on I_n, while the permutation σ is defined as a product of mutually disjunct subsets ordered from the left to right by the rules

$$\sigma = \left(i\, i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}\right)\left(i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}\right) \dots \left(i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}\right),$$
$$i_{k_t} < i_{k_t+s},\ i_{k_2} < i_{k_3} < \dots < i_{k_r},\ \forall\, t = 2, \dots, r,\ s = 1, \dots, l_t.$$

- For an arbitrary column index $j \in I_n$, *the jth $\mathfrak{C}$-determinant* of $\mathbf{A}$ is defined as the sum

$$\mathrm{cdet}_j\mathbf{A} = \sum_{\tau\in S_n} (-1)^{n-r}(a_{j_{k_r} j_{k_r+l_r}} \cdots a_{j_{k_r+1} j_{k_r}}) \cdots (a_{j\, j_{k_1+l_1}} \cdots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}),$$

in which a permutation τ is ordered from the right to left in the following way:

$$\tau = (j_{k_r+l_r} \cdots j_{k_r+1} j_{k_r}) \cdots (j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2}) \ (j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j),$$
$$j_{k_t} < j_{k_t+s},\ j_{k_2} < j_{k_3} < \cdots < j_{k_r}.$$

Due to the non-commutativity of quaternions, all $\mathfrak{R}$- and $\mathfrak{C}$-determinants are generally different. However, the following equalities are verified for a Hermitian matrix $\mathbf{A}$ in [11]:

$$\mathrm{rdet}_1\mathbf{A} = \cdots = \mathrm{rdet}_n\mathbf{A} = \mathrm{cdet}_1\mathbf{A} = \cdots = \mathrm{cdet}_n\mathbf{A} = \alpha \in \mathbb{R}.$$

It allows us to define the unique determinant of a Hermitian matrix $\mathbf{A}$ by putting $\det\mathbf{A} = \alpha$. We also will use the denotation $|\mathbf{A}| := \det\mathbf{A}$. For more details on quaternion column-row determinants see [17].

The next symbols related to $\mathfrak{D}$-representations will be used. The ith row and jth column of $\mathbf{A}$ are marked with $\mathbf{a}_{i.}$ and $\mathbf{a}_{.j}$, respectively. Let $\mathbf{A}_{.j}(\mathbf{c})$ (resp. $\mathbf{A}_{i.}(\mathbf{b})$) mean the matrices formed by replacing jth column (resp. ith row) of $\mathbf{A}$ by the column vector $\mathbf{c}$ (resp. by the row vector $\mathbf{b}$). Suppose $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ are subsets with $1 \leq k \leq \min\{m, n\}$. For $\mathbf{A} \in \mathbb{H}^{m\times n}$, the notation $\mathbf{A}^{\alpha}_{\beta}$ stands for a submatrix with rows and columns indexed by α and β, respectively. Further, $\mathbf{A}^{\alpha}_{\alpha}$ and $|\mathbf{A}|^{\alpha}_{\alpha}$ denote a principal submatrix and a principal minor of Hermitian $\mathbf{A} \in \mathbb{H}^{n\times n}$, respectively. The standard notation

$$L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k),\ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$$

will mean the set of strictly increasing sequences of $k \in \{1, \ldots, n\}$ integers elected from $\{1, \ldots, n\}$. In this respect, we put

$$I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\},\ J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$$

for some fixed $i \in \alpha$ and $j \in \beta$.

Denote by $\mathbf{a}^{(m)}_{.j}$ and $\mathbf{a}^{(m)}_{i.}$ the jth column and the ith row of $\mathbf{A}^m$, and by $\hat{\mathbf{a}}_{.s}$ and $\check{\mathbf{a}}_{t.}$ the sth column of $(\mathbf{A}^{2k+1})^*\mathbf{A}^k =: \hat{\mathbf{A}} = (\hat{a}_{ij}) \in \mathbb{H}^{n\times n}$ and the tth row of $\mathbf{A}^k(\mathbf{A}^{2k+1})^* =: \check{\mathbf{A}} = (\check{a}_{ij}) \in \mathbb{H}^{n\times n}$, respectively, for all $s, t = 1, \ldots .n$. The next lemmas give $\mathfrak{D}$-representations of the Drazin inverse over the quaternion skew field.

Lemma 2.1. [13] *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *with* $\mathrm{rank}(\mathbf{A}^{\mathrm{k}}) = r$, *then the Drazin inverse* A^{D} *possesses the determinantal representations*

$$a^{\mathrm{D}}_{ij} = \frac{\hat{\phi}_{ij}}{\sum\limits_{\beta \in J_{r,\,n}} |(\mathbf{A}^{2k+1})^*\,\mathbf{A}^{2k+1}|^{\beta}_{\beta}} \tag{2.1}$$

$$= \frac{\hat{\psi}_{ij}}{\sum\limits_{\alpha \in I_{r,\,n}} |\mathbf{A}^{2k+1}\,(\mathbf{A}^{2k+1})^*|^{\alpha}_{\alpha}}, \tag{2.2}$$

where $\hat{\mathbf{\Phi}} = \left(\hat{\phi}_{ij}\right) = \mathbf{A}^k\mathbf{\Phi}$ *and* $\hat{\mathbf{\Psi}} = \left(\hat{\psi}_{ij}\right) = \mathbf{\Psi}\mathbf{A}^k$. *The matrices* $\mathbf{\Phi} = \left(\phi_{tj}\right)$ *and* $\mathbf{\Psi} = (\psi_{is})$ *are determined by*

$$\phi_{tj} = \sum_{\beta \in J_{r,\,n}\{t\}} \mathrm{cdet}_t\left(\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{.t}(\hat{\mathbf{a}}_{.j})\right)^{\beta}_{\beta}, \tag{2.3}$$

$$\psi_{is} = \sum_{\alpha \in I_{r,\,n}\{s\}} \mathrm{rdet}_s\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{s.}(\check{\mathbf{a}}_{i.})\right)^{\alpha}_{\alpha}. \tag{2.4}$$

In the special case when $\mathbf{A} \in \mathbb{H}^{n\times n}$ is Hermitian, we can obtain simpler determinantal representations of the Drazin inverse.

Lemma 2.2. [13] *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *is Hermitian with* $\mathrm{rank}(\mathbf{A}^{\mathrm{k}}) = r$, *then the Drazin inverse* $\mathbf{A}^{\mathrm{D}} = \left(a_{ij}^{\mathrm{D}}\right)$ *is represented as follows*

$$a_{ij}^{\mathrm{D}} = \frac{\sum\limits_{\beta \in J_{r,n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i} \left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,n}} |\mathbf{A}^{k+1}|_{\beta}^{\beta}} \tag{2.5}$$

$$= \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}^{k+1})_{j.} (\mathbf{a}_{i.}^{(k)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,n}} |\mathbf{A}^{k+1}|_{\alpha}^{\alpha}}. \tag{2.6}$$

The corresponding statement is valid in the case of complex matrices.

Lemma 2.3. [10] *If* $A \in \mathbb{C}^{(n)(k)}$ *with* $\mathrm{rank}(A^k) = r$, *then the Drazin inverse* $A^{\mathrm{D}} = \left(a_{ij}^{\mathrm{D}}\right)$ *by componentwise can be represented as follows*

$$a_{ij}^{\mathrm{D}} = \frac{\sum\limits_{\beta \in J_{r,n}\{i\}} \left| \left(A^{k+1}\right)_{.i} \left(a_{.j}^{(k)}\right)\right|_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,n}} |A^{k+1}|_{\beta}^{\beta}} \tag{2.7}$$

$$= \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \left| (A^{k+1})_{j.} (a_{i.}^{(k)})\right|_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,n}} |A^{k+1}|_{\alpha}^{\alpha}}. \tag{2.8}$$

Now, we derive determinantal representations of the Drazin-star and star-Drazin matrices for quaternion and complex matrices. $\mathfrak{D}$-representations of various generalized inverses expressed in terms of the $\mathfrak{R}$- and $\mathfrak{C}$-determinants can be found in [12, 15, 14].

Theorem 2.4. *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *with* $\mathrm{rank}(\mathbf{A}^{\mathrm{k}}) = r$, *then the Drazin-star matrix* $\mathbf{A}^{\mathrm{D},*} = \left(a_{ij}^{\mathrm{D},*}\right)$ *possesses the determinantal representation*

$$a_{ij}^{\mathrm{D},*} = \frac{\sum\limits_{t=1}^{n} a_{it}^{(k)} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \mathrm{cdet}_t \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1}\right)_{.t} (\tilde{\mathbf{a}}_{.j})\right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} |\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1}|_{\beta}^{\beta}}, \tag{2.9}$$

where $\tilde{\mathbf{a}}_{.j}$ *is the* j*th column of* $\tilde{\mathbf{A}} = (\mathbf{A}^{2k+1})^* \mathbf{A}^{k+1} \mathbf{A}^*$.

Proof. By (1.1),

$$a_{ij}^{\mathrm{D},*} = \sum_{l=1}^{n} \sum_{m=1}^{n} a_{il}^{\mathrm{D}} a_{lm} a_{mj}^*.$$

Suppose that $\mathbf{A}$ is not Hermitian. Then $\mathbf{A}^{\mathrm{D}} = (a_{ij}^{\mathrm{D}})$ is $\mathfrak{D}$-presentable by (2.1). Therefore,

$$a_{ij}^{\mathrm{D},*} = \frac{\sum\limits_{l=1}^{n} \sum\limits_{m=1}^{n} \hat{\phi}_{il} a_{lm} a_{mj}^*}{\sum\limits_{\beta \in J_{r,\,n}} |\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1}|_{\beta}^{\beta}},$$

where by (2.3)

$$\sum_{l=1}^{n}\sum_{m=1}^{n}\hat{\phi}_{il}a_{lm}a^{*}_{mj}=\sum_{t=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}a^{(k)}_{it}\sum_{\beta\in J_{r,\,n}\{t\}}\operatorname{cdet}_t\left(\left(\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right)_{.t}(\hat{\mathbf{a}}_{.l})\right)^{\beta}_{\beta}a_{lm}a^{*}_{mj}.$$

with $\hat{\mathbf{a}}_{.l}$ standing for the lth column of $\hat{\mathbf{A}}=(\mathbf{A}^{2k+1})^{*}\mathbf{A}^{k}$. Denote $\tilde{\mathbf{A}}=(\mathbf{A}^{2k+1})^{*}\mathbf{A}^{k+1}\mathbf{A}^{*}$. Since

$$\sum_{l=1}^{n}\sum_{m=1}^{n}\hat{\mathbf{a}}_{.l}a_{lm}a^{*}_{mj}=\tilde{\mathbf{a}}_{.j},$$

then (2.9) follows. ∎

Corollary 1. *If $\mathbf{A}\in\mathbb{H}^{(n)(k)}$ with $\operatorname{rank}(\mathbf{A}^{k})=1$, then the Drazin-star matrix $\mathbf{A}^{\mathrm{D},*}=\left(a^{\mathrm{D},*}_{ij}\right)$ can be componentwise expressed by*

$$a^{\mathrm{D},*}_{ij}=\frac{\tilde{\phi}_{ij}}{\operatorname{tr}\left(\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right)},\tag{2.10}$$

where $\tilde{\mathbf{\Phi}}=\left(\tilde{\phi}_{ij}\right)=\mathbf{A}^{k}(\mathbf{A}^{2k+1})^{*}\mathbf{A}^{k+1}\mathbf{A}^{*}$.

Proof. If $\operatorname{rank}(\mathbf{A}^{k+1})=\operatorname{rank}(\mathbf{A}^{k})=1$, then

$$\phi_{tl}=\sum_{\beta\in J_{r,\,n}\{t\}}\operatorname{cdet}_t\left(\left(\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right)_{.t}(\hat{\mathbf{a}}_{.l})\right)^{\beta}_{\beta}=\hat{a}_{tl}=\sum_{s=1}^{n}\left(a^{(2k+1)}_{ts}\right)^{*}a^{(k)}_{sl}$$

and

$$\begin{aligned}\tilde{\phi}_{ij}&:=\sum_{l}\sum_{m}\hat{\phi}_{il}a_{lm}a^{*}_{mj}=\sum_{l}\sum_{m}\sum_{t}a^{(k)}_{it}\phi_{tl}a_{lm}a^{*}_{mj}=\\&=\sum_{l}\sum_{m}\sum_{t}\sum_{s}a^{(k)}_{it}\left(a^{(2k+1)}_{ts}\right)^{*}a^{(k)}_{sl}a_{lm}a^{*}_{mj}=\sum_{m}\sum_{t}\sum_{s}a^{(k)}_{it}\left(a^{(2k+1)}_{ts}\right)^{*}a^{(k+1)}_{sm}a^{*}_{mj}.\end{aligned}$$

Hence, $\tilde{\mathbf{\Phi}}=\left(\tilde{\phi}_{ij}\right)=\mathbf{A}^{k}(\mathbf{A}^{2k+1})^{*}\mathbf{A}^{k+1}\mathbf{A}^{*}$. Since

$$\sum_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right|^{\beta}_{\beta}=\operatorname{tr}\left(\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right),$$

which implies (2.10). ∎

The $\mathfrak{D}$-representation of the complex Drazin-star matrix have its own features.

Theorem 2.5. *If $A\in\mathbb{C}^{(n)(k)}$ with $\operatorname{rank}(A^{k})=r$, then the Drazin-star matrix $A^{\mathrm{D},*}=\left(a^{\mathrm{D},*}_{ij}\right)$ possess the determinantal representation*

$$a^{\mathrm{D},*}_{ij}=\frac{\sum\limits_{\beta\in J_{r,n}\{i\}}\left|\left(A^{k+1}\right)_{.i}(\tilde{a}_{.j})\right|^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,n}}\left|A^{k+1}\right|^{\beta}_{\beta}},\tag{2.11}$$

where $\tilde{a}_{.j}$ is the jth column of $\tilde{A}=A^{k+1}A^{*}$.

Proof. By (1.1),

$$a_{ij}^{\mathrm{D},*}=\sum_{l=1}^{n}\sum_{m=1}^{n}a_{il}^{\mathrm{D}}a_{lm}a_{mj}^{*}.$$

The $\mathfrak{D}$-representation of the Drazin inverse $A^{\mathrm{D}}=(a_{ij}^{\mathrm{D}})$ is obtained by Lemma 2.3. We use the $\mathfrak{D}$-representation (2.7) of A^{D}. Then,

$$a_{ij}^{\mathrm{D},*}=\frac{\sum\limits_{l=1}^{n}\sum\limits_{m=1}^{n}\sum\limits_{\beta\in J_{r,n}\{i\}}\left|\left(A^{k+1}\right)_{.i}\left(a_{.l}^{(k)}\right)\right|_{\beta}^{\beta}a_{lm}a_{mj}^{*}}{\sum\limits_{\beta\in J_{r,\,n}}|A^{k+1}|_{\beta}^{\beta}}.$$

Denote $\tilde{A}=A^{k+1}A^{*}$. Since $\sum\limits_{l=1}^{n}\sum\limits_{m=1}^{n}a_{.l}^{(k)}a_{lm}a_{mj}^{*}=\tilde{a}_{.j}$, then we have (2.11). ∎

Now, we derive determinantal representations of the star-Drazin matrix.

Theorem 2.6. *If* $\mathbf{A}\in\mathbb{H}^{(n)(k)}$ *with* $\operatorname{rank}(\mathbf{A}^{k})=r$*, then the star-Drazin matrix* $\mathbf{A}^{*,\mathrm{D}}=\left(a_{ij}^{*,\mathrm{D}}\right)$ *possesses the determinantal representation*

$$a_{ij}^{*,\mathrm{D}}=\frac{\sum\limits_{t=1}^{n}\sum\limits_{\alpha\in I_{r,\,n}\{t\}}\mathrm{rdet}_{t}\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right)_{t.}(\bar{\mathbf{a}}_{i.})\right)_{\alpha}^{\alpha}a_{tj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}}|\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}|_{\alpha}^{\alpha}},\tag{2.12}$$

where $\bar{\mathbf{a}}_{i.}$ *is the* i*th row of* $\bar{\mathbf{A}}=\mathbf{A}^{*}\mathbf{A}^{k+1}(\mathbf{A}^{2k+1})^{*}$.

Proof. By (1.2),

$$a_{ij}^{*,\mathrm{D}}=\sum_{l=1}^{n}\sum_{m=1}^{n}a_{il}^{*}a_{lm}a_{mj}^{\mathrm{D}}.$$

Suppose that the matrix $\mathbf{A}$ is not Hermitian. Then the $\mathfrak{D}$-representation of the Drazin inverse $\mathbf{A}^{\mathrm{D}}=(a_{ij}^{\mathrm{D}})$ is obtained by Lemma 2.1. We use the $\mathfrak{D}$-representation (2.2) of $\mathbf{A}^{\mathrm{D}}$. Then,

$$a_{ij}^{*,\mathrm{D}}=\frac{\sum\limits_{l=1}^{n}\sum\limits_{m=1}^{n}a_{il}^{*}a_{lm}\hat{\psi}_{mj}}{\sum\limits_{\alpha\in I_{r,\,n}}|\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}|_{\alpha}^{\alpha}},$$

where by (2.4)

$$\sum_{l=1}^{n}\sum_{m=1}^{n}a_{il}^{*}a_{lm}\hat{\psi}_{mj}=\sum_{t=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}a_{il}^{*}a_{lm}\sum_{\alpha\in I_{r,\,n}\{t\}}\mathrm{rdet}_{t}\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right)_{t.}(\check{\mathbf{a}}_{m.})\right)_{\alpha}^{\alpha}a_{tj}^{(k)}.$$

with $\check{\mathbf{a}}_{m.}$ standing for the mth row of $\check{\mathbf{A}}=\mathbf{A}^{k}(\mathbf{A}^{2k+1})^{*}$. Denote $\bar{\mathbf{A}}=\mathbf{A}^{*}\mathbf{A}^{k+1}(\mathbf{A}^{2k+1})^{*}$. Since

$$\sum_{l=1}^{n}\sum_{m=1}^{n}a_{il}^{*}a_{lm}\check{\mathbf{a}}_{m.}=\bar{\mathbf{a}}_{i.},$$

then it follows (2.12). ∎

The following corollary can be proven similarly to Corollary 1.

Corollary 2. *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *and* $\operatorname{rank}(\mathbf{A}^k) = 1$*, then the star-Drazin matrix* $\mathbf{A}^{*,\mathrm{D}} = \left(a_{ij}^{*,\mathrm{D}}\right)$ *can be componentwise expressed as*

$$a_{ij}^{*,\mathrm{D}} = \frac{\tilde{\psi}_{ij}}{\operatorname{tr}\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)}, \tag{2.13}$$

where $\tilde{\mathbf{\Psi}} = \left(\tilde{\psi}_{ij}\right) = \mathbf{A}^*\mathbf{A}^{k+1}(\mathbf{A}^{2k+1})^*\mathbf{A}^k$.

If $\mathbf{A} \in \mathbb{H}^{n\times n}$ is Hermitian, then

$$\begin{aligned} \mathbf{A}^{\mathrm{D},*} &= \mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{A}^* = \mathbf{A}^{\mathrm{D}}\mathbf{A}^2 = \mathbf{A}^2\mathbf{A}^{\mathrm{D}} = \mathbf{A}^{*,\mathrm{D}}, \text{ when } \operatorname{Ind}\mathbf{A} = k \geq 2, \\ \mathbf{A}^{*,\#} &= \mathbf{A}^{\#,*} = \mathbf{A}^{\#}\mathbf{A}\mathbf{A}^* = \mathbf{A}^{\#}\mathbf{A}^2 = \mathbf{A}, \text{ when } \operatorname{Ind}\mathbf{A} = k < 2. \end{aligned} \tag{2.14}$$

Corollary 3. *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *is Hermitian with* $k \geq 2$ *and* $\operatorname{rank}(\mathbf{A}^k) = r$*, then its corresponding Drazin-star and star-Drazin matrices coincide and*

$$a_{ij}^{\mathrm{D},*} = a_{ij}^{*,\mathrm{D}} = \frac{\sum\limits_{\beta\in J_{r,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k+2)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,n}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta}} \tag{2.15}$$

$$= \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}^{k+1})_{j.}(\mathbf{a}_{i.}^{(k+2)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} \left|\mathbf{A}^{k+1}\right|_{\alpha}^{\alpha}}. \tag{2.16}$$

Proof. By (2.14),

$$a_{ij}^{\mathrm{D},*} = a_{ij}^{*,\mathrm{D}} = \sum_{l=1}^{n} a_{il}^{\mathrm{D}} a_{lj}^{(2)}.$$

Using (2.5) for the $\mathfrak{D}$-representation of $\mathbf{A}^{\mathrm{D}} = \left(a_{il}^{\mathrm{D}}\right)$, it can be derived

$$a_{ij}^{\mathrm{D},*} = a_{ij}^{*,\mathrm{D}} = \frac{\sum\limits_{l=1}^{n}\sum\limits_{\beta\in J_{r,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.l}^{(k)}\right)\right)_{\beta}^{\beta} a_{lj}^{(2)}}{\sum\limits_{\beta\in J_{r,n}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta}}.$$

From $\sum\limits_{l=1}^{n} \mathbf{a}_{.l}^{(k)} a_{lj}^{(2)} = \mathbf{a}_{.l}^{(k+2)}$, it follows (2.15).

Using (2.6) for the $\mathfrak{D}$-representation of $\mathbf{A}^{\mathrm{D}} = \left(a_{il}^{\mathrm{D}}\right)$,

$$a_{ij}^{\mathrm{D},*} = a_{ij}^{*,\mathrm{D}} = \frac{\sum\limits_{l=1}^{n} a_{il}^{(2)} \sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}^{k+1})_{j.}(\mathbf{a}_{l.}^{(k)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} \left|\mathbf{A}^{k+1}\right|_{\alpha}^{\alpha}}.$$

Because of $\sum\limits_{l=1}^{n} a_{il}^{(2)} \mathbf{a}_{l.}^{(k)} = \mathbf{a}_{i.}^{(k+2)}$, from this it follows (2.16). ∎

The $\mathfrak{D}$-representation of the complex star-Drazin matrix can derived with a similar procedure as in Theorem 2.5.

Theorem 2.7. *If $A \in \mathbb{C}^{(n)(k)}$ and $\operatorname{rank}(A^k) = r$, then the star-Drazin matrix $A^{*,\mathrm{D}} = \left(a_{ij}^{*,\mathrm{D}}\right)$ componentwise can be expressed as*

$$a_{ij}^{*,\mathrm{D}} = \frac{\sum\limits_{\alpha \in I_{r,n}\{j\}} \left| \left(A^{k+1}\right)_{j.} (\bar{a}_{i.}) \right|_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,n}} \left|A^{k+1}\right|_{\alpha}^{\alpha}}, \tag{2.17}$$

where $\bar{a}_{i.}$ is the ith row of $\bar{A} = A^* A^{k+1}$.

3. DRAZIN-STAR–STAR-DRAZIN SOLUTIONS TO Q-RMES

This section is devoted to the solvability of Q-RMEs (1.3)-(1.4) as well as their special types.

Theorem 3.1. *The Q-RME* (1.3) *is uniquely solvable by*

$$\mathbf{X} = \mathbf{A}^{\mathrm{D},*}\mathbf{C}\mathbf{B}^{*,\mathrm{D}}. \tag{3.1}$$

Proof. Recall that $\mathcal{C}_r(\mathbf{A}^{\mathrm{D}}) = \mathcal{C}_r(\mathbf{A}^k)$ and $\mathcal{R}_l(\mathbf{B}^{\mathrm{D}}) = \mathcal{R}_l(\mathbf{B}^q)$. Since $\mathbf{X} = \mathbf{A}^{\mathrm{D},*}\mathbf{C}\mathbf{B}^{*,\mathrm{D}}$ satisfies

$$\mathbf{X} = \mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{A}^*\mathbf{C}\mathbf{B}^*\mathbf{B}\mathbf{B}^{\mathrm{D}} \in \mathcal{O}_{\subset}(\mathbf{A}^{\mathrm{D}}, \mathbf{B}^{\mathrm{D}}) = \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q)$$

and

$$\begin{aligned} \mathbf{A}^k\mathbf{X}\mathbf{B}^q &= (\mathbf{A}^k\mathbf{A}^{\mathrm{D}}\mathbf{A})\mathbf{A}^*\mathbf{C}\mathbf{B}^*(\mathbf{B}\mathbf{B}^{\mathrm{D}}\mathbf{B}^q) \\ &= \mathbf{A}^k\mathbf{A}^*\mathbf{C}\mathbf{B}^*\mathbf{B}^q, \end{aligned}$$

we conclude that (1.3) has uniquely determined solution $\mathbf{X} = \mathbf{A}^{\mathrm{D},*}\mathbf{C}\mathbf{B}^{*,\mathrm{D}}$.

For two solutions $\mathbf{X}_1$ and $\mathbf{X}$ of (1.3), notice that $\mathbf{A}^k(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^q = \mathbf{0}$, $\mathcal{C}_r(\mathbf{X}_1) \subset \mathcal{C}_r(\mathbf{A}^k)$ and $\mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k)$ give

$$(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^q \in \mathcal{N}_r(\mathbf{A}^k) \cap \mathcal{C}_r(\mathbf{A}^k) = \{\mathbf{0}\}.$$

Then $\mathcal{R}_l(\mathbf{X}_1) \subset \mathcal{R}_l(\mathbf{B}^q)$, $\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q)$ and $(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^q = \mathbf{0}$ yield

$$\mathbf{X}_1 - \mathbf{X} \in \mathcal{N}_l(\mathbf{B}^q) \cap \mathcal{R}_l(\mathbf{B}^q) = \{\mathbf{0}\}.$$

So, (3.1) represents the unique solution to (1.3). ∎

Theorem 3.1 implies the next result in the case that $\mathbf{A} = \mathbf{I}_m$ or $\mathbf{B} = \mathbf{I}_n$.

Corollary 4. *Let $\mathbf{C} \in \mathbb{H}^{m\times n}$.*

(a) *If $\mathbf{A} \in \mathbb{H}^{(m)(k)}$, then*

$$\mathbf{X} = \mathbf{A}^{\mathrm{D},*}\mathbf{C} \tag{3.2}$$

is unique solution to

$$\mathbf{A}^k\mathbf{X} = \mathbf{A}^k\mathbf{A}^*\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k). \tag{3.3}$$

(b) *If $\mathbf{B} \in \mathbb{H}^{(n)(q)}$, then*

$$\mathbf{X} = \mathbf{C}\mathbf{B}^{*,\mathrm{D}} \tag{3.4}$$

is unique solution to

$$\mathbf{X}\mathbf{B}^q = \mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q). \tag{3.5}$$

Under additional assumptions on $\mathbf{C}$, we solve the following Q-RMEs when $\mathbf{A}$ and $\mathbf{B}$ are partial isometries.

Corollary 5. *Let $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ and $\mathbf{C} \in \mathbb{H}^{m\times n}$.*

(a) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are partial isometries, then*

$$\mathbf{X} = \mathbf{A}^{\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D}} \tag{3.6}$$

is unique solution to

$$\mathbf{A}^k\mathbf{X}\mathbf{B}^q = \mathbf{A}^k\mathbf{A}^*\mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q), \quad \mathbf{C} \in \mathcal{O}_{\subset}(\mathbf{A}, \mathbf{B}).$$

(b) *If* $\mathbf{A}$ *is a partial isometry, then* $\mathbf{X} = \mathbf{A}^{\mathrm{D}}\mathbf{C}$ *is unique solution to*

$$\mathbf{A}^k\mathbf{X} = \mathbf{A}^k\mathbf{A}^*\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k), \quad \mathcal{C}_r(\mathbf{C}) \subset \mathcal{C}_r(\mathbf{A}).$$

(c) *If* $\mathbf{B}$ *is a partial isometry, then* $\mathbf{X} = \mathbf{C}\mathbf{B}^{\mathrm{D}}$ *is unique solution to*

$$\mathbf{X}\mathbf{B}^q = \mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q), \quad \mathcal{R}_l(\mathbf{C}) \subset \mathcal{R}_l(\mathbf{B}).$$

The Q-RME (1.4) is solvable in terms of the star-Drazin matrix of $\mathbf{A}$ and the Drazin-star matrix of $\mathbf{B}$.

Theorem 3.2. *The Q-RME* (1.4) *has the unique solution presented by*

$$\mathbf{X} = \mathbf{A}^{*,\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D},*}. \tag{3.7}$$

Proof. We observe that $\mathbf{X} = \mathbf{A}^{*,\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D},*} = \mathbf{A}^*\mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D}}\mathbf{B}\mathbf{B}^* \in \mathcal{O}_{\subset}(\mathbf{A}^*, \mathbf{B}^*)$ and

$$(\mathbf{A}^\dagger)^*\mathbf{X}(\mathbf{B}^\dagger)^* = (\mathbf{A}^\dagger)^*\mathbf{A}^*\mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D}}\mathbf{B}\mathbf{B}^*(\mathbf{B}^\dagger)^* = \mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D}}\mathbf{B},$$

i.e. the Q-RME (1.4) has a solution of the form (3.7).

In the case if (1.4) has two solutions $\mathbf{X}_1$ and $\mathbf{X}$, from $(\mathbf{A}^\dagger)^*(\mathbf{X}_1 - \mathbf{X})(\mathbf{B}^\dagger)^* = \mathbf{0}$, $\mathcal{C}_r(\mathbf{X}_1) \subset \mathcal{C}_r(\mathbf{A}^*)$ and $\mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^*)$, we deduce that

$$(\mathbf{X}_1 - \mathbf{X})(\mathbf{B}^\dagger)^* \in \mathcal{N}_r((\mathbf{A}^\dagger)^*) \cap \mathcal{C}_r(\mathbf{A}^*) = \mathcal{N}_r(\mathbf{A}) \cap \mathcal{C}_r(\mathbf{A}^*) = \{\mathbf{0}\}.$$

Because $\mathcal{R}_l(\mathbf{X}_1) \subset \mathcal{R}_l(\mathbf{B}^*)$, $\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^*)$ and $(\mathbf{X}_1 - \mathbf{X})(\mathbf{B}^\dagger)^* = \mathbf{0}$, then

$$\mathbf{X}_1 - \mathbf{X} \in \mathcal{N}_l((\mathbf{B}^\dagger)^*) \cap \mathcal{R}_l(\mathbf{B}^*) = \mathcal{N}_l(\mathbf{B}) \cap \mathcal{R}_l(\mathbf{B}^*) = \{\mathbf{0}\}$$

implies that (1.4) has the unique solution given by (3.7). ∎

Under additional restriction $\mathbf{C} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q)$ for the equation (3.7), we obtain the following consequence of Theorem 3.2.

Corollary 6. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ *and* $\mathbf{C} \in \mathbb{H}^{m\times n}$. *Then the Q-RME*

$$(\mathbf{A}^\dagger)^*\mathbf{X}(\mathbf{B}^\dagger)^* = \mathbf{C}, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^*, \mathbf{B}^*), \quad \mathbf{C} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q)$$

has unique solution presented by

$$\mathbf{X} = \mathbf{A}^*\mathbf{C}\mathbf{B}^*.$$

Proof. The hypothesis $\mathbf{C} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q)$ yields $\mathbf{C} = \mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{C} = \mathbf{C}\mathbf{B}\mathbf{B}^{\mathrm{D}}$. The rest follows by Theorem 3.2. ∎

Remark that, if $\mathbf{A}$ and $\mathbf{B}$ are partial isometries in Corollary 6, then

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{C}, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^*, \mathbf{B}^*), \quad \mathbf{C} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q)$$

has unique solution presented by $\mathbf{X} = \mathbf{A}^*\mathbf{C}\mathbf{B}^* = \mathbf{A}^\dagger\mathbf{C}\mathbf{B}^\dagger$.

The solution of one more Q-RME can be represented by (3.7).

Theorem 3.3. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ *and* $\mathbf{C} \in \mathbb{H}^{m\times n}$. *Then the Q-RME*

$$\mathbf{A}^k(\mathbf{A}^\dagger)^*\mathbf{X}(\mathbf{B}^\dagger)^*\mathbf{B}^q = \mathbf{A}^k\mathbf{C}\mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^*\mathbf{A}^k, \mathbf{B}^q\mathbf{B}^*), \tag{3.8}$$

has unique solution presented by (3.7).

Proof. By Theorem 3.2, (3.7) is a solution to (1.4), which leads to the conclusion that it is a solution to (3.8):

$$\mathbf{A}^k(\mathbf{A}^\dagger)^*\mathbf{X}(\mathbf{B}^\dagger)^*\mathbf{B}^q = \mathbf{A}^k\mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{C}\mathbf{B}^{\mathrm{D}}\mathbf{B}\mathbf{B}^q = \mathbf{A}^k\mathbf{C}\mathbf{B}^q.$$

Let $\mathbf{X}_1$ and $\mathbf{X}$ be two solutions of (3.8). Now, by $\mathbf{A}^k(\mathbf{A}^\dagger)^*(\mathbf{X}_1 - \mathbf{X})(\mathbf{B}^\dagger)^*\mathbf{B}^q = \mathbf{0}$,

$$(\mathbf{X}_1 - \mathbf{X})(\mathbf{B}^\dagger)^*\mathbf{B}^q \in \mathcal{N}_r(\mathbf{A}^k(\mathbf{A}^\dagger)^*) \cap \mathcal{C}_r(\mathbf{A}^*\mathbf{A}^k) = \mathcal{N}_r(\mathbf{A}^{*,\mathrm{D}}(\mathbf{A}^\dagger)^*) \cap \mathcal{C}_r(\mathbf{A}^{*,\mathrm{D}}(\mathbf{A}^\dagger)^*) = \{\mathbf{0}\}.$$

Further,

$$\mathbf{X}_1 - \mathbf{X} \in \mathcal{N}_l((\mathbf{B}^\dagger)^*\mathbf{B}^q) \cap \mathcal{R}_l(\mathbf{B}^q\mathbf{B}^*) = \mathcal{N}_l(\mathbf{B}^q\mathbf{B}^{\mathrm{D},*}) \cap \mathcal{R}_l(\mathbf{B}^q\mathbf{B}^{\mathrm{D},*}) = \{\mathbf{0}\},$$

that is, (3.7) is uniquely determined solution to (3.8). ∎

As a consequence of Theorem 3.2 and Theorem 3.3 for $\mathbf{A} = \mathbf{I}_m$ or $\mathbf{B} = \mathbf{I}_n$, we get solvability of the next Q-RMEs.

Corollary 7. *Let* $\mathbf{C} \in \mathbb{H}^{m\times n}$.

(a) *If* $\mathbf{A} \in \mathbb{H}^{(m)(k)}$*, then*

$$\mathbf{X} = \mathbf{A}^{*,\mathrm{D}}\mathbf{C} \tag{3.9}$$

is unique solution to

(i) $(\mathbf{A}^\dagger)^*\mathbf{X} = \mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^*)$;

(ii) $\mathbf{A}^k(\mathbf{A}^\dagger)^*\mathbf{X} = \mathbf{A}^k\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^*\mathbf{A}^k)$.

(b) *If* $(\mathbf{B}) \in \mathbb{H}^{(n)(q)}$*, then*

$$\mathbf{X} = \mathbf{C}\mathbf{B}^{\mathrm{D},*} \tag{3.10}$$

is unique solution to

(i) $\mathbf{X}(\mathbf{B}^\dagger)^* = \mathbf{C}\mathbf{B}^{\mathrm{D}}\mathbf{B}, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^*)$;

(ii) $\mathbf{X}(\mathbf{B}^\dagger)^*\mathbf{B}^q = \mathbf{C}\mathbf{B}^q, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q\mathbf{B}^*)$.

Similarly, we solve the following Q-RMEs utilizing Drazin-star matrices of $\mathbf{A}$ and $\mathbf{B}$ or by star-Drazin matrices of $\mathbf{A}$ and $\mathbf{B}$.

Theorem 3.4. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ *and* $\mathbf{C} \in \mathbb{H}^{m\times n}$.

(a) *Then*

$$\mathbf{X} = \mathbf{A}^{\mathrm{D},*}\mathbf{C}\mathbf{B}^{\mathrm{D},*} \tag{3.11}$$

is unique solution to

$$\text{(i) } \mathbf{A}^k\mathbf{X}(\mathbf{B}^\dagger)^* = \mathbf{A}^k\mathbf{A}^*\mathbf{C}\mathbf{B}^D\mathbf{B}, \quad \mathbf{X} \in \mathcal{O}_\subset(\mathbf{A}^k, \mathbf{B}^*); \tag{3.12}$$

$$\text{(ii) } \mathbf{A}^k\mathbf{X}(\mathbf{B}^\dagger)^*\mathbf{B}^q = \mathbf{A}^k\mathbf{A}^*\mathbf{C}\mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_\subset(\mathbf{A}^k, \mathbf{B}^q\mathbf{B}^*). \tag{3.13}$$

(b) *Then*

$$\mathbf{X} = \mathbf{A}^{*,\mathrm{D}}\mathbf{C}\mathbf{B}^{*,\mathrm{D}} \tag{3.14}$$

is unique solution to

$$\text{(i) } (\mathbf{A}^\dagger)^*\mathbf{X}\mathbf{B}^q = \mathbf{A}\mathbf{A}^D\mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_\subset(\mathbf{A}^*, \mathbf{B}^q); \tag{3.15}$$

$$\text{(ii) } \mathbf{A}^k(\mathbf{A}^\dagger)^*\mathbf{X}\mathbf{B}^q = \mathbf{A}^k\mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_\subset(\mathbf{A}^*\mathbf{A}^k, \mathbf{B}^q). \tag{3.16}$$

Consequently, by Theorem 3.4, we obtain solvability of several Q-RMEs as follows.

Corollary 8. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ *and* $\mathbf{C} \in \mathbb{H}^{m\times n}$.

(a) *Then*

$$\mathbf{X} = \mathbf{A}^{\mathrm{D},*}\mathbf{C} \tag{3.17}$$

is unique solution for

$$\begin{aligned}&\text{(i) } \mathbf{A}^k\mathbf{X} = \mathbf{A}^k\mathbf{A}^*\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k);\\ &\text{(ii) } \mathbf{A}^k\mathbf{X} = \mathbf{A}^k\mathbf{A}^*\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k).\end{aligned}$$

(b) *Then*

$$\mathbf{X} = \mathbf{C}\mathbf{B}^{\mathrm{D},*} \tag{3.18}$$

is unique solution for

$$\begin{aligned}&\text{(i) } \mathbf{X}(\mathbf{B}^\dagger)^* = \mathbf{C}\mathbf{B}^{\mathrm{D}}\mathbf{B}, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^*);\\ &\text{(ii) } \mathbf{X}(\mathbf{B}^\dagger)^*\mathbf{B}^q = \mathbf{C}\mathbf{B}^q, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q\mathbf{B}^*).\end{aligned}$$

(c) *Then*

$$\mathbf{X} = \mathbf{A}^{*,\mathrm{D}}\mathbf{C} \tag{3.19}$$

is unique solution for

$$\begin{aligned}&\text{(i) } (\mathbf{A}^\dagger)^*\mathbf{X} = \mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^*);\\ &\text{(ii) } \mathbf{A}^k(\mathbf{A}^\dagger)^*\mathbf{X} = \mathbf{A}^k\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^*\mathbf{A}^k).\end{aligned}$$

(d) *Then*

$$\mathbf{X} = \mathbf{C}\mathbf{B}^{*,\mathrm{D}} \tag{3.20}$$

is unique solution for

$$\begin{aligned}&\text{(i) } \mathbf{X}\mathbf{B}^q = \mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q);\\ &\text{(ii) } \mathbf{X}\mathbf{B}^q = \mathbf{C}\mathbf{B}^*\mathbf{B}^q, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q).\end{aligned}$$

4. CRAMER'S RULES TO OBTAINED SOLUTIONS

In this section, we establish Cramer's rules for QRMEs considered in Section 3.

Theorem 4.1. *Let* $\mathbf{C} \in \mathbb{H}^{m\times n}$, $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ *with* $\operatorname{rank}(\mathbf{A}^k) = r$, *and* $\operatorname{rank}(\mathbf{B}^q) = s$. *The unique solution* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{m\times n}$ *from* (3.1) *can be expressed componentwise as follows.*
(i) *If the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{t=1}^{n}\sum\limits_{\alpha\in I_{s,\,n}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^*\right)_{t.}(\widetilde{\boldsymbol{\phi}}_{i.})\right)_\alpha^\alpha b_{tj}^{(q)}}{\sum\limits_{\beta\in J_{r,\,m}} \left|\left(\mathbf{A}^{2k+1}\right)^*\mathbf{A}^{2k+1}\right|_\beta^\beta \sum\limits_{\alpha\in I_{s,\,n}} \left|\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^*\right|_\alpha^\alpha} \tag{4.1}$$

$$= \frac{\sum\limits_{l=1}^{m} a_{il}^{(k)} \sum\limits_{\beta\in J_{r,\,m}\{l\}} \operatorname{cdet}_l\left(\left(\left(\mathbf{A}^{2k+1}\right)^*\mathbf{A}^{2k+1}\right)_{.l}\left(\widetilde{\boldsymbol{\psi}}_{.j}\right)\right)_\beta^\beta}{\sum\limits_{\beta\in J_{r,\,m}} \left|\left(\mathbf{A}^{2k+1}\right)^*\mathbf{A}^{2k+1}\right|_\beta^\beta \sum\limits_{\alpha\in I_{s,\,n}} \left|\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^*\right|_\alpha^\alpha}, \tag{4.2}$$

where $\widetilde{\boldsymbol{\phi}}_{i.}$ is the ith row of $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}\mathbf{B}^*\mathbf{B}^{q+1}(\mathbf{B}^{2q+1})^*$ and $\widetilde{\boldsymbol{\psi}}_{.j}$ is the jth column of $\widetilde{\boldsymbol{\Psi}} = (\mathbf{A}^{2k+1})^*\mathbf{A}^{k+1}\mathbf{A}^*\boldsymbol{\Psi}$. Here $\boldsymbol{\Phi} = (\phi_{ij})$ and $\boldsymbol{\Psi} = (\psi_{ij})$ are determined, respectively, by

$$\phi_{ip} = \sum_{l=1}^{m} a_{il}^{(k)} \sum_{\beta\in J_{r,\,m}\{l\}} \operatorname{cdet}_l \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1}\right)_{.l} (\tilde{\mathbf{c}}_{.p})\right)_{\beta}^{\beta}, \tag{4.3}$$

$$\psi_{gj} = \sum_{t=1}^{n} \sum_{\alpha\in I_{s,\,n}\{t\}} \operatorname{rdet}_t \left(\left(\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^*\right)_{t.} (\bar{\mathbf{c}}_{g.})\right)_{\alpha}^{\alpha} b_{tj}^{(q)}, \tag{4.4}$$

where $\tilde{\mathbf{c}}_{.p}$ is the pth column of $\tilde{\mathbf{C}} = (\mathbf{A}^{2k+1})^*\mathbf{A}^{k+1}\mathbf{A}^*\mathbf{C}$ and $\bar{\mathbf{c}}_{g.}$ is the gth row of $\bar{\mathbf{C}} = \mathbf{C}\mathbf{B}^*\mathbf{B}^{q+1}(\mathbf{B}^{2q+1})^*$.
(ii) *If both matrices $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, $k \geq 2$ and $q \geq 2$, then*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{s,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{B}^{q+1})_{j.}(\widetilde{\boldsymbol{\phi}}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{r,\,m}} |\mathbf{A}^{k+1}|_{\beta}^{\beta} \sum\limits_{\alpha\in I_{s,\,n}} |\mathbf{B}^{q+1}|_{\alpha}^{\alpha}} \tag{4.5}$$

$$= \frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\widetilde{\boldsymbol{\psi}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}} |\mathbf{A}^{k+1}|_{\beta}^{\beta} \sum\limits_{\alpha\in I_{s,\,n}} |\mathbf{B}^{q+1}|_{\alpha}^{\alpha}}, \tag{4.6}$$

where $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}\mathbf{B}^{q+1}$ and $\widetilde{\boldsymbol{\Psi}} = \mathbf{A}^{k+1}\boldsymbol{\Psi}$. Here $\boldsymbol{\Phi} = (\phi_{ij})$ and $\boldsymbol{\Psi} = (\psi_{ij})$ are determined, respectively, by

$$\phi_{ip} = \sum_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i} (\tilde{\mathbf{c}}_{.p})\right)_{\beta}^{\beta}, \tag{4.7}$$

$$\psi_{gj} = \sum_{\alpha\in I_{s,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{B}^{q+1})_{j.}(\bar{\mathbf{c}}_{g.})\right)_{\alpha}^{\alpha}, \tag{4.8}$$

where $\tilde{\mathbf{c}}_{.p}$ is the pth column of $\tilde{\mathbf{C}} = \mathbf{A}^{k+2}\mathbf{C}$ and $\bar{\mathbf{c}}_{g.}$ is the gth row of $\bar{\mathbf{C}} = \mathbf{C}\mathbf{B}^{q+2}$.
(iii) *If the matrix $\mathbf{A}$ is Hermitian with $k \geq 2$, and $\mathbf{B}$ is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\widetilde{\boldsymbol{\psi}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}} |\mathbf{A}^{k+1}|_{\beta}^{\beta} \sum\limits_{\alpha\in I_{s,\,n}} |\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^*|_{\alpha}^{\alpha}}, \tag{4.9}$$

where $\widetilde{\boldsymbol{\Psi}} = \mathbf{A}^{k+1}\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}$ is determined by (4.4).
(iv) *If the matrix $\mathbf{B}$ is Hermitian and $q \geq 2$, and $\mathbf{A}$ is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{s,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{B}^{q+1})_{j.}(\widetilde{\boldsymbol{\phi}}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{r,\,m}} |\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1}|_{\beta}^{\beta} \sum\limits_{\alpha\in I_{s,\,n}} |\mathbf{B}^{q+1}|_{\alpha}^{\alpha}} \tag{4.10}$$

where $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}\mathbf{B}^{q+1}$ and $\boldsymbol{\Phi}$ is determined by (4.3).

Proof. (i) According to (3.1) and $\mathfrak{D}$-representations (2.9) and (2.12) for the Drazin-star matrix $\mathbf{A}^{\mathrm{D},*} = (a_{ij}^{\mathrm{D},*})$ and the star-Drazin matrix $\mathbf{B}^{*,\mathrm{D}} = (b_{ij}^{*,\mathrm{D}})$, respectively, it is derived

$$\begin{aligned}
x_{ij} &= \sum_{g=1}^{m}\sum_{p=1}^{n} a_{ig}^{\mathrm{D},*} c_{gp} b_{pj}^{*,\mathrm{D}} \\
&= \sum_{g=1}^{m}\sum_{p=1}^{n} \frac{\sum\limits_{l=1}^{m} a_{il}^{(k)} \sum\limits_{\beta\in J_{r,\,m}\{l\}} \operatorname{cdet}_l \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right)_{.l} (\tilde{\mathbf{a}}_{.g}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}} \left| \left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right|_{\beta}^{\beta}} c_{gp} \\
&\times \frac{\sum\limits_{t=1}^{n} \sum\limits_{\alpha\in I_{r,\,n}\{t\}} \operatorname{rdet}_t \left(\left(\mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1}\right)^* \right)_{t.} (\bar{\mathbf{b}}_{p.}) \right)_{\alpha}^{\alpha} b_{tj}^{(q)}}{\sum\limits_{\alpha\in I_{r,\,n}} \left| \mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1}\right)^* \right|_{\alpha}^{\alpha}},
\end{aligned} \tag{4.11}$$

where $\tilde{\mathbf{a}}_{.g}$ is the gth column of $\tilde{\mathbf{A}} = (\mathbf{A}^{2k+1})^* \mathbf{A}^{k+1} \mathbf{A}^*$ and $\bar{\mathbf{b}}_{p.}$ is the pth row of $\bar{\mathbf{B}} = \mathbf{B}^* \mathbf{B}^{q+1} (\mathbf{B}^{2q+1})^*$.

To obtain expressive formulas, we make some convolutions of (4.11).

Denote $\tilde{\mathbf{C}} = (\mathbf{A}^{2k+1})^* \mathbf{A}^{k+1} \mathbf{A}^* \mathbf{C}$ and $\bar{\mathbf{C}} = \mathbf{C}\mathbf{B}^* \mathbf{B}^{q+1} (\mathbf{B}^{2q+1})^*$. Then,

$$\sum_{g=1}^{m} \tilde{\mathbf{a}}_{.g} c_{gp} = \tilde{\mathbf{c}}_{.p}, \ \sum_{p=1}^{n} c_{gp} \bar{\mathbf{b}}_{p.} = \bar{\mathbf{c}}_{g.}$$

If we denote by

$$\phi_{ip} = \sum_{l=1}^{m} a_{il}^{(k)} \sum_{\beta\in J_{r,\,m}\{l\}} \operatorname{cdet}_l \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right)_{.l} (\tilde{\mathbf{c}}_{.p}) \right)_{\beta}^{\beta}$$

the (ip)th element of $\mathbf{\Phi} \in \mathbb{H}$ and put $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}\bar{\mathbf{B}}$, then from

$$\begin{aligned}
&\sum_{p=1}^{n} \phi_{ip} \sum_{t=1}^{n} \sum_{\alpha\in I_{r,\,n}\{t\}} \operatorname{rdet}_t \left(\left(\mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1}\right)^* \right)_{t.} (\bar{\mathbf{b}}_{p.}) \right)_{\alpha}^{\alpha} b_{tj}^{(q)} \\
&\quad = \sum_{t=1}^{n} \sum_{\alpha\in I_{r,\,n}\{t\}} \operatorname{rdet}_t \left(\left(\mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1}\right)^* \right)_{t.} (\widetilde{\boldsymbol{\phi}}_{i.}) \right)_{\alpha}^{\alpha} b_{tj}^{(q)}
\end{aligned}$$

it follows (4.1). If we denote by

$$\psi_{gj} = \sum_{t=1}^{n} \sum_{\alpha\in I_{r,\,n}\{t\}} \operatorname{rdet}_t \left(\left(\mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1}\right)^* \right)_{t.} (\bar{\mathbf{c}}_{g.}) \right)_{\alpha}^{\alpha} b_{tj}^{(q)}$$

the (gj)th element of $\mathbf{\Psi} \in \mathbb{H}$ and put $\widetilde{\mathbf{\Psi}} = \tilde{\mathbf{A}}\mathbf{\Psi}$, then the equality

$$\begin{aligned}
&\sum_{g=1}^{m}\sum_{l=1}^{m} a_{il}^{(k)} \sum_{\beta\in J_{r,\,m}\{l\}} \operatorname{cdet}_l \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right)_{.l} (\tilde{\mathbf{a}}_{.g}) \right)_{\beta}^{\beta} \psi_{gj} \\
&\quad = \sum_{l=1}^{m} a_{il}^{(k)} \sum_{\beta\in J_{r,\,m}\{l\}} \operatorname{cdet}_l \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right)_{.l} \left(\widetilde{\boldsymbol{\psi}}_{.j}\right) \right)_{\beta}^{\beta}
\end{aligned}$$

gives (4.2)

(ii) Using $\mathfrak{D}$-representations (2.15) and (2.16) for the Drazin-star matrix $\mathbf{A}^{\mathrm{D},*} = (a_{ij}^{\mathrm{D},*})$ and the star-Drazin matrix $\mathbf{B}^{*,\mathrm{D}} = (b_{ij}^{*,\mathrm{D}})$, respectively, one derives

$$x_{ij} = \sum_{g=1}^{m}\sum_{p=1}^{n} a_{ig}^{\mathrm{D},*} c_{gp} b_{pj}^{*,\mathrm{D}} = \sum_{g=1}^{m}\sum_{p=1}^{n} \frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.g}^{(k+2)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,m}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta}} c_{gp}$$
$$\times \frac{\sum\limits_{\alpha\in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{B}^{q+1})_{j.}(\mathbf{b}_{p.}^{(q+2)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{s,n}} \left|\mathbf{B}^{q+1}\right|_{\alpha}^{\alpha}}.$$

Let us do the following designations

$$\tilde{\mathbf{c}}_{.p} := \sum_{g=1}^{m} \mathbf{a}_{.g}^{(k+2)} c_{gp},\ \phi_{ip} = \sum_{\beta\in J_{r,m}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i}(\tilde{\mathbf{c}}_{.p})\right)_{\beta}^{\beta},$$

and construct the matrix $\mathbf{\Phi} = (\phi_{ip}) \in \mathbb{H}^{m\times n}$. Then from putting $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}\mathbf{B}^{q+2}$, and

$$\sum_{p=1}^{n} \phi_{ip} \sum_{\alpha\in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{B}^{q+1})_{j.}(\mathbf{b}_{p.}^{(q+2)})\right)_{\alpha}^{\alpha} = \sum_{\alpha\in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{B}^{q+1})_{j.}(\widetilde{\boldsymbol{\phi}}_{i.})\right)_{\alpha}^{\alpha},$$

where $\widetilde{\boldsymbol{\phi}}_{i.}$ is the ith row of $\widetilde{\mathbf{\Phi}}$, it follows (4.5).

By denoting

$$\bar{\mathbf{c}}_{g.} := \sum_{p=1}^{n} c_{gp}\bar{\mathbf{b}}_{p.},\ \psi_{gj} = \sum_{\alpha\in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{B}^{q+1})_{j.}(\bar{\mathbf{c}}_{g.})\right)_{\alpha}^{\alpha}$$

as the (gj)th element of $\mathbf{\Psi} \in \mathbb{H}^{m\times n}$ and $\widetilde{\mathbf{\Psi}} = \mathbf{A}^{k+2}\mathbf{\Psi}$, then the equality

$$\sum_{g=1}^{m}\sum_{\beta\in J_{r,m}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.g}^{(k+2)}\right)\right)_{\beta}^{\beta}\psi_{gj} = \sum_{\beta\in J_{r,m}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\widetilde{\boldsymbol{\psi}}_{.j}\right)\right)_{\beta}^{\beta}$$

gives (4.6).

The proofs of the cases (iii) and (iv) are similar to above by using corresponding $\mathfrak{D}$-representations of the Drazin-star and star-Drazin inverses. ∎

Remark 1. *Since for a Hermitian matrix their corresponding Drazin-star and star-Drazin matrices coincide, then forward the case of both Hermitian matrix could be represented only by Cramer's rules* (4.5)-(4.6) *as the most optimal.*

The proofs of the next corollaries evidently follow from Theorem 4.1 by putting $\mathbf{A} = \mathbf{I}_m$ or $\mathbf{B} = \mathbf{I}_n$, respectively.

Corollary 9. *If* $\mathbf{A} \in \mathbb{H}^{(m)(k)}$ *and* $\mathrm{rank}(\mathbf{A}^k) = r$. *The unique solution* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{m\times n}$ *from* (3.2) *can be expressed componentwise as follows.*
(i) *If the matrix* $\mathbf{A}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{l=1}^{m} a_{il}^{(k)} \sum\limits_{\beta\in J_{r,\,m}\{l\}} \mathrm{cdet}_l \left(\left(\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right)_{.l}(\tilde{\mathbf{c}}_{.j})\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}} \left|\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right|_{\beta}^{\beta}}, \tag{4.12}$$

where $\tilde{\mathbf{c}}_{.j}$ *is the* j*th column of* $\tilde{\mathbf{C}} = (\mathbf{A}^{2k+1})^* \mathbf{A}^{k+1} \mathbf{A}^* \mathbf{C}$.
(ii) *If the matrix* $\mathbf{A}$ *is Hermitian and* $k \geq 2$*, then*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,m}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1} \right)_{.i} (\widetilde{\mathbf{c}}_{.j}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} \left| \mathbf{A}^{k+1} \right|_{\beta}^{\beta}}, \tag{4.13}$$

where $\tilde{\mathbf{c}}_{.j}$ *is the* j*th column of* $\tilde{\mathbf{C}} = \mathbf{A}^{k+2} \mathbf{C}$.

Corollary 10. *If* $\mathbf{B} \in \mathbb{H}^{(n)(q)}$ *and* $\operatorname{rank}(\mathbf{B}^q) = s$*. The unique solution* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{m \times n}$ *from* (3.4) *can be expressed componentwise as follows.*
(i) *If the matrix* $\mathbf{B}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} \sum\limits_{\alpha \in I_{s,\,n}\{t\}} \operatorname{rdet}_t \left(\left(\mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1} \right)^* \right)_{t.} (\bar{\mathbf{c}}_{i.}) \right)_{\alpha}^{\alpha} b_{tj}^{(q)}}{\sum\limits_{\alpha \in I_{s,\,n}} \left| \mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1} \right)^* \right|_{\alpha}^{\alpha}}, \tag{4.14}$$

where $\bar{\mathbf{c}}_{i.}$ *is the* i*th row of* $\bar{\mathbf{C}} = \mathbf{C}\mathbf{B}^* \mathbf{B}^{q+1} (\mathbf{B}^{2q+1})^*$.
(ii) *If the matrix* $\mathbf{B}$ *is Hermitian and* $q \geq 2$*, then*

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{s,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{B}^{q+1})_{j.} (\bar{\mathbf{c}}_{i.}) \right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{s,\,n}} \left| \mathbf{B}^{q+1} \right|_{\alpha}^{\alpha}}, \tag{4.15}$$

where $\bar{\mathbf{c}}_{i.}$ *is the* i*th row of* $\bar{\mathbf{C}} = \mathbf{C}\mathbf{B}^{q+2}$.

Remark 2. *Note that Cramer's rules for one-side right and left equations with Drazin-star and star-Drazin matrices of Hermitian matrices* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively, are optimally described by Eqs.* (4.13)-(4.15)*. So, we will consider henceforth one-side right and left equations with Drazin-star and Star-Drazin matrices only for arbitrary matrices* $\mathbf{A}$ *and* $\mathbf{B}$*.*

The case of complex matrices can be proven similarly using the $\mathfrak{D}$-representations (2.11) of the Drazin-star matrix $A^{\mathrm{D},*}$ and (2.17) of the star-Drazin matrix $B^{*,\mathrm{D}}$.

Theorem 4.2. *Let* $C \in \mathbb{C}^{m \times n}$, $(A|B) \in \mathbb{C}^{(m|n)(k|q)}$ *with* $\operatorname{rank}(A^k) = r$ *and* $\operatorname{rank}(B^q) = s$*. The unique solution* $X = (x_{ij}) \in \mathbb{C}^{m \times n}$ *can be expressed componentwise as follows.*
(i) *For Eq.* (3.1),

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{s,n}\{j\}} \left| (B^{q+1})_{j.} (\widetilde{\phi}_{i.}) \right|_{\alpha}^{\alpha}}{\sum\limits_{\beta \in J_{r,\,m}} \left| A^{k+1} \right|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{s,\,n}} \left| B^{q+1} \right|_{\alpha}^{\alpha}} = \frac{\sum\limits_{\beta \in J_{r,m}\{i\}} \left| \left(A^{k+1} \right)_{.i} \left(\widetilde{\psi}_{.j} \right) \right|_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} \left| A^{k+1} \right|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{s,\,n}} \left| B^{q+1} \right|_{\alpha}^{\alpha}}$$

where $\widetilde{\Phi} = \Phi B^* B^{q+1}$ *and* $\widetilde{\Psi} = A^{k+1} A^* \Psi$*. Here* $\Phi = (\phi_{ij})$ *and* $\Psi = (\psi_{ij})$ *are determined by*

$$\phi_{ip} = \sum_{\beta \in J_{r,m}\{i\}} \left| \left(A^{k+1} \right)_{.i} (\tilde{c}_{.p}) \right|_{\beta}^{\beta}, \tag{4.16}$$

$$\psi_{gj} = \sum_{\alpha \in I_{s,n}\{j\}} \left| (B^{q+1})_{j.} (\bar{c}_{g.}) \right|_{\alpha}^{\alpha}, \tag{4.17}$$

where $\tilde{c}_{.p}$ *is the* p*th column of* $\tilde{C} = A^{k+1} A^* C$ *and* $\bar{c}_{g.}$ *is the* g*th row of* $\bar{C} = C B^* B^{q+1}$.

(ii) *For Eq.* (3.2),

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,m}\{i\}} \left|\left(A^{k+1}\right)_{.i}(\widetilde{c}_{.j})\right|_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} |A^{k+1}|_{\beta}^{\beta}}. \tag{4.18}$$

(iii) *For Eq.* (3.4),

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{s,n}\{j\}} |(B^{q+1})_{j.}(\bar{c}_{i.})|_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{s,\,n}} |B^{q+1}|_{\alpha}^{\alpha}}. \tag{4.19}$$

Remark 3. *Cramer's rules in the framework of the theory of row-column determinants for Eq.* (3.6) *and its one-side consequences have been derived in* [13].

Theorem 4.3. *Let* $\mathbf{C} \in \mathbb{H}^{m\times n}$, $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ *with* $\mathrm{rank}(\mathbf{A}^k) = r$ *and* $\mathrm{rank}(\mathbf{B}^q) = s$. *The unique solution* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{m\times n}$ *from* (3.7) *can be expressed componentwise as follows.*
(i) *If the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary, then*

$$x_{ij} = \frac{\tilde{c}_{ij}}{\sum\limits_{\alpha \in I_{r,\,m}} |\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}|_{\alpha}^{\alpha} \sum\limits_{\beta \in J_{s,\,n}} |(\mathbf{B}^{2q+1})^{*}\,\mathbf{B}^{2q+1}|_{\beta}^{\beta}}, \tag{4.20}$$

where $\tilde{c}_{ij}$ *is the* (ij)*th element of* $\widetilde{\mathbf{C}} = \mathbf{\Phi}\mathbf{A}^k\mathbf{C}\mathbf{B}^q\mathbf{\Psi}$. *Here* $\mathbf{\Phi} = (\phi_{ij})$ *and* $\mathbf{\Psi} = (\psi_{ij})$ *are determined, respectively, by*

$$\phi_{ip} = \sum_{\alpha \in I_{r,\,m}\{t\}} \mathrm{rdet}_t\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right)_{t.}(\bar{\mathbf{a}}_{i.})\right)_{\alpha}^{\alpha}, \tag{4.21}$$

$$\psi_{lj} = \sum_{\beta \in J_{s,\,n}\{l\}} \mathrm{cdet}_l\left(\left(\left(\mathbf{B}^{2q+1}\right)^{*}\mathbf{B}^{2q+1}\right)_{.l}\left(\tilde{\mathbf{b}}_{.j}\right)\right)_{\beta}^{\beta}, \tag{4.22}$$

where $\bar{\mathbf{a}}_{i.}$ *is the* i*th row of* $\bar{\mathbf{A}} = \mathbf{A}^{*}\mathbf{A}^{k+1}(\mathbf{A}^{2k+1})^{*}$ *and* $\tilde{\mathbf{b}}_{.j}$ *is the* j*th column of* $\tilde{\mathbf{B}} = (\mathbf{B}^{2q+1})^{*}\mathbf{B}^{q+1}\mathbf{B}^{*}$.
(ii) *If the matrix* $\mathbf{A}$ *is Hermitian with* $k \geq 2$, *and* $\mathbf{B}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,m}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\widetilde{\boldsymbol{\psi}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} |\mathbf{A}^{k+1}|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{s,\,n}} |\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^{*}|_{\alpha}^{\alpha}}, \tag{4.23}$$

where $\widetilde{\mathbf{\Psi}} = \mathbf{A}^{k+2}\mathbf{C}\mathbf{B}^q\mathbf{\Psi}$ *and* $\mathbf{\Psi}$ *is determined by (4.8).*
(iii) *If the matrix* $\mathbf{B}$ *is Hermitian with* $q \geq 2$, *and* $\mathbf{A}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{q+1})_{j.}(\widetilde{\boldsymbol{\phi}}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta \in J_{r,\,m}} |(\mathbf{A}^{2k+1})^{*}\,\mathbf{A}^{2k+1}|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{s,\,n}} |\mathbf{B}^{q+1}|_{\alpha}^{\alpha}}, \tag{4.24}$$

where $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}\mathbf{A}^k\mathbf{C}\mathbf{B}^{q+2}$ *and* $\mathbf{\Phi}$ *is determined by* (4.7).

Proof. (i) According to (3.7) and $\mathfrak{D}$-representations (2.12) and (2.9) for the star-Drazin matrix $\mathbf{A}^{*,\mathrm{D}} = (a_{ij}^{*,\mathrm{D}})$ and the Drazin-star matrix $\mathbf{B}^{\mathrm{D},*} = (b_{ij}^{\mathrm{D},*})$, respectively, we have

$$x_{ij} = \sum_{g=1}^{m}\sum_{p=1}^{n} a_{ig}^{*,\mathrm{D}} c_{gp} b_{pj}$$

$$= \sum_{g=1}^{m}\sum_{p=1}^{n} \frac{\sum\limits_{t=1}^{m}\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \mathrm{rdet}_t\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{t.}(\bar{\mathbf{a}}_{i.})\right)_{\alpha}^{\alpha} a_{tg}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right|_{\alpha}^{\alpha}} c_{gp}$$

$$\times \frac{\sum\limits_{l=1}^{n} b_{pl}^{(q)} \sum\limits_{\beta\in J_{s,\,n}\{l\}} \mathrm{cdet}_l\left(\left(\left(\mathbf{B}^{2q+1}\right)^*\mathbf{B}^{2q+1}\right)_{.l}\left(\tilde{\mathbf{b}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s,\,n}} \left|\left(\mathbf{B}^{2q+1}\right)^*\mathbf{B}^{2q+1}\right|_{\beta}^{\beta}},$$

where $\bar{\mathbf{a}}_{i.}$ is the ith row of $\bar{\mathbf{A}} = \mathbf{A}^*\mathbf{A}^{k+1}(\mathbf{A}^{2k+1})^*$ and $\tilde{\mathbf{b}}_{.j}$ is the jth column of $\tilde{\mathbf{B}} = (\mathbf{B}^{2q+1})^*\mathbf{B}^{q+1}\mathbf{B}^*$.

To obtain an expressive formula, we make the following denotations

$$\bar{c}_{tl} = \sum_{t=1}^{m}\sum_{l=1}^{n} a_{tg}^{(k)} c_{gp} b_{pl}^{(q)},$$

$$\phi_{ip} = \sum_{\alpha\in I_{r,\,m}\{t\}} \mathrm{rdet}_t\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{t.}(\bar{\mathbf{a}}_{i.})\right)_{\alpha}^{\alpha},$$

$$\psi_{lj} = \sum_{\beta\in J_{s,\,n}\{l\}} \mathrm{cdet}_l\left(\left(\left(\mathbf{B}^{2q+1}\right)^*\mathbf{B}^{2q+1}\right)_{.l}\left(\tilde{\mathbf{b}}_{.j}\right)\right)_{\beta}^{\beta},$$

and construct the matrices $\bar{\mathbf{C}} = \mathbf{A}^k\mathbf{C}\mathbf{B}^q = (\bar{c}_{tl}) \in \mathbb{H}^{m\times n}$, $\mathbf{\Phi} = (\phi_{ip}) \in \mathbb{H}^{m\times m}$, and $\mathbf{\Psi} = (\psi_{lj}) \in \mathbb{H}^{n\times n}$. Then from putting $\widetilde{\mathbf{C}} = \mathbf{\Phi}\bar{\mathbf{C}}\mathbf{\Psi} = \mathbf{\Phi}\mathbf{A}^k\mathbf{C}\mathbf{B}^q\mathbf{\Psi} = (\tilde{c}_{ij}) \in \mathbb{H}^{m\times n}$, it follows (4.20).

(ii) Using $\mathfrak{D}$-representations (2.15) and (2.9) for the Drazin-star matrix $\mathbf{A}^{\mathrm{D},*} = (a_{ij}^{\mathrm{D},*})$ and the star-Drazin matrix $\mathbf{B}^{*,\mathrm{D}} = (b_{ij}^{*,\mathrm{D}})$, respectively, it is derived

$$x_{ij} = \sum_{g=1}^{m}\sum_{p=1}^{n} a_{ig}^{*,\mathrm{D}} c_{gp} b_{pj} = \sum_{g=1}^{m}\sum_{p=1}^{n} \frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.g}^{(k+2)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,m}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta}} c_{gp}$$

$$\times \frac{\sum\limits_{l=1}^{n} b_{pl}^{(q)} \sum\limits_{\beta\in J_{s,\,n}\{l\}} \mathrm{cdet}_l\left(\left(\left(\mathbf{B}^{2q+1}\right)^*\mathbf{B}^{2q+1}\right)_{.l}\left(\tilde{\mathbf{b}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s,\,n}} \left|\left(\mathbf{B}^{2q+1}\right)^*\mathbf{B}^{2q+1}\right|_{\beta}^{\beta}},$$

where $\tilde{\mathbf{b}}_{.j}$ is the jth column of $\tilde{\mathbf{B}} = (\mathbf{B}^{2q+1})^*\mathbf{B}^{q+1}\mathbf{B}^*$.

By $\bar{\mathbf{c}}_{.l} = \sum\limits_{t=1}^{m}\sum\limits_{l=1}^{n} \mathbf{a}_{.g}^{(k+2)} c_{gp} b_{pl}^{(q)}$ denote the lth column of $\bar{\mathbf{C}} = \mathbf{A}^{k+2}\mathbf{C}\mathbf{B}^q$ and determine the matrix $\mathbf{\Psi} = (\psi_{lj})$ by (4.8). Then (4.23) follows from putting $\widetilde{\bar{\mathbf{\Psi}}} = \mathbf{A}^{k+2}\mathbf{C}\mathbf{B}^q\mathbf{\Psi}$.

(iii) The proof is similar to the proof of the item (ii). ∎

Representations in the next corollaries follows as consequences of Theorem 4.3 for $\mathbf{A} = \mathbf{I}_m$ or $\mathbf{B} = \mathbf{I}_n$.

Corollary 11. *Let $\mathbf{C} \in \mathbb{H}^{m\times n}$ and arbitrary $\mathbf{A} \in \mathbb{H}^{(m)(k)}$. Then the unique solution $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{m\times n}$ from* (3.9) *is represented as*

$$x_{ij} = \frac{\tilde{c}_{ij}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right|_{\alpha}^{\alpha}}, \tag{4.25}$$

where $\tilde{c}_{ij}$ is the (ij)th element of $\widetilde{\mathbf{C}} = \mathbf{\Phi}\mathbf{A}^k\mathbf{C}$ and $\mathbf{\Phi} = (\phi_{ij})$ is determined by (4.21).

Corollary 12. *Let $\mathbf{C} \in \mathbb{H}^{m\times n}$ and arbitrary $(\mathbf{B}) \in \mathbb{H}^{(n)(q)}$. Then the unique solution $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{m\times n}$ from* (3.10) *is*

$$x_{ij} = \frac{\tilde{c}_{ij}}{\sum\limits_{\beta\in J_{s,\,n}} \left|\left(\mathbf{B}^{2q+1}\right)^*\mathbf{B}^{2q+1}\right|_{\beta}^{\beta}}, \tag{4.26}$$

where $\tilde{c}_{ij}$ is the (ij)th element of $\widetilde{\mathbf{C}} = \mathbf{C}\mathbf{B}^q\mathbf{\Psi}$, and $\mathbf{\Psi} = (\psi_{ij})$ is determined by (4.22).

Theorem 4.4. *Let $C \in \mathbb{C}^{m\times n}$, $(A|B) \in \mathbb{C}^{(m|n)(k|q)}$ with $\operatorname{rank}(A^k) = r$ and $\operatorname{rank}(B^q) = s$. The unique solution $X = (x_{ij}) \in \mathbb{C}^{m\times n}$ can be expressed componentwise as follows.*
(i) *For equation* (3.7),

$$x_{ij} = \frac{\tilde{c}_{ij}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|A^{k+1}\right|_{\alpha}^{\alpha} \sum\limits_{\beta\in J_{s,\,n}} \left|B^{q+1}\right|_{\beta}^{\beta}}$$

where $\tilde{c}_{ij}$ is the (ij)th element of $\widetilde{C} = \Phi C\Psi$. Here $\Phi = (\phi_{ij})$ and $\Psi = (\psi_{ij})$ are determined, respectively, by

$$\phi_{ig} = \sum_{\alpha\in I_{r,m}\{g\}} \left|\left(A^{k+1}\right)_{g.}\left(\bar{a}_{i.}\right)\right|_{\alpha}^{\alpha}, \tag{4.27}$$

$$\psi_{pj} = \sum_{\beta\in J_{s,n}\{p\}} \left|\left(B^{q+1}\right)_{.p}\left(\tilde{b}_{.j}\right)\right|_{\beta}^{\beta}, \tag{4.28}$$

where $\bar{a}_{i.}$ is the ith row of $\bar{A} = A^*A^{k+1}$ and $\tilde{b}_{.j}$ is the jth column of $\tilde{B} = B^{q+1}B^*$.
(ii) *For* (3.9),

$$x_{ij} = \frac{c_{ij}^{(1)}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|A^{k+1}\right|_{\alpha}^{\alpha}} \tag{4.29}$$

where $c_{ij}^{(1)}$ is the (ij)th element of $C_1 = \Phi C$ and $\Phi = (\phi_{ij})$ is determined by (4.27).
(iii) *For* (3.10),

$$x_{ij} = \frac{c_{ij}^{(2)}}{\sum\limits_{\beta\in J_{s,\,n}} \left|B^{q+1}\right|_{\beta}^{\beta}} \tag{4.30}$$

where $c_{ij}^{(2)}$ is the (ij)th element of $C_2 = C\Psi$ and $\Psi = (\psi_{ij})$ is determined by (4.28).

Theorem 4.5. *Let $\mathbf{C} \in \mathbb{H}^{m\times n}$, $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(m|n)(k|q)}$ with $\operatorname{rank}(\mathbf{A}^k) = r$ and $\operatorname{rank}(\mathbf{B}^q) = s$. The unique solution $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{m\times n}$ from* (3.11) *can be expressed componentwise as follows.*

(i) *If the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{l=1}^{m} a_{il}^{(k)} \sum\limits_{\beta \in J_{r,\,m}\{l\}} \mathrm{cdet}_l \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right)_{.l} \left(\widetilde{\boldsymbol{\psi}}_{.j} \right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} \left| \left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{s,\,n}} \left| \mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1}\right)^* \right|_{\alpha}^{\alpha}}, \tag{4.31}$$

where $\widetilde{\boldsymbol{\psi}}_{.j}$ *is the* j*th column of* $\widetilde{\boldsymbol{\Psi}} = (\mathbf{A}^{2k+1})^* \mathbf{A}^{k+1} \mathbf{A}^* \boldsymbol{\Psi}$ *and* $\boldsymbol{\Psi} = (\psi_{lj})$ *by* (4.22).

(ii) *If the matrix* $\mathbf{A}$ *is Hermitian with* $k \geq 2$*, and* $\mathbf{B}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,m}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^{k+1}\right)_{.i} \left(\widetilde{\boldsymbol{\psi}}_{.j} \right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} \left| \mathbf{A}^{k+1} \right|_{\beta}^{\beta} \sum\limits_{\beta \in J_{s,\,n}} \left| \left(\mathbf{B}^{2q+1}\right)^* \mathbf{B}^{2q+1} \right|_{\beta}^{\beta}}, \tag{4.32}$$

where $\widetilde{\boldsymbol{\Psi}} = \mathbf{A}^{k+2} \mathbf{C} \mathbf{B}^q \boldsymbol{\Psi}$ *and* $\boldsymbol{\Psi}$ *is determined by* (4.22).

(iii) *If the matrix* $\mathbf{B}$ *is Hermitian with* $q \geq 2$*, and* $\mathbf{A}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{B}^{q+1})_{j.} (\tilde{\boldsymbol{\phi}}_{.j}) \right)_{\alpha}^{\alpha}}{\sum\limits_{\beta \in J_{r,\,m}} \left| \left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{s,\,n}} \left| \mathbf{B}^{q+1} \right|_{\alpha}^{\alpha}}, \tag{4.33}$$

where $\tilde{\boldsymbol{\phi}}_{.j}$ *is the* j*th column of* $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi} \mathbf{C} \mathbf{B}^{q+2}$ *and* $\boldsymbol{\Phi}$ *is determined by* (4.3).

Proof. According to (3.11) and $\mathfrak{D}$-representations (2.9) for the Drazin-star matrices $\mathbf{A}^{\mathrm{D},*} = (a_{ij}^{\mathrm{D},*})$ and $\mathbf{B}^{\mathrm{D},*} = (b_{ij}^{\mathrm{D},*})$, it follows

$$x_{ij} = \sum_{g=1}^{m} \sum_{p=1}^{n} a_{ig}^{\mathrm{D},*} c_{gp} b_{pj}^{\mathrm{D},*} = \sum_{g=1}^{m} \sum_{p=1}^{n} \frac{\sum\limits_{t=1}^{m} a_{it}^{(k)} \sum\limits_{\beta \in J_{r,\,m}\{t\}} \mathrm{cdet}_t \left(\left(\left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right)_{.t} (\tilde{\mathbf{a}}_{.g}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} \left| \left(\mathbf{A}^{2k+1}\right)^* \mathbf{A}^{2k+1} \right|_{\beta}^{\beta}} c_{gp}$$
$$\times \frac{\sum\limits_{l=1}^{n} b_{pl}^{(q)} \sum\limits_{\beta \in J_{s,\,n}\{l\}} \mathrm{cdet}_l \left(\left(\left(\mathbf{B}^{2q+1}\right)^* \mathbf{B}^{2q+1} \right)_{.l} \left(\tilde{\mathbf{b}}_{.j} \right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{s,\,n}} \left| \left(\mathbf{B}^{2q+1}\right)^* \mathbf{B}^{2q+1} \right|_{\beta}^{\beta}},$$

where $\tilde{\mathbf{a}}_{.g}$ is the gth column of $\tilde{\mathbf{A}} = (\mathbf{A}^{2k+1})^* \mathbf{A}^{k+1} \mathbf{A}^*$ and $\tilde{\mathbf{b}}_{.j}$ is the jth column of $\tilde{\mathbf{B}} = (\mathbf{B}^{2q+1})^* \mathbf{B}^{q+1} \mathbf{B}^*$.

Construct the matrix $\boldsymbol{\Psi} = (\psi_{lj})$ defined by (4.22) and $\widetilde{\boldsymbol{\Psi}} = \tilde{\mathbf{A}} \mathbf{C} \mathbf{B}^q \boldsymbol{\Psi} = (\mathbf{A}^{2k+1})^* \mathbf{A}^{k+1} \mathbf{A}^* \mathbf{C} \mathbf{B}^q \boldsymbol{\Psi}$. Then, from

$$\sum_{g=1}^{m} \sum_{p=1}^{n} \sum_{l=1}^{n} \tilde{\mathbf{a}}_{.g} c_{gp} b_{pl}^{(q)} \psi_{lj} = \tilde{\boldsymbol{\psi}}_{.j},$$

it follows (4.31).

(ii) Using $\mathfrak{D}$-representations (2.15) and (2.9) for the Drazin-star matrices $\mathbf{A}^{\mathrm{D},*} = (a_{ij}^{\mathrm{D},*})$ and $\mathbf{B}^{\mathrm{D},*} = (b_{ij}^{\mathrm{D},*})$, we have

$$x_{ij}=\sum_{g=1}^{m}\sum_{p=1}^{n}a_{ig}^{*,\mathrm{D}}c_{gp}b_{pj}^{\mathrm{D},*}=\sum_{g=1}^{m}\sum_{p=1}^{n}\frac{\sum\limits_{\beta\in J_{r,m}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.g}^{(k+2)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,m}}\left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta}}c_{gp}$$
$$\times\frac{\sum\limits_{l=1}^{n}b_{pl}^{(q)}\sum\limits_{\beta\in J_{s,\,n}\{l\}}\mathrm{cdet}_l\left(\left(\left(\mathbf{B}^{2q+1}\right)^{*}\mathbf{B}^{2q+1}\right)_{.l}\left(\tilde{\mathbf{b}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s,\,n}}\left|\left(\mathbf{B}^{2q+1}\right)^{*}\mathbf{B}^{2q+1}\right|_{\beta}^{\beta}},$$

where $\tilde{\mathbf{b}}_{.j}$ is the jth column of $\tilde{\mathbf{B}}=(\mathbf{B}^{2q+1})^{*}\mathbf{B}^{q+1}\mathbf{B}^{*}$.

By $\bar{\mathbf{c}}_{.l}=\sum\limits_{t=1}^{m}\sum\limits_{l=1}^{n}\mathbf{a}_{.g}^{(k+2)}c_{gp}b_{pl}^{(q)}$ denote the lth column of $\bar{\mathbf{C}}=\mathbf{A}^{k+2}\mathbf{C}\mathbf{B}^{q}$ and determine the matrix $\mathbf{\Psi}=(\psi_{lj})$ defined in (4.22). Then from putting $\widetilde{\mathbf{\Psi}}=\mathbf{A}^{k+2}\mathbf{C}\mathbf{B}^{q}\mathbf{\Psi}$, it follows (4.32).

(iii) Using $\mathfrak{D}$-representations (2.9) and (2.16) for the Drazin-star matrices $\mathbf{A}^{\mathrm{D},*}=(a_{ij}^{\mathrm{D},*})$ and $\mathbf{B}^{\mathrm{D},*}=(b_{ij}^{\mathrm{D},*})$, it is derived

$$x_{ij}=\sum_{g=1}^{m}\sum_{p=1}^{n}a_{ig}^{*,\mathrm{D}}c_{gp}b_{pj}^{\mathrm{D},*}=\sum_{g=1}^{m}\sum_{p=1}^{n}\frac{\sum\limits_{t=1}^{m}a_{it}^{(k)}\sum\limits_{\beta\in J_{r,\,m}\{t\}}\mathrm{cdet}_t\left(\left(\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right)_{.t}(\tilde{\mathbf{a}}_{.g})\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}}\left|\left(\mathbf{A}^{2k+1}\right)^{*}\mathbf{A}^{2k+1}\right|_{\beta}^{\beta}}c_{gp}$$
$$\times\frac{\sum\limits_{\alpha\in I_{s,n}\{j\}}\mathrm{rdet}_j\left((\mathbf{B}^{q+1})_{j.}(\mathbf{b}_{p.}^{(q+2)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{s,n}}\left|\mathbf{B}^{q+1}\right|_{\alpha}^{\alpha}},$$

where $\tilde{\mathbf{a}}_{.g}$ is the gth column of $\tilde{\mathbf{A}}=(\mathbf{A}^{2k+1})^{*}\mathbf{A}^{k+1}\mathbf{A}^{*}$.

Determine the matrix $\mathbf{\Phi}=(\phi_{ig})$ by (4.3). Then from the denotation $\tilde{\boldsymbol{\phi}}_{.j}=\sum\limits_{g=1}^{m}\sum\limits_{l=1}^{n}\phi_{ig}c_{gp}\mathbf{a}_{p.}^{(k+2)}$ by the jth column of $\widetilde{\mathbf{\Phi}}=\mathbf{\Phi}\mathbf{C}\mathbf{B}^{q+2}$, it follows (4.33) ∎

Corollary 13. *Let* $\mathbf{C}\in\mathbb{H}^{m\times n}$ *and arbitrary* $\mathbf{A}\in\mathbb{H}^{(m)(k)}$. *Then the unique solutions* $\mathbf{X}=(x_{ij})\in\mathbb{H}^{m\times n}$ *from* (3.17) *and* (3.19) *are represented by* (4.12) *and* (4.25), *respectively.*

Corollary 14. *Let* $\mathbf{C}\in\mathbb{H}^{m\times n}$ *and arbitrary* $\mathbf{B}\in\mathbb{H}^{(n)(q)}$. *Then the unique solutions* $\mathbf{X}=(x_{ij})\in\mathbb{H}^{m\times n}$ *from* (3.18) *and* (3.20) *are represented by* (4.26) *and* (4.14), *respectively.*

Theorem 4.6. *Let* $\mathbf{C}\in\mathbb{H}^{m\times n}$, $(\mathbf{A}|\mathbf{B})\in\mathbb{H}^{(m|n)(k|q)}$ *with* $\mathrm{rank}(\mathbf{A}^{k})=r$ *and* $\mathrm{rank}(\mathbf{B}^{q})=s$. *The unique solution* $\mathbf{X}=(x_{ij})\in\mathbb{H}^{m\times n}$ *from* (3.14) *can be expressed componentwise as follows.*

(i) *If the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary, then*

$$x_{ij}=\frac{\sum\limits_{l=1}^{n}\sum\limits_{\alpha\in I_{s,\,n}\{l\}}\mathrm{rdet}_l\left(\left(\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^{*}\right)_{l.}(\tilde{\boldsymbol{\phi}}_{i.})\right)_{\alpha}^{\alpha}b_{lj}^{(q)}}{\sum\limits_{\alpha\in I_{r,\,m}}\left|\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right|_{\alpha}^{\alpha}\sum\limits_{\alpha\in I_{s,\,n}}\left|\mathbf{B}^{2q+1}\left(\mathbf{B}^{2q+1}\right)^{*}\right|_{\alpha}^{\alpha}},\tag{4.34}$$

where $\tilde{\boldsymbol{\phi}}_{i.}$ *is the* i*th row of* $\widetilde{\mathbf{\Phi}}=\mathbf{\Phi}\mathbf{A}^{k}\mathbf{C}\mathbf{B}^{*}\mathbf{B}^{q+1}(\mathbf{B}^{2q+1})^{*}$ *and* $\mathbf{\Phi}=(\phi_{it})$ *by* (4.21).

(ii) *If the matrix* $\mathbf{A}$ *is Hermitian with* $k \geq 2$ *and* $\mathbf{B}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{r,m}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1} \right)_{.i} \left(\widetilde{\boldsymbol{\psi}}_{.j} \right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,m}} \left| \mathbf{A}^{k+1} \right|_{\beta}^{\beta} \sum\limits_{\beta \in J_{s,\,n}} \left| \left(\mathbf{B}^{2q+1} \right)^* \mathbf{B}^{2q+1} \right|_{\beta}^{\beta}}, \tag{4.35}$$

where $\widetilde{\boldsymbol{\Psi}} = \mathbf{A}^{k+2}\mathbf{C}\boldsymbol{\Psi}$ *and* $\boldsymbol{\Psi}$ *is determined by* (4.4).

(iii) *If the matrix* $\mathbf{B}$ *is Hermitian with* $q \geq 2$ *and* $\mathbf{A}$ *is arbitrary, then*

$$x_{ij} = \frac{\sum\limits_{\alpha \in I_{s,n}\{j\}} \operatorname{rdet}_j \left((\mathbf{B}^{q+1})_{j.} (\tilde{\boldsymbol{\phi}}_{.j}) \right)_{\alpha}^{\alpha}}{\sum\limits_{\beta \in J_{r,\,m}} \left| \left(\mathbf{A}^{2k+1} \right)^* \mathbf{A}^{2k+1} \right|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{s,\,n}} \left| \mathbf{B}^{q+1} \right|_{\alpha}^{\alpha}} \tag{4.36}$$

where $\tilde{\boldsymbol{\phi}}_{.j}$ *is the* j*th column of* $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}\mathbf{C}\mathbf{B}^{q+2}$ *and* $\boldsymbol{\Phi}$ *is determined by* (4.3).

Proof. According to (3.11) and $\mathfrak{D}$-representations (2.12) for the star-Drazin matrices $\mathbf{A}^{*,\mathrm{D}} = (a_{ij}^{*,\mathrm{D}})$ and $\mathbf{B}^{*,\mathrm{D}} = (b_{ij}^{*,\mathrm{D}})$, we have

$$x_{ij} = \sum_{g=1}^{m} \sum_{p=1}^{n} a_{ig}^{*,\mathrm{D}} c_{gp} b_{pj}^{*,\mathrm{D}} = \sum_{g=1}^{m} \sum_{p=1}^{n} \frac{\sum\limits_{t=1}^{m} \sum\limits_{\alpha \in I_{r,\,m}\{t\}} \operatorname{rdet}_t \left(\left(\mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1} \right)^* \right)_{t.} (\bar{\mathbf{a}}_{i.}) \right)_{\alpha}^{\alpha} a_{tg}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,m}} \left| \mathbf{A}^{2k+1} \left(\mathbf{A}^{2k+1} \right)^* \right|_{\alpha}^{\alpha}} c_{gp}$$
$$\times \frac{\sum\limits_{l=1}^{n} \sum\limits_{\alpha \in I_{s,\,n}\{l\}} \operatorname{rdet}_l \left(\left(\mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1} \right)^* \right)_{l.} (\bar{\mathbf{b}}_{p.}) \right)_{\alpha}^{\alpha} b_{lj}^{(q)}}{\sum\limits_{\alpha \in I_{s,\,n}} \left| \mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1} \right)^* \right|_{\alpha}^{\alpha}},$$

where $\bar{\mathbf{a}}_{i.}$ is the ith row of $\bar{\mathbf{A}} = \mathbf{A}^*\mathbf{A}^{k+1}(\mathbf{A}^{2k+1})^*$ and $\bar{\mathbf{b}}_{p.}$ is the pth row of $\bar{\mathbf{B}} = \mathbf{B}^*\mathbf{B}^{q+1}(\mathbf{B}^{2q+1})^*$.

Construct the matrix $\boldsymbol{\Phi} = (\phi_{it})$ by (4.21) and $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}\mathbf{A}^k\mathbf{C}\bar{\mathbf{B}} = \boldsymbol{\Phi}\mathbf{A}^k\mathbf{C}\mathbf{B}^*\mathbf{B}^{q+1}(\mathbf{B}^{2q+1})^*$. Then, from

$$\sum_{g=1}^{m} \sum_{p=1}^{n} \sum_{l=1}^{n} \phi_{it} a_{tg}^{(k)} c_{gp} \bar{\mathbf{b}}_{p.} = \tilde{\boldsymbol{\phi}}_{i.},$$

it follows (4.34).

(ii) Using $\mathfrak{D}$-representations (2.15) and (2.12) for the star-Drazin matrices $\mathbf{A}^{*,\mathrm{D}} = (a_{ij}^{*,\mathrm{D}})$ and $\mathbf{B}^{*,\mathrm{D}} = (b_{ij}^{*,\mathrm{D}})$, we derive

$$x_{ij} = \sum_{g=1}^{m} \sum_{p=1}^{n} a_{ig}^{*,\mathrm{D}} c_{gp} b_{pj}^{*,\mathrm{D}} = \sum_{g=1}^{m} \sum_{p=1}^{n} \frac{\sum\limits_{\beta \in J_{r,m}\{i\}} \operatorname{cdet}_i \left(\left(\mathbf{A}^{k+1} \right)_{.i} \left(\mathbf{a}_{.g}^{(k+2)} \right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,m}} \left| \mathbf{A}^{k+1} \right|_{\beta}^{\beta}} c_{gp}$$
$$\times \frac{\sum\limits_{l=1}^{n} \sum\limits_{\alpha \in I_{s,\,n}\{l\}} \operatorname{rdet}_l \left(\left(\mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1} \right)^* \right)_{l.} (\bar{\mathbf{b}}_{p.}) \right)_{\alpha}^{\alpha} b_{lj}^{(q)}}{\sum\limits_{\alpha \in I_{s,\,n}} \left| \mathbf{B}^{2q+1} \left(\mathbf{B}^{2q+1} \right)^* \right|_{\alpha}^{\alpha}},$$

where $\bar{\mathbf{b}}_{p.}$ is the pth row of $\bar{\mathbf{B}} = \mathbf{B}^*\mathbf{B}^{q+1}(\mathbf{B}^{2q+1})^*$. Determine the matrix $\boldsymbol{\Psi} = (\psi_{lj})$ by (4.4). Then from putting $\widetilde{\boldsymbol{\Psi}} = \mathbf{A}^{k+2}\mathbf{C}\boldsymbol{\Psi}$, it follows (4.35)

(iii) Using $\mathfrak{D}$-representations (2.12) and (2.16) for the star-Drazin matrices $\mathbf{A}^{*,\mathrm{D}} = (a_{ij}^{*,\mathrm{D}})$ and $\mathbf{B}^{*,\mathrm{D}} = (b_{ij}^{*,\mathrm{D}})$, we have

$$x_{ij} = \sum_{g=1}^{m}\sum_{p=1}^{n} a_{ig}^{*,\mathrm{D}} c_{gp} b_{pj}^{*,\mathrm{D}} = \sum_{g=1}^{m}\sum_{p=1}^{n} \frac{\sum\limits_{t=1}^{m}\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \mathrm{rdet}_t\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{t.}(\bar{\mathbf{a}}_{i.})\right)_{\alpha}^{\alpha} a_{tg}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}} |\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*|_{\alpha}^{\alpha}} c_{gp}$$
$$\times \frac{\sum\limits_{\alpha\in I_{s,n}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{q+1})_{j.}(\mathbf{b}_{p.}^{(q+2)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{s,n}} |\mathbf{B}^{q+1}|_{\alpha}^{\alpha}},$$

where $\bar{\mathbf{a}}_{i.}$ is the ith row of $\bar{\mathbf{A}} = \mathbf{A}^*\mathbf{A}^{k+1}(\mathbf{A}^{2k+1})^*$. Determine the matrix $\mathbf{\Phi} = (\phi_{ig})$ by (4.21). Then (4.36) follows from the denotation $\tilde{\boldsymbol{\phi}}_{i.} = \sum\limits_{t=1}^{m}\sum\limits_{g=1}^{m}\sum\limits_{l=1}^{n} \phi_{it} a_{tg}^{(k)} c_{gp} \mathbf{b}_{p.}^{(q+2)}$ by the ith row of $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}\mathbf{A}^k\mathbf{C}\mathbf{B}^{q+2}$. ∎

Theorem 4.7. *Let $C \in \mathbb{C}^{m\times n}$, $(A|B) \in \mathbb{C}^{(m|n)(k|q)}$ with $\mathrm{rank}(A^k) = r$, and $\mathrm{rank}(B^q) = s$. The unique solution $X = (x_{ij}) \in \mathbb{C}^{m\times n}$ can be expressed componentwise as follows.*
(i) *For* (3.11),

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \left|\left(A^{k+1}\right)_{.i}\left(\widetilde{\psi}_{.j}\right)\right|_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}} |A^{k+1}|_{\beta}^{\beta} \sum\limits_{\beta\in J_{s,\,n}} |B^{q+1}|_{\beta}^{\beta}},$$

where $\widetilde{\Psi} = A^*A^{k+1}C\Psi$ and Ψ is determined by (4.28).
(ii) *For* (3.14),

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{s,n}\{j\}} \left|(B^{q+1})_{j.}(\tilde{\phi}_{.j})\right|_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} |A^{k+1}|_{\alpha}^{\alpha} \sum\limits_{\alpha\in I_{s,\,n}} |B^{q+1}|_{\alpha}^{\alpha}},$$

where $\tilde{\phi}_{.j}$ is the jth column of $\widetilde{\Phi} = \Phi C B^{q+1} B^*$ and Φ is determined by (4.27).

Corollary 15. *Let $C \in \mathbb{C}^{m\times n}$ and $A \in \mathbb{C}^{(m)(k)}$. Then the unique solutions $X = (x_{ij}) \in \mathbb{C}^{m\times n}$ from* (3.17) *and* (3.19) *are represented by* (4.18) *and* (4.29)*, respectively.*

Corollary 16. *Let $C \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{(n)(q)}$. Then the unique solutions $X = (x_{ij}) \in \mathbb{C}^{m\times n}$ from* (3.18) *and* (3.20) *are represented by* (4.30) *and* (4.19)*, respectively.*

5. AN ILLUSTRATIVE EXAMPLE

To explain derived results and representations, subsequent examples are conducted.
Let's derive Cramer's rule for the solution (3.1) to Eq. (1.3) with input matrices

$$\mathbf{A} = \begin{bmatrix} -\mathbf{k} & -\mathbf{j} & 0 & \mathbf{i} \\ -1-\mathbf{j} & \mathbf{i}+\mathbf{k} & \mathbf{j} & 1+\mathbf{j} \\ \mathbf{k} & 0 & \mathbf{i} & 0 \\ -\mathbf{i}+\mathbf{k} & 1-\mathbf{j} & \mathbf{i} & \mathbf{i}-\mathbf{k} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4\mathbf{k} & 4\mathbf{i} & -5\mathbf{i} \\ -2\mathbf{j} & 2\mathbf{k} & 3 \\ \mathbf{i} & -1 & \mathbf{k} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -\mathbf{i} & 0 & -1 \\ 0 & -\mathbf{k} & 0 \\ \mathbf{k} & 0 & \mathbf{j} \\ 0 & -\mathbf{j} & 0 \end{bmatrix}. \tag{5.1}$$

One can find

$$\mathbf{A}^3(\mathbf{A}^3)^* = \begin{bmatrix} 3 & 6\mathbf{i}+4\mathbf{k} & -4-3\mathbf{j} & -4-6\mathbf{j} \\ -6\mathbf{i}-4\mathbf{k} & 19 & 4\mathbf{i}+13\mathbf{k} & 19\mathbf{k} \\ -4+3\mathbf{j} & -4\mathbf{i}-13\mathbf{k} & 10 & 13+4\mathbf{j} \\ -4+6\mathbf{j} & -19\mathbf{k} & 13-4\mathbf{j} & 19 \end{bmatrix},$$

$$\left(\mathbf{B}^3\right)^* \mathbf{B}^3 = \begin{bmatrix} 11 & 11\mathbf{i} & 11\mathbf{j} \\ -11\mathbf{i} & 11 & -11\mathbf{k} \\ -11\mathbf{j} & 11\mathbf{k} & 11 \end{bmatrix}.$$

Since $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^*\mathbf{A}) = 3$, $\operatorname{rank}(\mathbf{A}^3) = \operatorname{rank}(\mathbf{A}^2) = 2$, $\operatorname{rank}(\mathbf{B}) = \operatorname{rank}(\mathbf{B}^*\mathbf{B}) = 2$, $\operatorname{rank}(\mathbf{B}^3) = \operatorname{rank}(\mathbf{B}^2) = 1$, it follows $k = \operatorname{Ind}(\mathbf{A}) = 2$, $q = \operatorname{Ind}(\mathbf{B}) = 2$.

Based on Theorem 4.1 in the case (4.1), the Cramer's rule to the solution (3.1) is expressed in the subsequent way.

1. Compute the matrices $\tilde{\mathbf{C}} = (\mathbf{A}^5)^*\mathbf{A}^3\mathbf{A}^*\mathbf{C}$, $\mathbf{\Phi}$ by (4.3), and $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}\mathbf{B}^*\mathbf{B}^3(\mathbf{B}^5)^*$,

$$\tilde{\mathbf{C}} = \begin{bmatrix} -29-72\mathbf{j} & 111\mathbf{i}+97\mathbf{k} & -29\mathbf{i}+72\mathbf{k} \\ 35\mathbf{i}+55\mathbf{k} & 76+98\mathbf{j} & -35+55\mathbf{j} \\ 17+6\mathbf{j} & -\mathbf{i}-35\mathbf{k} & 17\mathbf{i}-6\mathbf{k} \\ -55+35\mathbf{j} & -98\mathbf{i}+76\mathbf{k} & -55\mathbf{i}-35\mathbf{k} \end{bmatrix}, \ \mathbf{\Phi} = 25 \begin{bmatrix} 1+3\mathbf{j} & -2\mathbf{i}-4\mathbf{k} & \mathbf{i}-3\mathbf{k} \\ -2\mathbf{i}-\mathbf{k} & -3+\mathbf{j} & 2-\mathbf{j} \\ 2+\mathbf{j} & 5\mathbf{i}+\mathbf{k} & 2\mathbf{i}-\mathbf{k} \\ -1+2\mathbf{j} & \mathbf{i}+3\mathbf{k} & -\mathbf{i}-2\mathbf{k} \end{bmatrix},$$

$$\widetilde{\mathbf{\Phi}} = 25 \begin{bmatrix} 171+945\mathbf{i}+513\mathbf{j}-135\mathbf{k} & 315-45\mathbf{i}+45\mathbf{j}+171\mathbf{k} & 171-45\mathbf{i}-57\mathbf{j}-315\mathbf{k} \\ -342\mathbf{i}-135\mathbf{j}-171\mathbf{k} & -114+57\mathbf{j}-45\mathbf{k} & -45-57\mathbf{i}+114\mathbf{k} \\ 342+171\mathbf{j}+945\mathbf{k} & -114\mathbf{i}-315\mathbf{j}+57\mathbf{k} & 57+315\mathbf{i}-114\mathbf{j} \\ -171-135\mathbf{i}+342\mathbf{j} & -45+57\mathbf{i}+114\mathbf{k} & 114+57\mathbf{j}+45\mathbf{k} \end{bmatrix}.$$

2. Taking into account

$$\sum_{\beta \in J_{2,\,4}} \left| \left(\mathbf{A}^5\right)^* \mathbf{A}^5 \right|_\beta^\beta = 25, \ \sum_{\alpha \in I_{1,\,3}} \left| \mathbf{B}^5 \left(\mathbf{B}^5\right)^* \right|_\alpha^\alpha = \operatorname{tr}\left(\mathbf{B}^5 \left(\mathbf{B}^5\right)^*\right) = 33,$$

from (4.1) it follows that

$$\mathbf{X} = \begin{bmatrix} -19-105\mathbf{i}-57\mathbf{j}+15\mathbf{k} & 105-19\mathbf{i}+15\mathbf{j}+57\mathbf{k} & 57-15\mathbf{i}-19\mathbf{j}-105\mathbf{k} \\ 38\mathbf{i}+15\mathbf{j}+19\mathbf{k} & -38+19\mathbf{j}-15\mathbf{k} & -15-19\mathbf{i}+38\mathbf{k} \\ -38-19\mathbf{j}-105\mathbf{k} & -38\mathbf{i}-105\mathbf{j}+19\mathbf{k} & 19+105\mathbf{i}-38\mathbf{j} \\ 19+15\mathbf{i}-38\mathbf{j} & -15+19\mathbf{i}+38\mathbf{k} & 38+19\mathbf{j}+15\mathbf{k} \end{bmatrix}$$

is the solution to the Q-RME (1.3).

Similarly:

$$\mathbf{X} = \begin{bmatrix} 13-39\mathbf{i}+13\mathbf{j}-65\mathbf{k} & -21-7\mathbf{i}+35\mathbf{j}+7\mathbf{k} & 3-15\mathbf{i}+3\mathbf{j}+9\mathbf{k} \\ -52-13\mathbf{j}-13\mathbf{k} & 28\mathbf{i}+7\mathbf{j}-7\mathbf{k} & -3-3\mathbf{i}+12\mathbf{j} \\ -52\mathbf{i}+13\mathbf{j}-13\mathbf{k} & -28+7\mathbf{j}+7\mathbf{k} & 3-3\mathbf{i}+12\mathbf{k} \\ 13+13\mathbf{i}-52\mathbf{k} & 7-7\mathbf{i}+28\mathbf{j} & -12\mathbf{i}-3\mathbf{j}-3\mathbf{k} \end{bmatrix}$$

is the solution to the Q-RMEs (1.4)-(3.8);

$$\mathbf{X} = \begin{bmatrix} 13-65\mathbf{i}+39\mathbf{j}-65\mathbf{k} & -35-7\mathbf{i}+35\mathbf{j}+21\mathbf{k} & 9-15\mathbf{i}-3\mathbf{j}+15\mathbf{k} \\ -52-26\mathbf{i}-117\mathbf{j}-13\mathbf{k} & -14+28\mathbf{i}+7\mathbf{j}-63\mathbf{k} & -27-3\mathbf{i}+12\mathbf{j}+6\mathbf{k} \\ 26-52\mathbf{i}+13\mathbf{j}-13\mathbf{k} & -28-14\mathbf{i}+7\mathbf{j}+7\mathbf{k} & 3-3\mathbf{i}-6\mathbf{j}+12\mathbf{k} \\ -13-117\mathbf{i}+26\mathbf{j}+52\mathbf{k} & -63+7\mathbf{i}-28\mathbf{j}+14\mathbf{k} & 6+12\mathbf{i}+3\mathbf{j}+27\mathbf{k} \end{bmatrix}$$

is the solution to the Q-RMEs (3.12)-(3.13);

$$\mathbf{X} = \begin{bmatrix} -19-15\mathbf{i}-19\mathbf{j}+15\mathbf{k} & 15-19\mathbf{i}+15\mathbf{j}+19\mathbf{k} & 19-15\mathbf{i}-19\mathbf{j}-15\mathbf{k} \\ 15\mathbf{j}+19\mathbf{k} & 19\mathbf{j}-15\mathbf{k} & -15-19\mathbf{i} \\ -19\mathbf{j}+15\mathbf{k} & 15\mathbf{j}+19\mathbf{k} & 19-15\mathbf{i} \\ -19-15\mathbf{i} & 15-19\mathbf{i} & -19\mathbf{j}-15\mathbf{k} \end{bmatrix}$$

is the solution to the Q-RMEs (3.15)-(3.16).

6. CONCLUSION

Our principal outcomes are related with solving the quaternion restricted two-sided matrix equation $\mathbf{AXB} = \mathbf{C}$. The study encompasses all possible two-sided quaternion matrix equations with restrictions on matrix spaces, such as ranges and kernels, wherein their solutions are uniquely determined by the Drazin-star and star-Drazin matrices. The obtained determinantal representations of these matrices are then utilized to solve the equations using Cramer's rules with noncommutative row-column determinants in the case of quaternion matrix equations. A numerical example is provided to illustrate the obtained results.

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