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NEW FAST EXTRAGRADIENT-LIKE METHODS FOR NON-LIPSCHITZIAN PSEUDO-MONOTONE VARIATIONAL INEQUALITIES

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ABSTRACT. An efficient double-projection method, with a new search strategy, is designed for solving variational inequalities in real Hilbert spaces with pseudo-monotone cost operator. Our proposed method uses a computationally inexpensive simple line search procedure based on local information of the operator and very weak conditions of parameters to obtain larger step sizes. A description of the algorithm along with its weak convergence is provided without assuming Lipschitz continuity. Also, a modification to the proposed method is presented, wherein the second projection onto the closed and convex subset is replaced with the one onto a subgradient half space. Numerical experiments and comparisons with related methods demonstrate the reliability and benefits of the proposed schemes.

Key words and phrases: Variational inequality problem; Pseudo-monotonicity; Non-Lipschitz continuity; extragradient method; Weak convergence.

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1. INTRODUCTION

In this work, we are interested in the investigation of a double-projection method for discovering a solution to the variational inequality problem(VIP), which is formulated as follows:

$$\text{Find } x^* \in \mathcal{C} \text{ such that } \langle \mathcal{F}(x^*), y - x^* \rangle \geq 0,\ \forall y \in \mathcal{C}, \tag{1.1}$$

where $\mathcal{C}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Here, $\langle .,. \rangle$ denotes the usual inner product in $\mathcal{H}$ and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is a given operator. This problem has been widely used in mathematical programming to model optimization and decision-making problems in various fields such as finance, economics, network analysis, transportation, elasticity, optimal control and so on [1, 3, 9, 19, 32]. Over the years, there have been numerous studies focusing on algorithmic development in this area [10, 21, 27, 30].

The projection method, with its different forms, is a fundamental technique for obtaining a solution to the VIP due to its simplicity and applicability. This method is inspired by a well-known theorem stating that x^* is a solution of (1.1) if and only if it be a fixed point of $\mathcal{P}_\mathcal{C}(x - \alpha\mathcal{F}(x))$, i.e., $x^* = \mathcal{P}_\mathcal{C}(x^* - \alpha\mathcal{F}(x^*))$, where $\mathcal{P}_\mathcal{C}$ is the (metric) projection of $\mathcal{H}$ onto $\mathcal{C}$ and α is any positive real number. Based on this alternative formulation, the simplest solution method for VIPs constructed as the following:

$$x_{n+1} = \mathcal{P}_\mathcal{C}(x_n - \alpha\mathcal{F}(x_n)), \quad \forall n \geq 1.$$

However, the convergence of this method requires an $\mathcal{L}$-Lipschitz continuity and strong (or inverse strong) monotonicity on $\mathcal{F}$. These assumptions can be quite restrictive in practice.

To overcome this drawback, Korpelevich [20] proposed a double-projection method, so-called the extragradient method, as

$$\begin{cases} y_n = \mathcal{P}_\mathcal{C}(x_n - \alpha\mathcal{F}(x_n)), \\ x_{n+1} = \mathcal{P}_\mathcal{C}(x_n - \alpha\mathcal{F}(y_n)), \end{cases}$$

for all $n \geq 1$, where $\alpha \in (0, \frac{1}{\mathcal{L}})$ and $\mathcal{F}$ is the $\mathcal{L}$-Lipschitz continuous and monotone mapping.

Because of its importance, some new ideas and results in the development of extragradient method have been proposed in various ways [5, 7, 10, 24, 27], including line search procedures, avoiding Lipschitz continuity assumptions, extension in monotonicity of the assigned mapping, etc.

One of the primary considerations affecting the efficiency of the projection methods is the selection of the appropriate step size. Motivated and inspired by the extragradient method and some related works, in this paper, we develop a line search strategy to select the proper step size and introduce a new fast double-projection algorithm, comparable with the related well-known algorithms. This strategy at least theoretically allows for an increase in step size from the first projection to the second projection, which reduces the distance to the solution. Figure 1 illustrates the benefits of selecting a larger step size for the second projection.

The main features and qualities of this work are: (1) We propose a new double-projection method which is developed for solving pseudo-monotone VIPs. (2) In contrast to some methods proposed in the literature (for example see [17, 24, 27]), the new method works independently of the Lipschitz constant of the involving mapping $\mathcal{F}$, if be Lipschitz. This feature is particularly interesting when the Lipschitz constant is either unknown or difficult to approximate. For example, considering the following constrained optimization

$$\begin{aligned} &\min f(x) \\ &s.t.\ h_i(x) \leq 0, i = 1, ..., m, \end{aligned}$$

where $h_i : \mathcal{H} \to \mathcal{R}, i = 1, ..., m$, are given convex and continuous functions and the objective function $f : \mathcal{H} \to \mathcal{R}$ is a convex and twice continuously differentiable function. Then, the

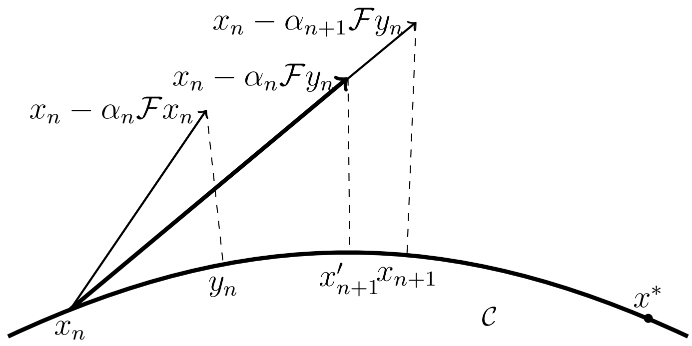

Figure 1: Benefits of employing the larger stepsize α_{n+1} for the second projection.

above constrained optimization problem is equivalent to (1.1) with $\mathcal{F} = \bigtriangledown f$. It is not difficult to see that $\bigtriangledown f$ is a Lipschitz continuous operator with constant

$$\mathcal{L} = \max\{\bigtriangledown^2 f(x) : h_i(x) \leq 0, i = 1, ..., m\}.$$

This means that finding or estimating $\mathcal{L}$ results in an optimization problem, which is an unexpected time-consuming task. (3) To overcome the knowledge of the Lipschitz constant difficulty, some new methods are proposed in which convergence of these methods has been shown under the assumption of Lipschitz continuity of $\mathcal{F}$ [2, 14, 15, 18, 22, 23, 31]. In contrast to these methods, the convergence analysis of our algorithm only requires a locally Lipschitz assumption on $\mathcal{F}$, which is weaker than Lipschitz continuity of $\mathcal{F}$. (4) To the best of our knowledge, the step size sequences of the double-projection methods are nonincreasing from iteration to iteration. In this work, after some iterations, the step size sequence $\{\alpha_n\}_{n\geq\mathcal{N}_0}$ is updated in an increasing way, which can significantly enhances the convergence rate and yields better numerical results in terms of both iteration count and execution time. In particular, this method performs a new search strategy based on a computationally inexpensive simple line search to compute the next larger step size.

The rest of the paper is organized as follows. In the next section, we provide some useful lemmas and notations for reference. In Section 3, we introduce a new double-projection method. We also analyze the convergence of the sequence generated by this method. At the end of the section, we propose a modified version of our algorithm which needs only one projection onto the feasible set. In Section 4, we present some numerical experiments to showcase the efficiency of the algorithms and provide a computational overview through a comparison with the performance of some related methods. The last section, Section 5, concludes the paper.

2. PRELIMINARIES

In this section, we collect some notations, definitions and lemmas for reference. The weak convergence of $\{x_n\}$ to x^* is denoted by $x_n \rightharpoonup x^*$ as $n \to \infty$.

The following lemma states some useful properties of the projection [4].

Lemma 2.1. *Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Then for each $y \in \mathcal{C}$*

(i) $\langle x - \mathcal{P}_\mathcal{C}(x), y - \mathcal{P}_\mathcal{C}(x)\rangle \leq 0$;
(ii) $\|\mathcal{P}_\mathcal{C}(x) - y\|^2 \leq \|x - y\|^2 - \|x - \mathcal{P}_\mathcal{C}(x)\|^2$.

We review some pertinent for future reference.

Definition 2.1. The mapping $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is said to be

a) pseudo-monotone if

$$\langle \mathcal{F}(x), y - x\rangle \geq 0 \Rightarrow \langle \mathcal{F}(y), y - x\rangle \geq 0, \quad \forall x, y \in \mathcal{H};$$

b) monotone if

$$\langle \mathcal{F}(x) - \mathcal{F}(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H};$$

c) Lipschitz continuous if there exists $\mathcal{L} > 0$ such that

$$\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq \mathcal{L}\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

d) locally Lipschitz continuous if for any $u \in \mathcal{H}$ there is $\rho > 0$ and $\mathcal{L} \geq 0$ such that

$$\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq \mathcal{L}\|x - y\|, \quad \forall x, y \in \mathcal{B}(u, \rho);$$

e) sequentially weakly continuous if for any sequence $\{x_n\}$ weakly converging to x^*, the sequence $\{\mathcal{F}(x_n)\}$ converges weakly to $\mathcal{F}(x^*)$.

Remark 2.1. **(i)** It is clear that monotonicity implies pseudo-monotonicity, yet the converse is not true in general. For instance, define $\mathcal{F} : \mathcal{R}^+ \to \mathcal{R}^+$ by $\mathcal{F}(x) = \dfrac{v}{x+v}$ with $v > 0$. Suppose $x, y \in \mathcal{R}^+$, $x \neq y$, then

$$\langle \mathcal{F}(x) - \mathcal{F}(y), x - y\rangle = \frac{-v(x-y)^2}{(x+v)(y+v)} < 0.$$

This shows that $\mathcal{F}$ is not monotone, but it is pseudo-monotone. Indeed, let $x, y \in \mathcal{R}^+$. Clearly, if $\langle \mathcal{F}(x), y - x\rangle \geq 0$ then $y \geq x$. Since $\mathcal{F}(y) > 0$, so $\langle \mathcal{F}(y), y - x\rangle \geq 0$. Hence, $\mathcal{F}$ is pseudo-monotone.
(ii) Lipschitz continuity implies locally Lipschitz continuity, but the converse does not generally hold. For example, it is not difficult to show that the real-valued function $\mathcal{F}(x) = x^2$ is locally Lipschitz continuous but not Lipschitz continuous on $\mathcal{R}$.

The following lemma is a standard result in mathematical analysis.

Lemma 2.2. *A locally Lipschitz continuous mapping $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ on a bounded set $\mathcal{D} \subset \mathcal{H}$ is Lipschitz continuous.*

The well-known Minty lemma is as follows[6].

Lemma 2.3. *Consider the VIP* (1.1) *with $\mathcal{F} : \mathcal{C} \to \mathcal{H}$ being pseudo-monotone and continuous. Then, x^* is a solution of* (1.1) *if and only if*

$$\langle \mathcal{F}(y), y - x^*\rangle \geq 0, \ \forall y \in \mathcal{C}.$$

The proof of the following lemma, which will be used for the subsequent convergence analysis, is straightforward.

Lemma 2.4. *Suppose that $\{a_n\}$ and $\{b_n\}$ are two nonnegative real sequences that*

$$a_{n+1} \leq a_n - b_n,$$

for all n. Then $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n = 0$.

3. THE PROPOSED METHODS AND CONVERGENCE ANALYSIS

In this section, we introduce our proposed algorithms and show that the sequences produced by these algorithms are weakly convergent to a solution of the VIP (1.1). For the convergence analysis of the proposed algorithms, we assume the following:

A1: $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz, pseudo-monotone and sequentially weakly continuous on $\mathcal{C}$.

A2: The solution set of (1.1), denoted by $\mathcal{S}$, is nonempty.

The construction of our first algorithm is as follows.

Algorithm 1

Initialization: Choose $\sigma \in (0, 1), \theta \in (0, 1), \alpha_0 > 0, \alpha_{\max} > 0$ and $x_0 \in \mathcal{C}$.
Iterative Steps: Given the current iterates x_n and α_n, calculate the next one x_{n+1} and α_{n+1} as follows:
Step 1. Set $j = 0$ and run the following line search:

1.a. Take $\beta_n = \theta^j$ and compute

$$y_n = \mathcal{P}_{\mathcal{C}}(x_n - \beta_n \alpha_n \mathcal{F}(x_n)).$$

1.b. Choose the largest $\alpha_{n+1} \leq \min\{\frac{1+\beta_n}{\beta_n}\alpha_n, \alpha_{\max}\}$ such that

$$\|\alpha_{n+1}\mathcal{F}(y_n) - \beta_n \alpha_n \mathcal{F}(x_n)\| \leq \sigma \|y_n - x_n\|. \tag{3.1}$$

1.c. Break line search if such α_{n+1} exists. Otherwise, set $j := j + 1$ and go to **1.a.**

If $x_n = y_n$, then stop; x_n belongs to $\mathcal{S}$. Otherwise,
Step 2. Compute

$$x_{n+1} = \mathcal{P}_{\mathcal{C}}(x_n - \alpha_{n+1}\mathcal{F}(y_n)).$$

Set $n := n + 1$ and go to **Step 1.**

Remark 3.1. **(i)** To ensure that the sequence $\{\alpha_n\}_{n \in \mathcal{N}}$ in Algorithm 1 has an upper bound, the constant $\alpha_{\max}$ is given. Therefore, it makes sense to choose this constant quite large. **(ii)** The largest α_{n+1} that satisfies (3.1) can be easily found by solving equivalent quadratic equation.

The following two lemmas show that Algorithm 1 is well defined.

Lemma 3.1. *The step size calculation in line search of Algorithm 1 terminates after finitely many inner loops.*

Proof. Suppose that the line search procedure in Algorithm 1 fails to terminate at n-th iteration i.e., for all $\alpha \in (0, \frac{1+\beta_n}{\beta_n}\alpha_n)$ with $\beta_n = \theta^j$ and $j = 0, 1, 2, ...$, we have

$$\sigma \|y_n - x_n\| < \|\alpha \mathcal{F}(y_n) - \beta_n \alpha_n \mathcal{F}(x_n)\|. \tag{3.2}$$

We show that this leads to a contradiction. Let $\mathcal{D} = conv\{x_n, y_n\}$. Since $\mathcal{D}$ is a bounded set it follows from Lemma 2.2 that $\mathcal{F}$ is Lipschitz continuous on $\mathcal{D}$. Hence, there exists a positive constant $\mathcal{L}$ such that

$$\|\mathcal{F}(y_n) - \mathcal{F}(x_n)\| \leq \mathcal{L}\sigma \|y_n - x_n\|. \tag{3.3}$$

Taking $\beta_n < \frac{1}{\mathcal{L}\alpha_n}$ and setting $\alpha = \beta_n \alpha_n$, from (3.2) and (3.3) we have

$$\sigma \|y_n - x_n\| < \|\alpha \mathcal{F}(y_n) - \beta_n \alpha_n \mathcal{F}(x_n)\| = \beta_n \alpha_n \|\mathcal{F}(y_n) - \mathcal{F}(x_n)\| < \sigma \|y_n - x_n\|,$$

a contradiction. This completes the proof. ∎

Lemma 3.2. *If* $\{x_n\}$*, the sequence generated by Algorithm 1, be bounded then* $\{\alpha_n\}$ *is bounded and* $\limsup_{n\to\infty} \alpha_n > 0$.

Proof. By Remark 3.1, $\{\alpha_n\}$ is upper bounded. As the sequences $\{x_n\}$ and $\{y_n\}$ are bounded, recall that $\mathcal{F}$ is a locally Lipschitz continuous mapping from assumption A1, there exists any positive constant $\mathcal{L}$ such that

$$\|\mathcal{F}(y_n) - \mathcal{F}(x_n)\| \leq \mathcal{L}\sigma\|y_n - x_n\| \quad \forall n \in \mathcal{N}.$$

Given the construction of $\{\alpha_n\}$, it is not difficult to see that if we have $\alpha_n < \dfrac{1}{\mathcal{L}}$, then $\beta_n = \theta^0 = 1$ and $\alpha = \alpha_n$ satisfy inequality

$$\|\alpha\mathcal{F}(y_n) - \beta_n\alpha_n\mathcal{F}(x_n)\| \leq \sigma\|y_n - x_n\|,$$

i.e., the line search procedure terminates after the first iteration. Since we seek the largest $\alpha \in (0, \dfrac{1+\beta_n}{\beta_n}\alpha_n)$, we have $\alpha_n \leq \alpha_{n+1}$.

Now, by a contradiction suppose that $\limsup_{n\to\infty} \alpha_n = 0$. So, there exists a positive integer $\mathcal{N}_0$ such that $\alpha_n < \dfrac{1}{\mathcal{L}}$ whenever $n \geq \mathcal{N}_0$. Let $n+1 > \mathcal{N}_0$. Considering $\alpha_n < \dfrac{1}{\mathcal{L}}$, we obtain $\alpha_n \leq \alpha_{n+1}$. Moreover, as $\alpha_{n+1} < \dfrac{1}{\mathcal{L}}$, we also have $\alpha_{n+1} \leq \alpha_{n+2}$. It follows by induction that $\{\alpha_n\}_{n\geq\mathcal{N}_0}$ is nondecreasing and can not converge to zero. Thus our initial assumption has led to a contradiction, proving the lemma. ∎

Now, we give and prove the following lemmas, which are important in the proof of the main theorem.

Lemma 3.3. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 1. If* $\{x_n\}$ *be bounded and* $\lim_{n\to\infty} \|x_n - y_n\| = 0$*, then it has a weak cluster point belongs to* $\mathcal{S}$.

Proof. By Lemma 3.2, there exists an increasing sequence of positive integers, $\{n_k\}$, such that $\{\alpha_{n_k}\}$, as well as $\{\beta_{n_k}\alpha_{n_k}\}$ according to Lemma 3.1, is separated from zero and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that for some $x^* \in \mathcal{C}$, $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$.

Now, we show that x^* belongs to $\mathcal{S}$. Indeed, since $x_{n_k} \rightharpoonup x^*$ and $\lim_{n\to\infty} \|y_n - x_n\| = 0$, thus $y_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Further, $x^* \in \mathcal{C}$ due to $\{x_n\} \subset \mathcal{C}$. Applying Lemma 2.1 (i), we deduce that

$$\langle x_{n_k} - \beta_{n_k}\alpha_{n_k}\mathcal{F}(x_{n_k}) - y_{n_k}, p - y_{n_k}\rangle \leq 0, \quad \forall p \in \mathcal{C},$$

i.e.,

$$\frac{1}{\beta_{n_k}\alpha_{n_k}}\langle x_{n_k} - y_{n_k}, p - y_{n_k}\rangle \leq \langle \mathcal{F}(x_{n_k}), p - y_{n_k}\rangle, \quad \forall p \in \mathcal{C}.$$

This implies that

$$\frac{1}{\beta_{n_k}\alpha_{n_k}}\langle x_{n_k} - y_{n_k}, p - y_{n_k}\rangle + \langle \mathcal{F}(x_{n_k}), y_{n_k} - x_{n_k}\rangle \leq \langle \mathcal{F}(x_{n_k}), p - x_{n_k}\rangle, \quad \forall p \in \mathcal{C}. \tag{3.4}$$

Fixing $p \in \mathcal{C}$ and taking $k \to \infty$ in (3.4), since $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and $\beta_{n_k}\alpha_{n_k} > 0$ for all k, we get

$$\liminf_{k\to\infty} \langle \mathcal{F}(x_{n_k}), p - x_{n_k}\rangle \geq 0. \tag{3.5}$$

Now, we choose a decreasing sequence of positive numbers, $\{\epsilon_k\}_k$, tending to zero. By (3.5), for each k, there exists the smallest positive integer $\mathcal{N}_k$ such that

$$\langle \mathcal{F}(x_{n_j}), p - x_{n_j}\rangle + \epsilon_k \geq 0, \quad \forall j \geq \mathcal{N}_k. \tag{3.6}$$

From the decreasing nature of sequence $\{\epsilon_k\}_k$, it follows that the sequence $\{\mathcal{N}_k\}_k$ is increasing. For each k, suppose $\mathcal{F}(x_{n_{\mathcal{N}_k}}) \neq 0$ (otherwise, $x_{n_{\mathcal{N}_k}}$ is a solution). Setting

$$\nu_{n_{\mathcal{N}_k}} = \frac{\mathcal{F}(x_{n_{\mathcal{N}_k}})}{\|\mathcal{F}(x_{n_{\mathcal{N}_k}})\|^2},$$

we have $\epsilon_k = \langle \mathcal{F}(x_{n_{\mathcal{N}_k}}), \epsilon_k \nu_{n_{\mathcal{N}_k}}\rangle$ for each k. So, from (3.6) we get

$$\langle \mathcal{F}(x_{n_{\mathcal{N}_k}}), p + \epsilon_k \nu_{n_{\mathcal{N}_k}} - x_{n_{\mathcal{N}_k}}\rangle \geq 0, \quad \forall k.$$

Since $\mathcal{F}$ is pseudo-monotone, then

$$\langle \mathcal{F}(p + \epsilon_k \nu_{n_{\mathcal{N}_k}}), p + \epsilon_k \nu_{n_{\mathcal{N}_k}} - x_{n_{\mathcal{N}_k}}\rangle \geq 0. \tag{3.7}$$

Taking the limit as $k \to \infty$ in (3.7), since $\{x_{n_{\mathcal{N}_k}}\} \subset \{x_{n_k}\}$ and $x_{n_k} \rightharpoonup x^*$, to obtain $\langle \mathcal{F}(p), p - x^*\rangle \geq 0$, i.e., to demonstrate that x^* belongs to $\mathcal{S}$, by Lemma 2.3, it suffices to show that $\lim_{k\to\infty} \epsilon_k \nu_{n_{\mathcal{N}_k}} = 0$.

Since $\mathcal{F}$ is sequentially weakly continuous on $\mathcal{C}$, then $\mathcal{F}(x_{n_k}) \rightharpoonup \mathcal{F}(x^*)$. Assuming that $\mathcal{F}(x^*) \neq 0$ (otherwise, x^* is a solution), from sequentially weakly lower semicontinuity of norm, it follows that

$$0 < \|\mathcal{F}(x^*)\| \leq \liminf_{k\to\infty} \|\mathcal{F}(x_{n_k})\|.$$

So,

$$0 \leq \lim_{k\to\infty} \|\epsilon_k \nu_{n_{\mathcal{N}_k}}\| = \lim_{k\to\infty} \frac{\epsilon_k}{\|\mathcal{F}(x_{n_k})\|} \leq \frac{0}{\|\mathcal{F}(x^*)\|} = 0.$$

It would imply that $\lim_{k\to\infty} \epsilon_k \nu_{n_{\mathcal{N}_k}} = 0$ and the proof is completed. ∎

Lemma 3.4. *Assume that $\{x_n\}$ and $\{y_n\}$ are generated by Algorithm 1 and let $u \in \mathcal{S}$. Then, for every $n \in \mathcal{N}$, we have*

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 - (1 - \sigma^2)\|y_n - x_n\|^2. \tag{3.8}$$

Proof. Take any $u \in \mathcal{S}$. By Lemma 2.1 (ii), we have

$$\begin{aligned} \|x_{n+1} - u\|^2 &\leq \|x_n - \alpha_{n+1}\mathcal{F}(y_n) - u\|^2 - \|x_n - \alpha_{n+1}\mathcal{F}(y_n) - x_{n+1}\|^2 \\ &= \|x_n - u\|^2 - \|x_n - x_{n+1}\|^2 + 2\alpha_{n+1}\langle \mathcal{F}(y_n), u - x_{n+1}\rangle \\ &= \|x_n - u\|^2 - \|x_n - x_{n+1}\|^2 + 2\alpha_{n+1}\langle \mathcal{F}(y_n), y_n - x_{n+1}\rangle \\ &\quad + 2\alpha_{n+1}\langle \mathcal{F}(y_n), u - y_n\rangle. \end{aligned} \tag{3.9}$$

Since $u \in \mathcal{S}$, by the pseudo-monotonicity of $\mathcal{F}$, we get $\langle \mathcal{F}(y_n), y_n - u\rangle \geq 0$. From this term and (3.9) we have

$$\begin{aligned} \|x_{n+1} - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - x_{n+1}\|^2 + 2\alpha_{n+1}\langle \mathcal{F}(y_n), y_n - x_{n+1}\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 \\ &\quad - 2\langle x_n - y_n, y_n - x_{n+1}\rangle + 2\alpha_{n+1}\langle \mathcal{F}(y_n), y_n - x_{n+1}\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 \\ &\quad + 2\langle x_n - \alpha_{n+1}\mathcal{F}(y_n) - y_n, x_{n+1} - y_n\rangle. \end{aligned} \tag{3.10}$$

Moreover, using Lemma 2.1 (i) yields

$$\begin{aligned}\langle x_n - \alpha_{n+1}\mathcal{F}(y_n) - y_n, x_{n+1} - y_n\rangle &= \langle x_n - \beta_n\alpha_n\mathcal{F}(x_n) - y_n, x_{n+1} - y_n\rangle \\ &\quad + \langle \beta_n\alpha_n\mathcal{F}(x_n) - \alpha_{n+1}\mathcal{F}(y_n), x_{n+1} - y_n\rangle \\ &\leq \langle \beta_n\alpha_n\mathcal{F}(x_n) - \alpha_{n+1}\mathcal{F}(y_n), x_{n+1} - y_n\rangle \\ &\leq \frac{1}{2}\|\alpha_{n+1}\mathcal{F}(y_n) - \beta_n\alpha_n\mathcal{F}(x_n)\|^2 + \frac{1}{2}\|x_{n+1} - y_n\|^2. \qquad (3.11)\end{aligned}$$

Combining (3.10) and (3.11), we thus have shown that

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 + \|\alpha_{n+1}\mathcal{F}(y_n) - \beta_n\alpha_n\mathcal{F}(x_n)\|^2.$$

Taking into account (3.1), we obtain

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 - (1 - \sigma^2)\|y_n - x_n\|^2.$$

That is the desired result. ∎

Now, we are prepared to state and prove our main convergence result of the proposed algorithm.

Theorem 3.5. *Suppose that $\{x_n\}$ is a sequence generated by Algorithm 1. Then $\{x_n\}$ converges weakly to a solution of* (1.1).

Proof. Take any $u \in \mathcal{S}$. Let $a_n = \|x_n - u\|^2$ and $b_n = (1 - \sigma^2)\|y_n - x_n\|^2$, so, (3.8) with Lemma 2.4 imply that the sequence $\{x_n\}$ is bounded and also

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

By Lemma 3.3, $\{x_n\}$ has a weak cluster point $x^* \in \mathcal{S}$.

Finally, we will prove that $x_n \rightharpoonup x^*$. For this purpose, we show that all subsequences of $\{x_n\}$, converge weakly to x^*. As we seen in the proof of Lemma 3.3, there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to a point $\overline{x}$. As demonstrated previously, $\overline{x} \in \mathcal{S}$. It follows from the standard monotone convergence theorem that the sequences $\{\|x_n - x^*\|\}$ and $\{\|x_n - \overline{x}\|\}$ converge, since, both are monotonically decreasing, applying Lemma 3.4. For each $n \in \mathcal{N}$, we can display $\langle x_n, \overline{x} - x^*\rangle$ as

$$\langle x_n, \overline{x} - x^*\rangle = \frac{1}{2}\|x_n - x^*\|^2 - \frac{1}{2}\|x_n - \overline{x}\|^2 + \frac{1}{2}\|\overline{x}\|^2 - \frac{1}{2}\|x^*\|^2.$$

This implies that the sequence $\{\langle x_n, \overline{x} - x^*\rangle\}$, as well as $\{\langle x_{n_k}, \overline{x} - x^*\rangle\}$ and $\{\langle x_{n_j}, \overline{x} - x^*\rangle\}$, converges to a limit point Λ. Thus

$$\lim_{n\to\infty} \langle x_{n_k}, \overline{x} - x^*\rangle = \lim_{n\to\infty} \langle x_{n_j}, \overline{x} - x^*\rangle = \Lambda,$$

i.e.,

$$\langle x^*, \overline{x} - x^*\rangle = \langle \overline{x}, \overline{x} - x^*\rangle = \Lambda.$$

This would impliy that $\|\overline{x} - x^*\|^2 = 0$ and therefore $\overline{x} = x^*$. So, $x_n \rightharpoonup x^*$ and the proof is completed. ∎

We next propose a modification of Algorithm 1 that simplifies the computation process by using only one projection onto the feasible set, while the second has a clear formula. This modification is advantageous, particularly in cases where the feasible set has a complex structure, leading to time-consuming projection computations.

The proof of the following convergence theorem is similar to that of Theorem 3.5, and is thus omitted.

Theorem 3.6. *The conclusion of Theorem 3.5 remains true for Algorithm 2.*

Algorithm 2

Initialization: Choose $\sigma \in (0,1), \theta \in (0,1), \alpha_0 > 0, \alpha_{\max} > 0$ and $x_0 \in \mathcal{C}$.
Iterative Steps: Given the current iterates x_n and α_n, calculate x_{n+1} and α_{n+1} as follows:
Step 1. Set $i = 0$ and run the following line search:

1.a. Take $\beta_n = \theta^i$ and compute

$$y_n = \mathcal{P}_{\mathcal{C}}(x_n - \beta_n \alpha_n \mathcal{F}(x_n)).$$

1.b. Choose the largest $\alpha_{n+1} \leq \min\{\frac{1+\beta_n}{\beta_n}\alpha_n, \alpha_{\max}\}$ such that

$$\|\alpha_{n+1}\mathcal{F}(y_n) - \beta_n \alpha_n \mathcal{F}(x_n)\| \leq \sigma \|y_n - x_n\|. \tag{3.12}$$

1.c. Break line search if such α_{n+1} exists. Otherwise, set $i := i+1$ and go to **1.a.**

If $x_n = y_n$, then stop; x_n belongs to $\mathcal{S}$. Otherwise,
Step 2. Compute

$$x_{n+1} = \mathcal{P}_{\mathcal{T}_n}(x_n - \alpha_{n+1}\mathcal{F}(y_n)),$$

where

$$\mathcal{T}_n = \{z \in \mathcal{H} : \langle x_n - \beta_n \alpha_n \mathcal{F}(x_n) - y_n, z - y_n\rangle \leq 0\}.$$

Set $n := n+1$ and go to **Step 1.**

4. NUMERICAL RESULTS

In this section, we perform some experiments to illustrate the effectiveness and implementation of the proposed algorithms, namely Algorithm 1 (called Alg.1) and Algorithm 2 (Alg.2). We use two recently proposed algorithms as the modified extragradient-like algorithm (called Alg.3) of Hieu et al. [12, Algorithm 1] and the modified projection-type method (called Alg.4) of Thong et al. [29, Algorithm 2] to compare with our algorithms. Alg.3 uses a step size rule without any line search procedure, while, Alg.4 uses a line search approach to find a proper step size over each iteration. As be seen in [12, Sect. 5] and [29, Sect. 4], the Alg.3 and Alg.4 work better than several well-known algorithms presented in [11, 13, 25, 28, 33]. We tested the effectiveness of these algorithms to solve two academic test problems. All experiments were implemented in Python 3.7 on a Lenovo laptop with the following specifications: an Intel(R) Core(TM) i5-5200U CPU at 2.20 GHz , 8.00 GB of RAM running 64-bit Windows 10 Enterprise.

For numerical results, we terminate the iterations if the error term $\mathcal{D}_n = \|x_{n+1} - y_n\| + \|y_n - x_n\| \leq 10^{-6}$. For the projection onto the feasible set $\mathcal{C}$, we use the algorithm from [8]. From [4, p. 133], the projection onto the half-space $\mathcal{T} = \{x \in \mathcal{H} : \langle c, x\rangle \leq d\}$ with $0 \neq c \in \mathcal{H}$ and $d \in \mathcal{R}$ is calculated as

$$\mathcal{P}_{\mathcal{T}}(x) = \begin{cases} x + \frac{d - \langle c,x\rangle}{\|c\|^2}c, & \text{if } \langle c, x\rangle > d; \\ x, & \text{if } \langle c, x\rangle \leq d. \end{cases}$$

We choose $\sigma = 0.7$ and $\theta = 0.9$ for Alg.1 and Alg.2, $\mu = 0.4$ for Alg.3, as in original paper, and $l = 0.1$, $\gamma = 2$ and $\mu = 0.9$ for Alg.4. For the initial parameter α_0 for Alg.1, Alg.2 and Alg.3 (also α_1 for Alg.3) we choose any $\hat{x}$ as a small perturbation of the initial point x_0 and take $\alpha_0 (= \alpha_1) = \frac{\sigma\|\hat{x} - x_0\|}{\|\mathcal{F}(\hat{x}) - \mathcal{F}(x_0)\|}$.

We report the number of iterations (Iter.) and the running CPU time (Time) measured in seconds. The bold letter indicates the best results in the following tables.

Example 4.1. ***(Sun's problem)*** *In this example, we apply our proposed algorithm to the classical nonlinear test problem, presented by Sun in* [26]*, with* $m = 10^3, 10^4, 10^5$*. The feasible set is* $\mathcal{C} = \{x \in \mathcal{R}_+^m : x_1+x_2+\cdots+x_m = m\}$ *and the operator* $\mathcal{F}$ *is given as* $\mathcal{F}(x) = \mathcal{F}_1(x)+\mathcal{F}_2(x)$*, where*

$$\begin{aligned}
\mathcal{F}_1(x) &= (f_1(x), f_2(x), \dots, f_m(x)),\\
\mathcal{F}_2(x) &= \mathcal{D}x + c,\\
f_i(x) &= x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_i x_{i+1},\ i = 1, 2, \dots, m,\\
x_0 &= x_{m+1} = 0.
\end{aligned}$$

In the above definition of the operator $\mathcal{F}$*,* $\mathcal{D}$ *is a* $m \times m$ *matrix defined by condition*

$$d_{ij} = \begin{cases} 4 & i = j,\\ 1 & i - j = 1,\\ -2 & i - j = -1,\\ 0 & \textit{otherwise}, \end{cases}$$

and $c = (-1, -1, \dots, -1)$*. The initial point* x_0 *is generated uniformly randomly from* $[0, 10]^m$*. The numerical results for this example with* $m = 10^3$*,* $m = 10^4$ *and* $m = 10^5$ *are described in Fig. 2, Fig. 3 and Fig. 4, respectively, and are presented in Table 4.1.*

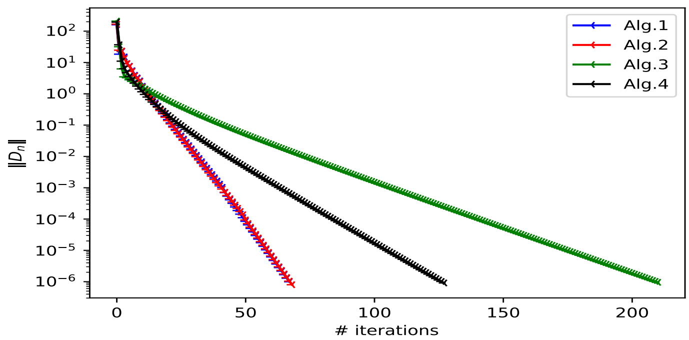

Figure 2: Convergence plot for Example 4.1 with $m = 10^3$. Execution times are 0.0469, 0.0312, 0.0625 and 0.1094, respectively.

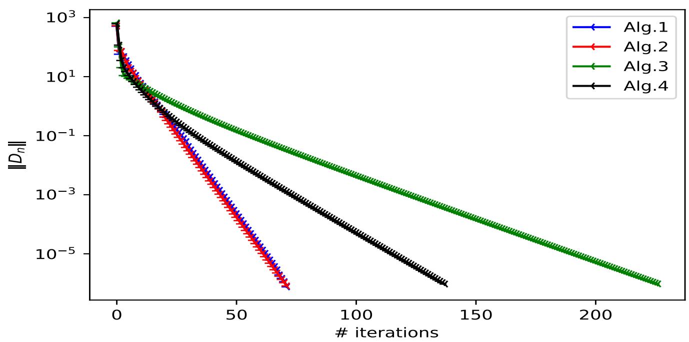

Figure 3: Convergence plot for Example 4.1 with $m = 10^4$. Execution times are 0.2753, 0.1829, 0.3948 and 0.6406, respectively.

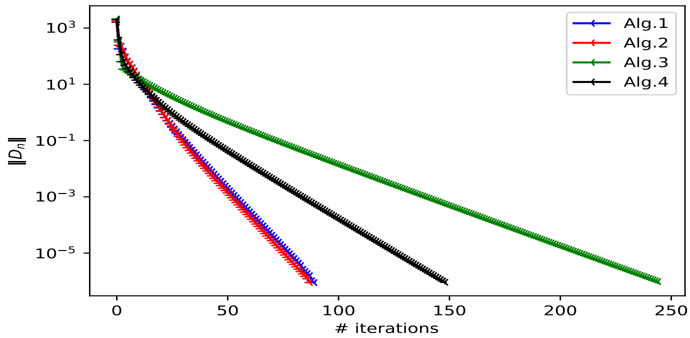

Figure 4: Convergence plot for Example 4.1 with $m = 10^5$. Execution times are 3.2758, 2.2186, 4.9854 and 7.7209, respectively.

Table 4.1: The numerical results for Example 4.1.

Algorithms	$m = 10^3$		$m = 10^4$		$m = 10^5$	
	Iter.	*Time*	*Iter.*	*Time*	*Iter.*	*Time*
Alg.1	*68*	*0.0469*	*72*	*0.2753*	*90*	*3.2758*
Alg.2	*69*	***0.0312***	*72*	***0.1829***	*88*	***2.2186***
Alg.3	*211*	*0.0625*	*227*	*0.3948*	*245*	*4.9854*
Alg.4	*128*	*0.1094*	*138*	*0.6406*	*149*	*7.7209*

Example 4.2. ***(Kojima–Shindo problem)*** *This problem is introduced in* [16]*, as a nonlinear complementarity problem, with the feasible set $\mathcal{C} = \{x \in \mathbb{R}^4_+ : x_1 + x_2 + x_3 + x_4 = 4\}$ and the operator $F : \mathcal{R}^4 \to \mathcal{R}^4$ as*

$$\mathcal{F}(x) = \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}.$$

The results, for this example, are presented in Table 4.2 and in Figures 5 and 6 for two different starting points $x_0 = (1, 1, 1, 1)$ and $x_0 = (4, 0, 0, 0)$, respectively.

Table 4.2: The numerical results for Example 4.2.

Algorithms	$x_0 = (1,1,1,1)$		$x_0 = (4,0,0,0)$	
	Iter.	*Time*	*Iter.*	*Time*
Alg.1	*66*	***0.0156***	*57*	***0.0156***
Alg.2	*69*	***0.0156***	*73*	***0.0156***
Alg.3	*114*	***0.0156***	*229*	*0.0312*
Alg.4	*138*	*0.0313*	*165*	*0.0469*

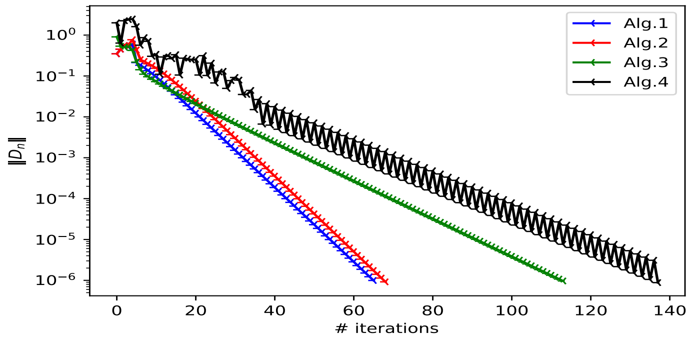

Figure 5: Convergence plot for Example 4.2 with $x_0 = (1, 1, 1, 1)$*. Execution times are* 0.0156*,* 0.0156*,* 0.0156 *and* 0.0313*, respectively.*

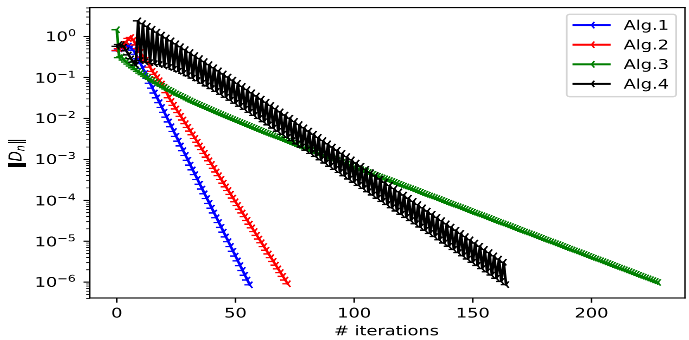

Figure 6: Convergence plot for Example 4.2 with $x_0 = (4, 0, 0, 0)$*. Execution times are* 0.0156*,* 0.0156*,* 0.0312 *and* 0.0469*, respectively.*

The reported results indicate that our proposed algorithms perform well, and have competitive advantages, compared to related algorithms. Fig.7, As an instance, shows the monotonically increasing of $\{\alpha_n\}_{n\geq n_0}$, mentioned in the Preliminaries, with respect to Iter. in example 4.2. The better numerical results may be due to the increasing step sizes of Alg.1 and Alg.2 from one iteration to another.

5. CONCLUSION

In this work, we have introdced a new double-projection method with a new step size rule to find the solution set of a pseudo-monotone VIP in real Hilbert spaces. The new method does not require Lipschitz continuity assumption and local information of the operator is used to construct a simple line search procedure. We have proved a weak convergence theorem of the proposed method under some mild conditions imposed on the parameters. We have introduced a modification to the proposed method which needs only one projection onto the feasible set. Furthermore, Numerical experiments have been performed to demonstrate the convergence of the algorithms and to show the superiority of our proposed schemes over some recently presented methods in the literature.

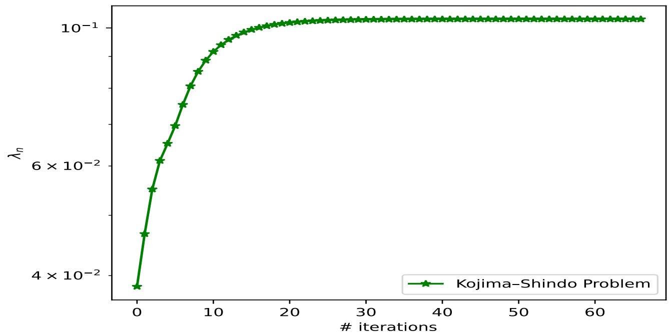

Figure 7: Evolution of $\{\alpha_n\}_{n\geq 0}$ *in Algorithm 1 for Kojima–Shindo problem with* $x_0 = (1, 1, 1, 1)$ *until meeting the termination criterion.*

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