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A GRADIENT ESTIMATE FOR RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper we introduce the Kato class and the non-linear Kato class, on a Riemannian manifold of dimension n. We also obtain a gradient estimate for the non-linear Kato class.

Key words and phrases: Fefferman's inequality, Morrey Space, Kato class.

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1. INTRODUCTION

In his celebrated paper C. Fefferman [5], proved the inequality

$$\int_{\mathbb{R}^n} |f(x)||u(x)|^2\,dx \leq C\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx, \tag{1.1}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$, assuming that the function f belongs to the Morrey space $L^{r,n-2r}(\mathbb{R}^n)$, with $1 < r \leq \frac{n}{2}$ (see [2] for the definition to the Morrey Space $L^{r,n-2r}(\mathbb{R}^n)$).

Later, Chiarenza and Frasca [2] extended Fefferman's result, with a different proof, assuming $f \in L^{r,n-2r}(\mathbb{R}^n)$, $1 < r \leq \frac{n}{p}$, $1 < p < n$.

A different approach to the inequality (1.1) was started with Schechter in [6], where he proved the inequality with f belonging to the Kato class (see definition 2.1 below). In [7], it was proved with $1 < p < n$ and f belonging to a more general class of potentials.

In this paper, we replace the Euclidean space $\mathbb{R}^n$ by Riemannian manifolds, then we investigate the relations between various functional inequalities and the geometry of the manifold.

Using the idea from [7], we provide a generalization of Schechter's result assuming $f \in \widetilde{\mathcal{P}}_p(M)$ (see definition 2.2 and Theorem 4.1).

2. DEFINITIONS AND NOTATION

Let M be a Riemannian manifold of dimension n. There is a canonical distance function associated to the Riemannian structure of M. We will denote it by $(x,y) \longrightarrow d(x,y)$. It can be defined as the shortest length of all piecewise C^1 curves from x to y. The topology of (M,d) as a metric space is the same as that of M as a manifold (see [3] §1.5): There is also a canonical measure on M which we denote by μ (see [3] §3,3).

Definition 2.1 (Kato Class)**.** Let $f \in L^1_{loc}(\mathbb{R}^n)$. For any $r > 0$, we set

$$\phi(f)(r) = \sup_{x\in\mathbb{R}^n} \int_{N(x,r)} \frac{|f(y)|}{|x-y|^{n-2}}\,dy,$$

where $N(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ is an open ball in $\mathbb{R}^n$ with radius r and center x. We say that f belongs to the Kato class $K_n(\mathbb{R}^n)$ if $\phi(f)(r) \to 0$ as $r \to 0^+$.

Definition 2.2 (Nonlinear Kato Class)**.** Let M be a Riemannian manifold of dimension n. We set

$$\phi(f)(r) = \sup_{x\in M}\left(\int_{B(x,r)} \frac{1}{|d(x,y)|^{n-1}}\left(\int_{B(x,r)} \frac{|f(z)|}{[d(y,z)]^{n-1}}\,dz\right)^{\frac{1}{p-1}} dy\right)^{p-1},$$

where $B(x,r) = \{y \in M : d(x,y) < r\}$ is an open ball in M with radius r and center x. We say that $f : M \longrightarrow \mathbb{R}^n$ belongs to the space $\widetilde{\mathcal{P}}_p(M)$ if and only if $\phi(f)(r)$ is finite for any $r > 0$. If, in addition, $\lim\limits_{r\to 0^+} \phi(f)(r) = 0$, then we say that f belongs to the space $\mathcal{P}_p(M)$.

Remark 2.1. Definition 2.2 gives back the classical Kato class (definition 2.1) if we take $p = 2$, $M = \mathbb{R}^n$ and $d(x,y) = |x-y|$ (see [1] and [4]).

3. RICCI CURVATURE

Let (M, g) be a complete Riemannian manifold, where g is the metric tensor. The Ricci curvature tensor $\mathcal{R}$ is a symmetric two-tensor obtained by contraction of the full curvature tensor (see [3]). Thus it can be compared with the metric tensor g.

The following lemma will play a crucial role in the proof of our main result, and it may be well-known to some readers or appear elsewhere in the literature. We have included its proof for the sake of completeness and the convenience of the reader.

Lemma 3.1. *Let (M, g) be a complete Riemannian manifold of dimension n having non-negative Ricci curvature. Then there exists a constant C dependig on n such that*

$$|u(x)| \leq C(n) \int\limits_{B(x_0,r)} \frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\, dy,$$

where $u \in C_0^\infty(M)$ is supported on $B(x_0, r)$.

Proof. Let $u \in C_0^\infty(M)$ supported on $B(x_0, r)$. Now, consider the integral

$$\int\limits_{B(x,R)} |u(x) - u(y)|\, dy \quad (R > 0).$$

To estimate this integral, we use polar (exponential) coordinates around x (see [2, Proposition 3.1]). This gives (in somewhat abusive notation)

$$\begin{aligned}\int\limits_{B(x,R)} |u(x) - u(y)|\, dy &= \int_0^R |u(x) - u(\rho,\theta)| \sqrt{g}(\rho,\theta)\, d\rho d\theta \\ &\leq \int_0^R \int_0^\rho |\partial_1 u(t,\theta)|\, dt \sqrt{g}(\rho,\theta)\, d\rho d\theta \\ &\leq \int_0^R \int_0^\rho |\nabla u(t,\theta)|\, dt \sqrt{g}(\rho,\theta)\, d\rho d\theta.\end{aligned}$$

Here, we have simply used the usual trick to control $u(x) - u(y)$ by integrating along the geodesic segment from x to y and used polar exponential coordinate $y = (\rho, \theta)$ around x. In particular, $\sqrt{g}(\rho,\theta)\, d\rho d\theta = dy$ is by definition the Riemannian volume element in polar coordinates.

We now use the hypothesis (M, g) has non-negative Ricci curvature. By Bishop's Theorem [2, theorem 3.8], the function $s \to \sqrt{g}(s,\theta)|s^{n-1}$ is non-increasing. It follows that

$$\begin{aligned}\int\limits_{B(x,R)} |u(x) - u(y)|\, dy &\leq \int_0^R |\nabla u(t,\theta)| t^{1-n} \sqrt{g}(\rho,\theta)\, dt \rho^{n-1}\, d\rho d\theta \\ &\leq \frac{R^n}{n} \int_0^R |\nabla u(t,\theta)| t^{1-n} \sqrt{g}(\rho,\theta)\, dt d\theta \\ &= \frac{R^n}{n} \int\limits_{B(x,R)} \frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\, dy,\end{aligned}$$

thus

$$\frac{1}{R^n} \int\limits_{B(x,R)} |u(x) - u(y)|\, dy \leq \frac{1}{n} \int\limits_{B(x,R)} \frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\, dy,$$

and

$$\frac{1}{\mu(B(x,R))}\int\limits_{B(x,R)}|u(x)-u(y)|\,dy \leq \frac{1}{\mu(B(0,1))^n}\int\limits_{B(x,R)}\frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\,dy. \tag{3.1}$$

Now, fix $x \in M$. We apply inequality (3.1) as follows

$$\begin{aligned}|u(x)| \leq & \frac{1}{\mu(B(x,R))}\int\limits_{B(x,R)}|u(x)-u(y)|\,dy + \frac{1}{\mu(B(x,R))}\int\limits_{B(x,R)}|u(y)|\,dy \\ \leq & C\int\limits_{B(x,R)}\frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\,dy + \frac{1}{\mu(B(x,R))}\int\limits_{B(x,R)}|u(y)|\,dy \\ \leq & C\int\limits_{B(x_0,R)}\frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\,dy + \frac{1}{\mu(B(x,R))}\int\limits_{B(x,R)}|u(y)|\,dy,\end{aligned}$$

finally,

$$|u(x)| \leq C\int\limits_{B(x_0,R)}\frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\,dy$$

as $R \to \infty$. The proof is complete. ∎

4. MAIN RESULT

With the previous results at hand, we are ready to state and prove our main result.

Theorem 4.1. *Assume $f \in \widetilde{\mathcal{P}}_p(M)$, where M is a complete Riemannian manifold of dimension n having non-negative Ricci curvature. Then, for any $r > 0$, there exists a positive constant $C(n,p)$ such that*

$$\int\limits_M |f(x)||u(x)|^p\,dx \leq C(n,p)\phi(2r)\int\limits_M |\nabla u(x)|^p\,dx,$$

for any $u \in C_0^\infty(M)$ supported in $B(x_0,r)$.

Proof. By lemma 3.1 and Fubini's theorem we have

$$\begin{aligned}I = \int\limits_M |f(x)||u(x)|^p\,dx = & \int\limits_{B(x,r)}|f(x)||u(x)|^p\,dx \\ \leq & C\int\limits_{B(x,r)}|f(x)||u(x)|^{p-1}\left(\int\limits_{B(x_0,r)}\frac{|\nabla u(y)|}{[d(x,y)]^{n-1}}\,dy\right)dx \\ \leq & C\int\limits_{B(x,r)}|\nabla u(y)|\left(\int\limits_{B(x_0,r)}|f(x)||u(x)|^{p-1}\frac{dx}{[d(x,y)]^{n-1}}\right)dy.\end{aligned}$$

Now, by Hölder's inequality, we obtain
(4.1)

$$I \leq C \left(\int\limits_{B(x,r)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}} \left\{ \underbrace{\int\limits_{B(x_0,r)} \left(\int\limits_{B(x_0,r)} |f(x)||u(x)|^{p-1} \frac{dx}{[d(x,y)^{n-1}]} \right)^{\frac{p}{p-1}} dy}_{A} \right\}^{\frac{p-1}{p}} .$$

For the above integral A, we have

$$\begin{aligned} A = & \int\limits_{B(x_0,r)} \left(\int\limits_{B(x_0,r)} |f(x)||u(x)|^{p-1} \frac{dx}{[d(x,y)]^{n-1}} \right)^{\frac{p}{p-1}} dy \\ \leq & \int\limits_{B(x_0,r)} \left(\int\limits_{B(x_0,r)} |f(x)||u(x)|^{p-1} \frac{|f(z)|}{[d(x,z)]^{n-1}} \, dz \right)^{\frac{p}{p-1}} \int\limits_{B(x_0,r)} \frac{|f(x)||u(x)|^p}{[d(x,y)]^{n-1}} \, dxdy \\ = & \int\limits_{B(x_0,r)} |f(x)||u(x)|^p \int\limits_{B(x_0,r)} \frac{1}{[d(y,z)]^{n-1}} \left(\int\limits_{B(x_0,r)} \frac{|f(z)|}{[d(y,z)]^{n-1}} \, dz \right)^{\frac{1}{p-1}} dydx \\ \leq & \{\phi(2r)\}^{\frac{1}{p-1}} \int\limits_{B(x_0,r)} |f(x)||u(x)|^p \, dx. \end{aligned}$$

Going back to (4.1), we obtain

$$\int\limits_M |f(x)||u(x)|^p \, dx \leq C \left(\int\limits_{B(x,r)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}} \left\{ \{\phi(2r)\}^{\frac{1}{p-1}} \int\limits_{B(x_0,r)} |f(x)||u(x)|^p \, dx. \right\}^{\frac{p-1}{p}},$$

from which the desired result easily follows. ∎

5. CONCLUSION

The inequality given in Theorem 4.1, in the literature is known as the Fefferman inequality, is a useful inequality, which is more scarce than, for instance, Poincaré inequality.

Among some applications, Fefferman inequality helps to prove unique continuation of the solution of an equation of this type:

$$\text{div} A(x, u, \nabla u) = B(x, u, \nabla u)$$

where

$$A(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

and

$$B(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

are continuous functions satisfying:

$$|A(x,u,\xi)| \leq a|\xi|^{p-1} + b(x)|u|^{p-1},$$
$$|B(x,u,\xi)| \leq C(x)|\xi|^{p-1} + d(x)|u|^{p-1},$$
$$A(x,u,\xi)\cdot\xi \geq |\xi|^{p} - d(x)|u|^{p}$$

for almost all $x \in \Omega$, $u \in \mathbb{R}$, and $\xi \in \mathbb{R}^n$, where $1 < p < \infty$, a is a positive constant and b, c, d are measurable functions in Ω whose extension with zero value outside Ω are such that

$$b^{\frac{1}{p}}, c^{p}, d \in \widetilde{\mathcal{P}}_p(M).$$

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