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OSCILLATION CRITERIA FOR SECOND ORDER DELAY DIFFERENCE EQUATIONS VIA CANONICAL TRANSFORMATIONS AND SOME NEW MONOTONIC PROPERTIES

R. DEEPALAKHMI, S. SARAVANAN, JOHN R. GRAEF, AND E. THANDAPANI

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DEPARTMENT OF INTERDISCIPLINARY STUDIES, TAMIL NADU DR. AMBEDKAR LAW UNIVERSITY, CHENNAI-600113, INDIA.
profdeepalakshmi@gmail.com

MADRAS SCHOOL OF ECONOMICS, CHENNAI-600025, INDIA.
profsaran11@gmail.come-mail@address.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE AT CHATTANOOGA, CHATTANOOGA,TN 37403, USA.
john-graef@utc.edu

RAMANUJAN INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, UNIVERSITY OF MADRAS, CHENNAI - 600 005, INDIA.
ethandapani@yahoo.co.in

ABSTRACT. This paper is concerned with second-order linear noncanonical delay difference equations of the form

$$\Delta(\mu(t)\Delta y(t)) + p(t)y(\phi(t)) = 0.$$

The authors prove new oscillation criteria by first transforming the equation into canonical form and then obtaining some new monotonic properties of the positive solutions of the transformed equation. By using a comparison with first-order delay difference equations and a generalization of a technique developed by Koplatadze, they obtain their main results. Examples illustrating the improvement over known results in the literature are presented.

Key words and phrases: Second-order difference equation; Delay; Canonical transformation; Oscillation.

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1. INTRODUCTION

Consider the linear second-order noncanonical delay difference equation

$$\Delta(\mu(t)\Delta y(t)) + p(t)y(\phi(t)) = 0 \tag{E}$$

where t_0 is a positive integer and $t \in \mathbb{N}(t_0) = \{t_0, t_0+1, t_0+2, \dots\}$. Throughout this paper we assume that the following conditions hold:

(H_1) $\{\mu(t)\}$ and $\{p(t)\}$ are positive real sequences for all $t \in \mathbb{N}(t_0)$;
(H_2) $\{\phi(t)\}$ is a sequence of integers with $\phi(t) \le t-1$, $\Delta\phi(t) > 0$, and $\lim_{t\to\infty}\phi(t) = \infty$;
(H_3) $A(t) = \sum_{s=t}^{\infty} \frac{1}{\mu(s)}$ with $A(t_0) < \infty$.

Recall that a *solution* of (E) is a nontrivial real-valued sequence $\{y(t)\}$ satisfying (E) for all $t \ge t_0 + \min_{t>t_0} \phi(t)$. A solution $\{y(t)\}$ of (E) is called *oscillatory* if it is neither eventually negative nor eventually positive, and it is called *nonoscillatory* otherwise. Equation (E) itself is called oscillatory if all its solutions are oscillatory.

Oscillatory phenomena occur in different models described by difference and differential equations that arise in real-world problems; for example, see [1, 7, 10, 12] and the references therein. The problem of obtaining oscillation criteria for difference equations with deviating arguments has received great interest among researchers over the past few decades (see [2, 3, 11, 4, 6, 8, 9, 5, 15, 16, 13, 14]; references to many known results can be found in the monographs by Agarwal *et al.* [2, 3].

In particular, for a canonical type delay difference equation

$$\Delta^2 y(t) + p(t)y(t-\sigma) = 0, \tag{1.1}$$

where σ is a positive integer, Koplatadze [11] proved that (1.1) is oscillatory if

$$\liminf_{n\to\infty} \frac{1}{t}\sum_{s=t_0}^{t-1} s^2 p(s) > \frac{1}{4}. \tag{1.2}$$

Also in [2], it can be seen that (1.1) is oscillatory if

$$\liminf_{t\to\infty} \sum_{s=t-\sigma}^{t-1} (s-\sigma)p(s) > \left(\frac{\sigma}{\sigma+1}\right)^{(\sigma+1)}. \tag{1.3}$$

Later, in [4, 6], the authors proved the following result to guarantees the oscillation of (E), namely,

$$\limsup_{t\to\infty} \left\{ A(\phi(t)) \sum_{s=t_0}^{\phi(t)-1} p(s) + \sum_{s=\phi(t)}^{t-1} A(s+1)p(s) + \frac{1}{A(\phi(t))}\sum_{s=t}^{\infty} A(s+1)A(\phi(s))p(s) \right\} > 1. \tag{1.4}$$

From a review of the literature, we see that there is a great interest in extending and improving conditions (1.2)–(1.4) to more general difference equations. Following this trend, in this paper we investigate the oscillatory behavior of (E) by transforming it into a canonical type equation, deriving some new monotonic properties of positive solutions of the canonical equation, and then using these to obtain oscillation criteria for (E). The results are demonstrated by some specific examples. Due to the linearity of (E), we only need to consider positive solutions of (E) in all proofs.

2. OSCILLATION CRITERIA

Throughout this paper, we use the following notation:

$$b(t) = \mu(t)A(t)A(t+1), \quad B(t) = \sum_{s=t_0}^{t-1} \frac{1}{b(s)}$$

$$\eta(t) = \frac{y(t)}{A(t)}, \quad Q(t) = A(t+1)A(\phi(t))p(t).$$

We begin with the following lemma which significantly simplifies the study of (E).

Lemma 2.1. *Assume that (H_1)–(H_3) hold. Then,*

$$\frac{1}{A(t+1)}\Delta\left(\mu(t)A(t)A(t+1)\Delta\left(\frac{y(t)}{A(t)}\right)\right) = \Delta(\mu(t)\Delta y(t)). \tag{2.1}$$

Proof. By a direct calculation, we can show that (2.1) holds for any sequence $\{y(t)\}$. Indeed,

$$\begin{aligned}\frac{1}{A(t+1)}&\Delta\left(\mu(t)A(t)A(t+1)\Delta\left(\frac{y(t)}{A(t)}\right)\right)\\ &= \frac{1}{A(t+1)}\Delta\left(\mu(t)A(t)\Delta y(t) + y(t)\right)\\ &= \frac{1}{A(t+1)}\left[A(t+1)\Delta\left(\mu(t)\Delta y(t)\right) - \Delta y(t) + \Delta y(t)\right]\\ &= \Delta(\mu(t)\Delta y(t)).\end{aligned}$$

Moreover,

$$\sum_{t=t_0}^{\infty} \frac{1}{\mu(t)A(t)A(t+1)} = \lim_{t\to\infty} \frac{1}{A(t)} - \frac{1}{A(t_0)} = \infty,$$

so the operator on the left side of (2.1) is in canonical form. This completes the proof of the lemma. ∎

In view of Lemma 2.1, we can see that (E) can be written in the equivalent form

$$\Delta\left(\mu(t)A(t)A(t+1)\Delta\left(\frac{y(t)}{A(t)}\right)\right) + A(t+1)p(t)y(\phi(t)) = 0,$$

or

$$\Delta(b(t)\Delta\eta(t)) + Q(t)\eta(\phi(t)) = 0. \tag{Ec}$$

Theorem 2.2. *The noncanonical difference equation* (E) *possesses a solution* $\{y(t)\}$ *if and only if the canonical equation* (Ec) *has the solution* $\{\eta(t)\} = \left\{\dfrac{y(t)}{A(t)}\right\}$.

Corollary 2.3. *The noncanonical difference equation* (E) *is oscillatory if and only if the canonical equation* (Ec) *is oscillatory.*

Corollary 2.3 simplifies the study of the noncanonical equation (E) since for (Ec), we see that any eventually positive solution of (Ec) satisfies (see [1, Theorem 1.8.11])

$$\eta(t) > 0, \quad \Delta\eta(t) > 0, \quad \text{and} \quad \Delta(b(t)\Delta\eta(t)) < 0 \tag{2.2}$$

for all $t \in \mathbb{N}(t_0)$. The proof is elementary and the details are omitted.

Lemma 2.4. *Let (H_1)–(H_3) hold and assume that*

$$\sum_{t=t_0}^{\infty} B(\phi(t))Q(t) = \infty. \tag{2.3}$$

Then for any positive solution $\{\eta(t)\}$ *of* (Ec),

$$\left\{\frac{\eta(t)}{B(t)}\right\} \text{ is decreasing and } \lim_{t\to\infty}\frac{\eta(t)}{B(t)} = 0. \tag{2.4}$$

Proof. Assume that $\{\eta(t)\}$ is a positive solution of (Ec). It is easy to see that $\{\eta(t)\}$ is increasing and

$$-\Delta\left(b(t)B(t)B(t+1)\Delta\left(\frac{\eta(t)}{B(t)}\right)\right) = B(t+1)Q(t)\eta(\phi(t)). \tag{2.5}$$

Let $M = \eta(t_0) - B(t_0)b(t_0)\Delta\eta(t_0)$. Then a summation of (2.5) from t_0 to $t-1$ yields

$$-b(t)B(t)B(t+1)\Delta\left(\frac{\eta(t)}{B(t)}\right) \geq M + \eta(\phi(t_0))\sum_{s=t_0}^{t-1} B(s+1)Q(s) \to \infty \tag{2.6}$$

by (2.3). Hence, $\Delta\left(\frac{\eta(t)}{B(t)}\right) < 0$ eventually, and so $\left\{\frac{\eta(t)}{B(t)}\right\}$ is decreasing and $M > 0$ for large t, say for $t \geq t_1 \geq t_0$.

Next, in order to obtain a contradiction, we assume that $\lim_{t\to\infty}\frac{\eta(t)}{B(t)} = l > 0$. In view of (2.6), the sequence $\{z(t)\} = \left\{\frac{\eta(t)}{B(t)}\right\}$ satisfies

$$\Delta z(t) + \frac{1}{b(t)B(t)B(t+1)}\sum_{s=t_1}^{t-1} B(s+1)Q(s)\eta(\phi(s)) = -\frac{M}{b(t)B(t)B(t+1)} \leq 0. \tag{2.7}$$

Summing (2.7) from t_1 to ∞, we obtain

$$z(t_0) - l \geq \sum_{s=t_1}^{\infty}\frac{1}{b(s)B(s)B(s+1)}\sum_{u=t_1}^{s-1} B(u+1)Q(u)B(\phi(u))z(\phi(u)) \geq l\sum_{s=t_1}^{\infty} B(\phi(s))Q(s),$$

which contradicts (2.3). Therefore, $l = 0$, and this proves the lemma. ∎

Theorem 2.5. *Let (H_1)–(H_3) and* (2.3) *hold. If* $\phi(t) = t - k$, *where* k *is a positive integer, and*

$$\liminf_{t\to\infty}\sum_{s=t-k}^{t-1}\frac{1}{d(s)}\sum_{u=0}^{s-1} B(u+1)B(u-k)Q(u) > \left(\frac{k}{k+1}\right)^{k+1} \tag{2.8}$$

where d(t)=b(t)B(t)B(t+1), then (E) *is oscillatory.*

Proof. Assume that $\{y(t)\}$ is a positive solution of (E). Then by Theorem 2.2, $\{\eta(t)\}$ is a positive solution of (Ec) and satisfies (2.2). Now, the sequence $\{z(t)\} = \left\{\frac{\eta(t)}{B(t)}\right\}$ satisfies

(2.6), that is,

$$\begin{aligned}0 &= \Delta z(t) + \frac{1}{d(t)}\left(\sum_{s=t_0}^{t-1} B(s+1)B(s-k)Q(s)z(s-k) + M\right)\\ &\geq \Delta z(t) + \frac{1}{d(t)}\left(z(t-k))\sum_{s=t_0}^{t-1} B(s+1)B(s-k)Q(s) + M\right)\\ &= \Delta z(t) + \frac{1}{d(t)}\left(z(t-k)\sum_{s=0}^{t-1} B(s+1)B(s-k)Q(s) + M\right.\\ &\quad\left. - z(t-k)\sum_{s=0}^{t_0-1} B(s+1)B(s-k)Q(s)\right).\end{aligned}$$

Since $\{z(t)\}$ is decreasing and $\lim_{t\to\infty} z(t) = 0$, we see that $\{z(t)\}$ is a positive solution of the difference inequality

$$\Delta z(t) + \left(\frac{1}{d(t)}\sum_{s=0}^{t-1} B(s+1)B(s-k)Q(s)\right) z(t-k) \leq 0, \tag{2.9}$$

which contradicts (2.8). This proves the theorem. ∎

In the following result, we improve the conclusion (2.4). For convenience, let

$$\alpha_1(t) = \frac{1}{d(t)}\sum_{s=t_0}^{t-1} B(s+1)B(s-k)Q(s) \ \text{ and } \ \beta_1(t) = \prod_{s=t_0}^{t-1}(1+\alpha_1(s)).$$

Lemma 2.6. *Let (H_1)–(H_3) and* (2.3) *hold. Then for every positive solution* $\{\eta(t)\}$ *of* (Ec)*, the sequence* $\{z(t)\} = \left\{\frac{\eta(t)}{B(t)}\right\}$ *satisfies*

$$\{z(t)\beta_1(t)\} \text{ is eventually decreasing.} \tag{2.10}$$

Proof. Assume that $\{\eta(t)\}$ is a positive, and so increasing, solution of (Ec). Then, the sequence $\{z(t)\}$ satisfies

$$\Delta z(t) + \alpha_1(t)z(t+1) \leq 0$$

by (2.9) and the fact that $\{z(t)\}$ is decreasing. It follows that

$$z(t+1)(1+\alpha_1(t)) - z(t) \leq 0,$$

and multiplying by $\prod_{s=t_0}^{t-1}(1+\alpha_1(s))$ gives

$$z(t+1)\prod_{s=t_0}^{t}(1+\alpha_1(s)) - \prod_{s=t_0}^{t-1}(1+\alpha_1(s))z(t) \leq 0.$$

Therefore, $\Delta(z(t)\beta_1(t)) \leq 0$, which proves the lemma. ∎

Using this new monotonic property given in (2.10), we are able to obtain some new and improved oscillation criteria for equation (E).

Theorem 2.7. *Let* (H_1)*–*(H_3) *and* (2.3) *hold, and let* $\phi(t) = t - k$*, where k is a positive integer. If*

$$\liminf_{t\to\infty} \sum_{s=t-k}^{t-1} \frac{\beta_1(s-k)}{d(s)} \left(\sum_{\zeta=t_0}^{s-1} \frac{B(\zeta+1)B(\zeta-k)Q(\zeta)}{\beta_1(\zeta-k)} \right) > \left(\frac{k}{k+1} \right)^{k+1}, \tag{2.11}$$

then (E) *is oscillatory.*

Proof. Assume that $\{y(t)\}$ is a positive solution of (E). Then by Theorem 2.2, $\{\eta(t)\} = \left\{ \frac{y(t)}{A(t)} \right\}$ is a positive solution of (Ec) and satisfies (2.2). As in the proof of Theorem 2.5, we see that $\{z(t)\} = \left\{ \frac{\eta(t)}{B(t)} \right\}$ satisfies (2.9), which in view of (2.10) implies that $\{z(t)\}$ is a positive solution of the difference inequality

$$\Delta z(t) + \left(\frac{\beta_1(t-k)}{d(t)} \sum_{s=t_0}^{t-1} \frac{B(s+1)B(s-k)Q(s)}{\beta_1(s-k)} \right) z(t-k) \le 0. \tag{2.12}$$

This contradicts (2.11) and completes the proof. ∎

Next, we present a result that improves the oscillation criterion given in (1.4).

Theorem 2.8. *Let* (H_1)*–*(H_3) *and* (2.3) *hold. If* $\phi(t) = t - k$*, where* k *is positive integer, and*

$$\limsup_{t\to\infty} \left\{ \frac{\beta_1(t-k)}{B(t-k)} \sum_{s=t_1}^{t-k-1} \frac{B(s+1)B(s-k)}{\beta_1(s-k)} Q(s) + \beta_1(t-k) \sum_{s=t-k}^{t-1} \frac{Q(s)B(s-k)}{\beta_1(s-k)} + B(t-k) \sum_{s=t}^{\infty} Q(s) \right\} > 1, \tag{2.13}$$

then (E) *is oscillatory.*

Proof. Assume that $\{y(t)\}$ is a positive solution of (E). Then by Theorem 2.2, $\{\eta(t)\}$ is a positive solution of (Ec) and satisfies (2.2). Summing (Ec) from t to ∞, yields

$$\Delta \eta(t) = \frac{1}{b(t)} \sum_{s=t}^{\infty} Q(s)\eta(s-k).$$

Summing this from $t_1 \ge t_0$ to $t-1$ and then using summation by parts gives

$$\eta(t) \ge \sum_{s=t_1}^{t-1} B(s+1)Q(s)\eta(s-k) + B(t) \sum_{s=t}^{\infty} Q(s)\eta(s-k).$$

So, we have

$$\eta(t-k) \ge \sum_{s=t_1}^{t-k-1} B(s+1)Q(s)\eta(s-k) + B(t-k) \sum_{s=t-k}^{t-1} Q(s)\eta(s-k) + B(t-k) \sum_{s=t}^{\infty} Q(s)\eta(s-k).$$

Taking into account that $\{\eta(t)\}$ is increasing and $\{z(t)\beta_1(t)\}| = \left\{\dfrac{\eta(t)}{B(t)}\beta_1(t)\right\}$ is decreasing by Lemma 2.6, we obtain

$$\eta(t-k) \geq \frac{\eta(t-k)\beta_1(t-k)}{B(t-k)} \sum_{s=t_1}^{t-k-1} \frac{B(s+1)B(s-k)Q(s)}{\beta_1(s-k)} + \eta(t-k)\beta_1(t-k) \sum_{s=t-k}^{t-1} \frac{Q(s)B(s-k)}{\beta_1(s-k)} + B(t-k)\eta(t-k) \sum_{s=t}^{\infty} Q(s),$$

so

$$1 \geq \left\{ \frac{\beta_1(t-k)}{B(t-k)} \sum_{s=t_1}^{t-k-1} \frac{B(s+1)B(s-k)Q(s)}{\beta_1(s-k)} + \beta_1(t-k) \sum_{s=t-k}^{t-1} \frac{Q(s)B(s-k)}{\beta_1(s-k)} + B(t-k) \sum_{s=t}^{\infty} Q(s) \right\}.$$

This contradicts (2.13) and completes the proof of the theorem. ∎

Theorem 2.9. *Let (H_1)–(H_3) and* (2.3) *hold. If $\phi(t) = t-k$, where k is a positive integer, and*

$$\liminf_{t\to\infty} B(t) \sum_{s=t}^{\infty} Q_1(s) > \frac{1}{4}, \tag{2.14}$$

where $Q_1(t) = Q(t)\dfrac{B(t-k)\beta_1(t)}{B(t)\beta_1(t-k)}$, then (E) *is oscillatory.*

Proof. Assume that $\{y(t)\}$ is a positive solution of (E). Then by Theorem 2.2, $\{\eta(t)\}$ is a positive solution of (Ec) and satisfies (2.2). Using the fact that $\left\{\dfrac{\eta(t)\beta_1(t)}{B(t)}\right\}$ is decreasing, from (Ec), we obtain

$$\Delta(b(t)\Delta\eta(t)) + Q(t)\frac{B(t-k)\beta_1(t-k)}{B(t-k)\beta_1(t-k)}\eta(t-k) = 0,$$

or

$$\Delta(b(t)\Delta\eta(t)) + Q(t)\frac{B(t-k)\beta_1(t)}{B(t)\beta_1(t-k)}\eta(t) \leq 0. \tag{2.15}$$

Define

$$w(t) = \frac{b(t)\Delta\eta(t)}{\eta(t)} > 0.$$

Then, using (2.15), we obtain

$$\Delta w(t) \leq -Q_1(t) - \frac{w(t)w(t+1)}{b(t)}.$$

Summing the last inequality from t to ∞, we find that

$$w(t) \geq \sum_{s=t}^{\infty} Q_1(s) + \sum_{s=t}^{\infty} \frac{w(s)w(s+1)}{b(s)},$$

and so

$$B(t)w(t) \geq B(t)\sum_{s=t}^{\infty} Q_1(s) + B(t)\sum_{s=t}^{\infty} \frac{B(s)w(s)B(s+1)w(s+1)}{b(s)B(s)B(s+1)}. \tag{2.16}$$

Let $\liminf_{t\to\infty} B(t)w(t) = M > 0$; then from (2.16),

$$M > \frac{1}{4} + M^2,$$

since $B(t)\sum_{s=t}^{\infty} \frac{1}{b(s)B(s)B(s+1)} = 1$. This is not possible for $M > 0$, and the proof of the theorem is complete. ∎

3. EXAMPLES

In this section we present two examples to illustrate the main results and compare our theorems to others in the literature.

Example 3.1. Consider the second-order Euler type difference equation

$$\Delta(t(t+1)\Delta y(t)) + (t+1)py(t-2) = 0, \quad t \geq 3, \tag{3.1}$$

where $p > 0$ is a constant. We have $\mu(t) = t(t+1)$, $\phi(t) = t-2$, $p(t) = (t+1)p$, $A(t) = \frac{1}{t}$, $b(t) = 1$, $B(t) \approx t$, $Q(t) = \frac{p}{t-2}$, and $k = 2$. Condition (2.3) becomes $\sum_{t=3}^{\infty} \frac{p}{t-2}(t-2) = \sum_{t=3}^{\infty} p = \infty$, that is, condition (2.3) holds. The transformed equation (Ec) becomes

$$\Delta^2\eta(t) + \frac{p}{t-2}\eta(t-2) = 0, \quad t \geq 3,$$

which is in canonical form. Moreover $d(t) = t(t+1)$ and condition (2.8) becomes

$$\liminf_{t\to\infty} \sum_{s=t-2}^{t-1} \frac{1}{s(s+1)} \sum_{u=0}^{s-1} p(u+1) = \liminf_{t\to\infty} \sum_{s=t-2}^{t-1} \frac{p}{2} = p > \left(\frac{2}{3}\right)^3,$$

that is, (2.8) holds if $p > \frac{8}{27}$. Therefore, by Theorem 2.5, equation (3.1) is oscillatory if $p > \frac{8}{27}$. Notice that in [4, Theorem 2.2] and [6, Theorem 3.2], equation (3.1) is shown to be oscillatory if $p > 0.5$, and so Theorem 2.5 improves both of those results.

Example 3.2. Consider the equation

$$\Delta(2^t\Delta y(t)) + p2^t y(t-1) = 0, \quad t \geq 1, \tag{3.2}$$

where $p > 0$ is a constant. In this case $\mu(t) = 2^t$, $p(t) = p2^t$, $\phi(t) = t-1$, and $k = 1$. Simple calculations show that $A(t) = \frac{2}{2^t}$, $b(t) = \frac{2}{2^t}$, $Q(t) = \frac{4p}{2^t}$, $B(t) \approx 2^{t-1}$, $d(t) \approx 2^t$, $\alpha_1(t) \approx p$, $\beta_1(t) = (1+p)^{t-1}$, and $Q_1(t) \approx \frac{2p(1+p)}{2^t}$. Condition (2.3) is clearly satisfied and (2.14) becomes $\liminf_{t\to\infty} 2^{t-1}\sum_{s=t}^{\infty} \frac{2p(p+1)}{2^s} \geq 2p(p+1) > \frac{1}{4}$, that is, (2.14) holds if $p(p+1) > \frac{1}{8}$. Therefore, by Theorem 2.9, we conclude that (3.2) is oscillatory if $p > 0.11237$. By [6, Theorem 3.6], equation (3.2) is found to be oscillatory if $p > 0.16666$ and so Theorem 2.9 improves Theorem 3.6 of [6].

4. CONCLUSION

In this paper, we obtained new oscillation criteria for second order retarded noncanonical difference equations by first transforming them into canonical form. We were then able to derive some new monotonic properties of their positive solutions. The oscillation results we then obtained are new and improve some results previously reported in the literature. We also illustrated our results with examples pointing out these improvements.

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