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RECURSIVE BOUNDS FOR THE EIGENVALUES OF SYMMETRIC POSITIVE DEFINITE MATRICES

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ABSTRACT. In this paper, we bound the extremal eigenvalues of a positive definite real symmetric matrix by considering a part of the characteristic equation in the region of the smallest and largest eigenvalues. An expansion around these values leads to a sequence of monotonic functions, whose zeros coincide with the extremal zeros of associated polynomials. The latter is shown to yield bounds that are fairly accurate.

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1. INTRODUCTION

The knowledge of the distribution of the spectrum $\sigma(\mathbf{A})$ of matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is vital to many applied mathematics and engineering problems. Their distribution in the complex plane determines the stability of the solution of a system of differential equations. For symmetric matrices these values are real and their extremal values serve an important aspect in determining the conditioning of an associated linear algebraic system. They are vital for the approximation of normal operators [5]. Some crude bounds are obtained by application of Gerschgorin's theorem [3] and the ovals of Cassini [1]. Trace bounds [13, 4] give reasonably good results, however the lower bound is not guaranteed to be positive as expected, for the class of positive definite real symmetric matrices. Also an improvement using trace bounds [8] requires much more effort as traces of powers of matrix $\mathbf{A}$ are required. The application of Rayleigh's theorem [3] provides good inner bounds, however the outer bounds are not so easily approximated. The solution of the characteristic equation of a matrix $\mathbf{A}$ is a difficult task for large dimensions, therefore many methods have been proposed for approximating the extremal eigenvalues. For positive definite symmetric matrices Dembo bounds [2] arise by examining the characteristic equation of $\mathbf{A}$ and relies on bounds of a principal submatrix. Ma and Zarowski [6] improved on Dembo's lower bound by ensuring that it was always positive. This idea was also used to further improve the lower bounds of the minimal eigenvalue [12] and to Toeplitz matrices by Melman [7] for both upper and lower bounds. All techniques are to be considered in their proper context to isolate the extremal eigenvalues. Recently there has been a resurgence in research into the bounding of the spectrum of real positive definite symmetric matrices [9, 10, 11].

2. THEORY

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, with spectrum $\sigma(\mathbf{A}) = \{\lambda_i\}_{i=1}^{n}$ arranged in ascending order

$$\lambda_1 \le \lambda_2 \cdots \le \lambda_n.$$

Partition $\mathbf{A}$ as follows

$$\mathbf{A} = \begin{bmatrix} c & \mathbf{b}^t \\ \mathbf{b} & \mathbf{B} \end{bmatrix}$$

where $\mathbf{B} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\mathbf{b} \neq \mathbf{0} \in \mathbb{R}^{n-1}$, with $c > 0$ (follows from positive definiteness).

Let $\sigma(\mathbf{B}) = \{\beta_i\}_{i=1}^{n-1}$ be arranged in ascending order

$$\beta_1 \le \beta_2 \cdots \le \beta_{n-1} \tag{2.1}$$

and note that by the interlacing theorem [3]

$$\lambda_1 < \beta_1 \le \lambda_2 \le \beta_2 \le \cdots \le \beta_{n-1} < \lambda_n$$

where we have assumed strict separation of the extremal eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. We examine the characteristic polynomial $\det(\lambda\mathbf{I} - \mathbf{A})$ in order to ascertain the eigenvalues.

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det(\lambda\mathbf{I} - \mathbf{B})[\lambda - c - \mathbf{b}^t(\lambda\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}] \tag{2.2}$$

Note that the resolvent $(\lambda\mathbf{I} - \mathbf{B})^{-1}$ exists for $\lambda \notin \sigma(\mathbf{B})$, hence (2.2) is valid for $\lambda \in (0,\, \beta_1) \cup (\beta_{n-1},\, \infty)$. It follows that λ_n must be a zero of the function.

$$f(\lambda) = \lambda - c - \mathbf{b}^t(\lambda\mathbf{I} - \mathbf{B})^{-1}\mathbf{b} \tag{2.3}$$

3. MAXIMUM EIGENVALUE BOUNDS

Lemma 3.1. *Let* $\mathbf{B} \in \mathbb{R}^{(n-1)\times(n-1)}$ *be a positive definite symmetric matrix with* $\sigma(\mathbf{B}) = \{\beta_i\}_{i=1}^{n-1}$ *arranged in ascending order* (2.1) *with* $\beta_l \le \beta_1$ *and* $\beta_{n-1} \le \beta_u$ *known lower and upper bounds for the extremal eigenvalues of* $\mathbf{B}$*. Then for* $\lambda > \sigma(\mathbf{B})$ *and non zero* $\mathbf{b} \in \mathbb{R}^{n-1}$ *we have*

$$\frac{\beta_l\langle \mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle}{\lambda^p(\lambda-\beta_l)} \le \sum \frac{\langle \mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^k} \le \frac{\beta_u\langle \mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle}{\lambda^p(\lambda-\beta_u)}$$

Proof. Let $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\cdots,\,\mathbf{v}_{n-1}\}$ be an orthogonal set of eigenvectors of $\mathbf{B}$, then $\mathbf{B}$ has the spectral decomposition

$$\begin{aligned}\mathbf{B} &= \sum_{i=1}^{n-1}\beta_i\mathbf{v}_i\mathbf{v}_i^t \\ &= \sum_{i=1}^{n-1}\beta_i\mathbf{G}_i,\end{aligned}$$

where $\mathbf{G}_i = \mathbf{v}_i\mathbf{v}_i^t$ are orthogonal projectors onto the nullspace $N(\beta_i\mathbf{I}-\mathbf{B})$

$$\begin{aligned}\sum_{k=p+1}^{\infty}\frac{\langle \mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^k} &= \sum_{k=1}^{\infty}\frac{\langle \mathbf{B}^{p+k}\mathbf{b},\,\mathbf{b}\rangle}{\lambda^{p+k}} \\ &= \sum_{k=1}^{\infty}\frac{\langle\sum_{i=1}^{n-1}\beta_i^{p+k}\mathbf{G}_i\mathbf{b},\,\mathbf{b}\rangle}{\lambda^{p+k}} \\ &= \frac{\left\langle\sum_{k=1}^{n-1}\beta_i^{p}\mathbf{G}_i\mathbf{b},\,\mathbf{b}\right\rangle}{\lambda^p}\sum_{k=1}^{\infty}\left(\frac{\beta_i}{\lambda}\right)^k \\ &= \frac{\langle \mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle}{\lambda^p}\sum_{k=1}^{\infty}\left(\frac{\beta_i}{\lambda}\right)^k \qquad (3.1)\end{aligned}$$

But

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{\beta_i}{\lambda}\right)^k &\le \sum_{k=1}^{\infty}\left(\frac{\beta_u}{\lambda}\right)^k \\ &= \frac{\beta_u}{\lambda-\beta_u} \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{\beta_i}{\lambda}\right)^k &\ge \sum_{k=1}^{\infty}\left(\frac{\beta_l}{\lambda}\right)^k \qquad (3.3) \\ &= \frac{\beta_l}{\lambda-\beta_l}\end{aligned}$$

The result then follows from (3.1), (3.2) and (3.3) ∎

Since the spectral radius $\rho\left(\frac{\mathbf{B}}{\lambda}\right)=\frac{\rho(\mathbf{B})}{\lambda}=\frac{\beta_{n-1}}{\lambda}<1$ we may write $f(\lambda)$ as

$$f^n(\lambda)=\lambda-c-\frac{\mathbf{b}^t}{\lambda}\left(\mathbf{I}-\frac{\mathbf{B}}{\lambda}\right)^{-1}\mathbf{b}$$

$$=\lambda-c-\frac{\mathbf{b}^t}{\lambda}\left(\sum_{k=0}^{\infty}\frac{\mathbf{B}^k}{\lambda^k}\right)\mathbf{b}$$

$$=\lambda-c-\frac{1}{\lambda}\sum_{k=0}^{p}\frac{\langle\mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^k}-\frac{1}{\lambda}\sum_{k=p+1}^{\infty}\frac{\langle\mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^k} \tag{3.4}$$

Apply 3.1 to (3.4) to obtain

$$f^n(\lambda)\le l_p^n(\lambda)=\lambda-c-\frac{1}{\lambda}\sum_{k=0}^{p}\frac{\langle\mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^k}-\frac{\beta_l\langle\mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle}{\lambda^{p+1}(\lambda-\beta_l)}$$

$$=\lambda-c-\sum_{k=0}^{p-1}\frac{\langle\mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^{k+1}}-\frac{\langle\mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle}{\lambda^p(\lambda-\beta_l)}$$

$$f^n(\lambda)\ge u_p^n(\lambda)=\lambda-c-\frac{1}{\lambda}\sum_{k=0}^{p}\frac{\langle\mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^k}-\frac{\beta_u\langle\mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle}{\lambda^{p+1}(\lambda-\beta_u)}$$

$$=\lambda-c-\sum_{k=0}^{p-1}\frac{\langle\mathbf{B}^k\mathbf{b},\,\mathbf{b}\rangle}{\lambda^{k+1}}-\frac{\langle\mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle}{\lambda^p(\lambda-\beta_u)}$$

Also

$$l_{p+1}^n(\lambda)-l_p^n(\lambda)=\frac{\beta_l\langle\mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle-\langle\mathbf{B}^{p+1}\mathbf{b},\,\mathbf{b}\rangle}{\lambda^{p+1}(\lambda-\beta_l)} \tag{3.5}$$

and

$$\beta_l\langle\mathbf{B}^p\mathbf{b},\,\mathbf{b}\rangle-\langle\mathbf{B}^{p+1}\mathbf{b},\,\mathbf{b}\rangle$$

$$=\left\langle\sum_{i=1}^{n-1}\beta_l\beta_i^p\mathbf{G}_i\mathbf{b},\,\mathbf{b}\right\rangle-\langle\mathbf{B}^{p+1}\mathbf{b},\,\mathbf{b}\rangle$$

$$\le\left\langle\sum_{i=1}^{n-1}\beta_i^{p+1}\mathbf{G}_i\mathbf{b},\,\mathbf{b}\right\rangle-\langle\mathbf{B}^{p+1}\mathbf{b},\,\mathbf{b}\rangle$$

$$=\langle\mathbf{B}^{p+1}\mathbf{b},\,\mathbf{b}\rangle-\langle\mathbf{B}^{p+1}\mathbf{b},\,\mathbf{b}\rangle$$

$$=0$$

Hence $l_{p+1}^n\le l_p^n(\lambda)$.

Similarly we may show that

$$u_{p+1}^n(\lambda) - u_p^n(\lambda) = \frac{\beta_u \langle \mathbf{B}^p \mathbf{b},\, \mathbf{b}\rangle - \langle \mathbf{B}^{p+1}\mathbf{b},\, \mathbf{b}\rangle}{\lambda^{p+1}(\lambda - \beta_u)}$$

$$\geq 0$$

so that $u_{p+1}^n(\lambda) \geq u_p^n(\lambda)$.

Hence we have the sequence of functions bounding $f^n(\lambda)$ given by

$$\begin{aligned} &u_0^n(\lambda) \leq u_1^n(\lambda) \leq \cdots \leq u_p^n(\lambda) \leq u_{p+1}^n(\lambda) \leq \cdots \leq f^n(\lambda) \\ &\leq \cdots l_{p+1}^n(\lambda) \leq l_p^n(\lambda) \leq l_1^n(\lambda) \leq l_0^n(\lambda) \end{aligned}$$

Note that $\frac{df^n}{d\lambda} > 0$ and $\frac{d^2 f^n}{d\lambda^2} < 0$ implies that $f^n(\lambda)$ is increasing and concave down. Also $f^n(\lambda)$ is asymptotic to $\lambda - c$.

We have $l_p^n(\lambda_n) \geq f^n(\lambda_n) = 0$ and $\lim\limits_{\lambda\to\infty} l_p^n(\lambda) = -\infty$, hence $l_p^n(\lambda)$ has a zero in $(-\infty,\, \lambda_n]$. Since $u_p^n(\lambda_n) \leq f^n(\lambda_n) = 0$ and $\lim\limits_{\lambda\to\infty} u_p^n(\lambda) = \infty$ it follows that $u_p^n(\lambda)$ has a zero in $[\lambda_n,\, \infty)$. Hence the maximal zero of $l_p^n(\lambda)$ is a lower bound for λ_n and the maximal zero of $u_p^n(\lambda)$ is an upper bound for λ_n.

4. MINIMUM EIGENVALUE BOUNDS

It folows that λ_1 is also a zero of (2.3).

Now consider $\lambda \in (0,\, \lambda_1)$, since $\rho(\lambda \mathbf{B}^{-1}) = \lambda \rho(\mathbf{B}^{-1}) = \frac{\lambda}{\lambda_1} < 1$ we may write (2.3) as

$$\begin{aligned} f^1(\lambda) &= \lambda - c + \mathbf{b}^t \mathbf{B}^{-1}(\mathbf{I} - \lambda \mathbf{B}^{-1})^{-1}\mathbf{b} \\ &= \lambda - c + \mathbf{b}^t \mathbf{B}^{-1} \sum_{k=0}^{\infty} \lambda^k \mathbf{B}^{-k}\mathbf{b} \\ &= \lambda - c + \sum_{k=0}^{\infty} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle \end{aligned}$$

$$= \lambda - c + \sum_{k=0}^{p} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle + \sum_{k=p+1}^{\infty} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle \tag{4.1}$$

Lemma 4.1. *For* $0 < \lambda < \lambda_1 \leq \beta_l$, *we have*

$$\frac{\lambda^{p+1}}{\beta_u - \lambda}\langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle \leq \sum_{k=p+1}^{\infty} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b}, \mathbf{b}\rangle \leq \frac{\lambda^{p+1}}{\beta_l - \lambda}\langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle$$

Proof. Note that $\sigma(\mathbf{B}^{-1}) = (\sigma(\mathbf{B}))^{-1}$ and that $\mathbf{B}$ and $\mathbf{B}^{-1}$ have the same eigenbasis. Hence

$$\sum_{k=p+1}^{\infty} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle$$

$$= \sum_{k=1}^{\infty} \lambda^{p+k} \langle \mathbf{B}^{-p-k-1}\mathbf{b},\, \mathbf{b}\rangle$$

$$= \sum_{k=1}^{\infty} \lambda^{p+k} \left\langle \sum_{i=1}^{n-1} \beta_i^{-p-k-1}\mathbf{G}_i\mathbf{b},\, \mathbf{b}\right\rangle$$

$$= \lambda^p \langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle \sum_{k=1}^{\infty} \left(\frac{\lambda}{\beta_i}\right)^k \tag{4.2}$$

But

$$\sum_{k=1}^{\infty} \left(\frac{\lambda}{\beta_i}\right)^k \le \sum_{k=1}^{\infty} \left(\frac{\lambda}{\beta_l}\right)^k$$

$$= \frac{\lambda}{\beta_l - \lambda} \tag{4.3}$$

and

$$\sum_{k=1}^{\infty} \left(\frac{\lambda}{\beta_i}\right)^k \ge \sum_{k=1}^{\infty} \left(\frac{\lambda}{\beta_u}\right)^k$$

$$= \frac{\lambda}{\beta_u - \lambda} \tag{4.4}$$

The result follows by substituting (4.3) and (4.4) into (4.1). ∎

Applying Lemma 4.1 to (4.1) we have

$$f^1(\lambda) \le l_p^1(\lambda) = \lambda - c + \sum_{k=0}^{p} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle + \frac{\lambda^{p+1}}{\beta_l - \lambda}\langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle$$

$$= \lambda - c + \sum_{k=0}^{p-1} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle + \frac{\beta_l\lambda^{p}}{\beta_l - \lambda}\langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle$$

and

$$f^1(\lambda) \ge u_p^1(\lambda) = \lambda - c + \sum_{k=0}^{p} \lambda^k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle + \frac{\lambda^{p+1}}{(\beta_u - \lambda)}\langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle$$

$$= \lambda - c + \sum_{k=0}^{p-1} \lambda_k \langle \mathbf{B}^{-k-1}\mathbf{b},\, \mathbf{b}\rangle + \frac{\beta_u\lambda^{p}}{(\beta_u - \lambda)}\langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle$$

As in the case for bounding λ_n, we may show that the sequence of functions bounding $f^1(\lambda)$ are given by

$$\begin{aligned} &u_0^1(\lambda) \leq u_1^1(\lambda) \leq \cdots \leq u_p^1(\lambda) \leq u_{p+1}^1(\lambda) \leq \cdots \leq f^1(\lambda) \\ &\leq \cdots l_{p+1}^1(\lambda) \leq l_p^1(\lambda) \leq l_1^1(\lambda) \leq l_0^1(\lambda) \end{aligned}$$

The zeros of the functions $l_p^n(\lambda)$, $u_p^n(\lambda)$, $l_p^1(\lambda)$ and $u_p^1(\lambda)$ are equivalent to the zeros of the corresponding polynomials $L_p^n(\lambda)$, $U_p^n(\lambda)$, $L_p^1(\lambda)$ and $U_p^1(\lambda)$. It can be readily shown that the following recurrence relations are satisfied.

$$L_{p+1}^n(\lambda) = \lambda L_p^n(\lambda) + \beta_l \langle \mathbf{B}^p \mathbf{b},\, \mathbf{b}\rangle - \langle \mathbf{B}^{p+1}\mathbf{b},\, \mathbf{b}\rangle \tag{4.5}$$

$$L_0^n(\lambda) = \lambda^2 - \lambda(\beta_l + c) + \beta_l c - \langle \mathbf{b},\, \mathbf{b}\rangle \tag{4.6}$$

$$U_{p+1}^n(\lambda) = \lambda U_p^n(\lambda) + \beta_u \langle \mathbf{B}^p \mathbf{b},\, \mathbf{b}\rangle - \langle \mathbf{B}^{p+1}\mathbf{b},\, \mathbf{b}\rangle \tag{4.7}$$

$$U_0^n(\lambda) = \lambda^2 - \lambda(\beta_u + c) + \beta_u c - \langle \mathbf{b},\, \mathbf{b}\rangle \tag{4.8}$$

$$L_{p+1}^1(\lambda) = \lambda L_p^1(\lambda) + \lambda^{p+1}(\beta_l \langle \mathbf{B}^{-p-2}\mathbf{b},\, \mathbf{b}\rangle - \langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle) \tag{4.9}$$

$$L_0^1(\lambda) = (\lambda - c)(\beta_l - \lambda) + \beta_l \langle \mathbf{B}^{-1}\mathbf{b},\, \mathbf{b}\rangle \tag{4.10}$$

$$U_{p+1}^1(\lambda) = \lambda U_p^1(\lambda) + \lambda^{p+1}(\beta_u \langle \mathbf{B}^{-p-2}\mathbf{b},\, \mathbf{b}\rangle - \langle \mathbf{B}^{-p-1}\mathbf{b},\, \mathbf{b}\rangle) \tag{4.11}$$

$$U_0^1(\lambda) = (\lambda - c)(\beta_u - \lambda) + \beta_u \langle \mathbf{B}^{-1}\mathbf{b},\, \mathbf{b}\rangle \tag{4.12}$$

We shall label the maximal zeros of (4.5)–(4.8) by $\lambda_n^{l,\,p}$ and $\lambda_n^{u,\,p}$ and the minimal zeros of (4.9)–(4.12) by $\lambda_1^{l,\,p}$ and $\lambda_1^{u,\,p}$.

The maximal zero of $L_0^n(\lambda)$ yields the Dembo lower bound

$$\lambda_n^{l,\,0} = \frac{\beta_l + c}{2} + \sqrt{\left(\frac{\beta_l - c}{2}\right) + \langle \mathbf{b},\, \mathbf{b}\rangle}$$

whilst the maximal zero of $U_0^n(\lambda)$ yields the Dembo upper bound

$$\lambda_n^{u,\,0} = \frac{\beta_u + c}{2} + \sqrt{\left(\frac{\beta_u - c}{2}\right) + \langle \mathbf{b},\, \mathbf{b}\rangle} \tag{4.13}$$

As

$$\frac{\langle \mathbf{b},\, \mathbf{b}\rangle}{\beta_u} \leq \langle \mathbf{B}^{-1}\mathbf{b},\, \mathbf{b}\rangle \leq \frac{\langle \mathbf{b},\, \mathbf{b}\rangle}{\beta_l}$$

it follows from (4.10) and(4.12) that

$$L_0^1(\lambda) \leq L_d^1(\lambda) = (\lambda - c)(\beta_l - \lambda) + \langle \mathbf{b},\, \mathbf{b}\rangle \tag{4.14}$$

$$U_0^1(\lambda) \geq U_d^1(\lambda) = (\lambda - c)(\beta_u - \lambda) + \langle \mathbf{b},\, \mathbf{b}\rangle \tag{4.15}$$

where we have used the subscript d to denote Dembo. It thus follows from (4.14) and (4.15) that $\lambda_1^{l,\,0}$ is larger than the Dembo lower bound which is the minimal zero of (4.14) given by

$$\lambda_1^{l,\,d} = \frac{\beta_l + c}{2} - \sqrt{\left(\frac{\beta_l - c}{2}\right) + \langle \mathbf{b},\, \mathbf{b}\rangle}$$

and that $\lambda_1^{u,\,0}$ is smaller than the Dembo lower bound which is the minimal zero of (4.15) given by

$$\lambda_1^{u,\,d} = \frac{\beta_u + c}{2} - \sqrt{\left(\frac{\beta_u - c}{2}\right) + \langle \mathbf{b},\, \mathbf{b}\rangle}$$

5. RESULTS

Consider the test matrix [13], which is symmetric positive definite.

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}$$

with minimum eigenvalue $\mathbf{1.425687}$ and maximum eigenvalue $\mathbf{9.375939}$ accurate to six decimal places. We use $\beta_1 = 4.585786$ and $\beta_u = 7.414214$, which are obtained from $\sigma(\mathbf{B})$ accurate to six digits in order to illustrate the efficacy of our method. While exact formula may be derived for these zeros of order up to four, it is easier to use the Newton method or a function root finder to locate these bounds. It is not necessary to evaluate powers of $\mathbf{B}$ or $\mathbf{B}^{-1}$ or even to determine $\mathbf{B}^{-1}$ explicitly. For example the computation of $L_4^n(\lambda)$ requires $\langle \mathbf{B}^k\mathbf{b},\, \mathbf{b}\rangle$ for $k = 1,\, 2,\, \cdots,\, 4$. Let $\mathbf{z}_1 = \mathbf{B}\mathbf{b}$ and $\mathbf{z}_2 = \mathbf{B}\mathbf{z}_1$ then $\langle \mathbf{B}\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{z}_1,\, \mathbf{b}\rangle$, $\langle \mathbf{B}^2\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{z}_1,\, \mathbf{z}_1\rangle$, $\langle \mathbf{B}^3\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{z}_2,\, \mathbf{z}_1\rangle$ and $\langle \mathbf{B}^4\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{z}_2,\, \mathbf{z}_2\rangle$. The computation of $U_4^1(\lambda)$ for example requires $\langle \mathbf{B}^{-k}\mathbf{b},\, \mathbf{b}\rangle$ for $k = 1,\, 2,\, \cdots,\, 5$. Let $\mathbf{B}\mathbf{y}_1 = \mathbf{b}$, $\mathbf{B}\mathbf{y}_2 = \mathbf{y}_1$ and $\mathbf{B}\mathbf{y}_3 = \mathbf{y}_2$, where $\mathbf{y}_1,\, \mathbf{y}_2,\, \mathbf{y}_3$ are determined by a linear solver (or LU decomposition for higher order polynomials). Then $\langle \mathbf{B}^{-1}\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{y}_1,\, \mathbf{b}\rangle$, $\langle \mathbf{B}^{-2}\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{y}_1,\, \mathbf{y}_1\rangle$, $\langle \mathbf{B}^{-3}\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{y}_2,\, \mathbf{y}_1\rangle$, $\langle \mathbf{B}^{-4}\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{y}_2,\, \mathbf{y}_2\rangle$ and $\langle \mathbf{B}^{-5}\mathbf{b},\, \mathbf{b}\rangle = \langle \mathbf{y}_3,\, \mathbf{y}_2\rangle$. We present result for orders up to six in table 5.1. It is noted that very good bounds are achieved for relatively low orders. From (3.5) it can be shown that

$$\begin{aligned} l_p^n(\lambda) - l_{p+1}^n(\lambda) &\le \frac{\beta_u^{p+1} - \beta_l^{p+1}}{\beta_u^{p+1}(\beta_u - \beta_l)}\langle \mathbf{b}, \mathbf{b}\rangle \\ &= \frac{1 - \left(\frac{\beta_l}{\beta_u}\right)^{p+1}}{\beta_u - \beta_l}\langle \mathbf{b}, \mathbf{b}\rangle. \end{aligned}$$

So for $\beta_l << \beta_u$ and for relatively small values of p, the zeros of $l_p^n(\lambda)$ and $l_{p+1}^n(\lambda)$ are close together and not much is gained by using very high orders of $l_p^n(\lambda)$ or $L_p^n(\lambda)$. A similar pattern is true in this case for the polynomials $U_p^n(\lambda)$, $L_p^1(\lambda)$ and $U_p^1(\lambda)$.

6. CONCLUSION

We have derived convenient recurrence relationships for the polynomials whose minimal zeros bound the smallest eigenvalue of positive definite matrices. Also the lower bound on this eigenvalue is guaranteed to be positive for relatively low orders of the polynomials as opposed to trace methods. Similarly we present polynomials whose maximal zeros bound the largest eigenvalue. These zeros are both easy and simple to compute using little computational effort.

p	$\lambda_1^{l,p}$	$\lambda_1^{u,p}$	$\lambda_n^{l,p}$	$\lambda_n^{u,p}$
0	2.852066	3.492517	7.910321	9.696369
1	1.350105	1.463952	8.338852	9.571525
2	1.363644	1.456318	8.636966	9.496576
3	1.365617	1.454813	8.842526	9.450860
4	1.365907	1.454512	8.985627	9.422709
5	1.365950	1.454450	9.086765	9.405258
6	1.365957	1.454438	9.159378	9.394385

Table 5.1: Extremal bounds for λ_1 *and* λ_n

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