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COMMUTATOR FOR SINGULAR OPERATORS ON VARIABLE EXPONENT SEQUENCE SPACES AND THEIR CORRESPONDING ERGODIC VERSION

A.M. ALPHONSE AND S.S.S. ANUPINDI

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DEPARTMENT OF MATHEMATICS, BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE- PILANI, HYDERABAD CAMPUS, JAWAHAR NAGAR, KAPRA MANDAL, DISTRICT.-MEDCHAL-500 078, TELANGANA, INDIA.
alphonse@hyderabad.bits-pilani.ac.in, p20180442@hyderabad.bits-pilani.ac.in
URL: https://www.bits-pilani.ac.in/hyderabad/a-michael-alphonse
https://www.bits-pilani.ac.in/research_scholars/sri-sakti-swarup-anupindi

ABSTRACT. In this paper, we prove strong type inequality for maximal commutator of singular operator on weighted lp spaces. Using these results we prove strong type inequality for the maximal commutator of singular operator on variable exponent sequence spaces.
Using Calderon-Coifman-Weiss transference principle we prove strong type inequality for maximal ergodic commutator of singular operator on a probability space equipped with measure preserving transformation U.

Key words and phrases: BMO; Commutator Of Maximal Ergodic Singular Operator; Variable Exponent Sequence Spaces; Ergodic Rectangles.

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1. Introduction

In [18] John and Nirenberg introduced the space BMO. A locally integrable function b defined on $\mathbb{R}$ is said to be in BMO($\mathbb{R}$) if

$$\sup_I \frac{1}{|I|}|b(x) - b_I|dx = \|b\|_{\star,\mathbb{R}} < \infty$$

where $b_I = \frac{1}{|I|}\int_I b(x)dx$ and the supremum is taken over all finite intervals in $\mathbb{R}$. $\| \quad \|_{\star,\mathbb{R}}$ is defined as the norm on BMO($\mathbb{R}$). With this norm BMO($\mathbb{R}$) is a Banach space. For standard results on BMO($\mathbb{R}$) see [23]. Most results on BMO($\mathbb{R}$) carry over to BMO($\mathbb{Z}$) and the proofs are more or less the same. For details we refer to [21].

In [8] Coifman, Rochberg and Weiss proveed that if $b \in BMO(\mathbb{R})$ then $[b, H]$ is a bounded operator on $L^p(\mathbb{R})$ for $1 < p < \infty$. Conversely, if $[b, H]$ is bounded on $L^p(\mathbb{R})$ for some $p, 1 < p < \infty$, then $b \in BMO(\mathbb{R})$.

Commutators were introduced by R. Coifman, R. Rochberg and G. Weiss in [9], who used them to extend the classical theory of H^p spaces to higher dimensions. They proved that $b \in BMO$ is sufficient for $[b, T]$ to be bounded and proved a partial converse. The full converse is due to S.Janson in [17] .

In this paper, we study the commutator $[b, T_\phi]$ of the operator of pointwise multiplication by a sequence $\{b(n) : n \in \mathbb{Z}\}$ and a singular operator T_ϕ. This is given by

$$[b, T_\phi]a(n) = b(n)T_\phi a(n) - T_\phi(ba)(n)$$

If $b \in BMO(\mathbb{Z})$ and T_ϕ is a discrete singular operator, we show that the maximal commutator

$$[b, T_\phi]^\star a(n) = \sup_N |\sum_{k=-N}^{N} \phi(k)[b(n) - b(n-k)]a(n-k)|$$

is bounded on $\ell^p_w(\mathbb{Z}), 1 < p < \infty, w \in A_p$.

For a probability space $(X, \mathcal{B}, \mu)$ and an invertible measure preserving transformation U on X, the space BMO(X) is defined as the space of those functions $b \in L^1(X)$ satisfying

$$\operatorname*{ess\,sup}_{x\in X}[\sup_{N\geq 1} \frac{1}{2N+1}\sum_{k=-N}^{N}|b(U^k x) - \frac{1}{2N+1}\sum_{j=-N}^{N} b(U^j x)|] = \|b\|_\star < \infty$$

In [9] Coifman and Weiss showed that this space is the dual space of the ergodic Hardy space $H^1(X) = \left\{f \in L^1(X) : \tilde{H}f \in L^1(X)\right\}$. Let $(X, \mathcal{B}, \mu)$ be a probability space and U a measure preserving transformation. We also study the commutator of the operator of pointwise multiplication by a function b on X and an ergodic singular operator $\tilde{T}_\phi$,

$$[b, \tilde{T}_\phi]f(x) = b(x)\tilde{T}_\phi f(x) - \tilde{T}_\phi(bf)(x), \quad f \in L^p(X)$$

To prove the almost everywhere existence of $[b, \tilde{T}_\phi]f$, for $f \in L^p(X)$ and the boundedness of this operator, we again consider the maximal operator

$$[b, \tilde{T}_\phi]^\star f(x) = \sup_N |\sum_{k=-N}^{N} \phi(k)[b(x) - b(U^{-k}x)]f(U^{-k}x)|$$

For $1 < p < \infty, b \in BMO(X)$, we show that this maximal commutator is bounded on $L^p_w(X, \mathcal{B}, \mu)$. Here w is an ergodic A_p weight. The commutator of the Hilbert transform on sequence spaces $\ell^p(\mathbb{Z}), 1 < p < \infty$ and their corresponding ergodic verion are studied in [21].

2. DEFINITIONS AND NOTATION

Throughout this paper, $\mathbb{Z}$ denotes set of all integers and $\mathbb{Z}_+$ denotes set of all positive integers. For a given interval I in $\mathbb{Z}$ (We always mean finite interval of integers) , $|I|$ always denotes the cardinality of I. For each positive integer N, consider collection of disjoint intervals of cardinality 2^N,

$$\{I_{N,j}\}_{j\in\mathbb{Z}} = \left\{[(j-1)2^N+1,\dots,j2^N]\right\}_{j\in\mathbb{Z}}$$

The set of intervals which are of the form $I_{N,j}$ where $N\in\mathbb{Z}_+$ and $j\in\mathbb{Z}$ are called dyadic intervals. For fixed N, $I_{N,j}$ are disjoint.

Let I be an interval in $\mathbb{Z}$ with center j_0. If I is an interval of length $2N$, by centre we mean

$$I = [j_0-N-1, j_0, j_0+1,\dots,j_0+N]$$

If I is an interval of length $2N+1$, by centre we mean

$$I = [j_0-N,\dots,j_0,\dots,j_0+N]$$

With j_0 be center of a dyadic interval, if j_0 is odd, $2I$ denotes an interval $[j_0-2N, j_0+2N]$. If j_0 is even, the left half would be slightly less than the right half and this slight difference will be observed in the constant derived in the inequalities.

For a given sequence $\{a(n): n\in\mathbb{Z}\}$ and an interval I, $a(I) = \sum_{k\in I} a(k)$. For a sequence $\{p(n): n\in\mathbb{Z}, p(n)\geq 1\}$, define $p_- = \inf\{p(n): n\in\mathbb{Z}\}, p_+ = \sup\{p(n): n\in\mathbb{Z}\}$. Throughout this paper, we assume $p_+<\infty$ and

$$1\leq p_-\leq p(n)\leq p_+<\infty, n\in\mathbb{Z}.$$

We denote set of all such sequences $\{p(n): n\in\mathbb{Z}\}$ by $\mathcal{S}$.
Throughout the paper $C, C_1, C_p\dots$ denote the constants which may change from one line to the next.

3. MAXIMAL OPERATORS

Let $\{a(n): n\in\mathbb{Z}\}$ be a sequence. We define the following three types of Hardy-Littlewood maximal operators as follows:

Definition 3.1. Given a sequence $\{a(n): n\in\mathbb{Z}\}$ and an interval I, let a_I denote average of $\{a(n): n\in\mathbb{Z}\}$ on I. Let, $a_I = \frac{1}{|I|}\sum_{m\in I} a(m)$. Define the sharp maximal operator $M^\#$ as follows

$$M^\# a(m) = \sup_{m\in I}\frac{1}{|I|}\sum_{n\in I}|a(n)-a_I|$$

where the supremum is taken over all intervals I containing m.
A sequence $\mathbf{b} = \{b(n)\}$ is said to be in $\mathbf{BMO}(\mathbb{Z})$ if $M^\# b\in\ell^\infty$. We define an norm on BMO($\mathbb{Z}$) as $\|b\|_\star = \left\|M^\# b\right\|_\infty$. Then the space BMO($\mathbb{Z}$) is a Banach space.

If I_r is the interval $\{-r,-r+1,\dots,0,1,2,\dots,r-1,r\}$, define centered Hardy-Littlewood maximal operator

$$M' a(m) = \sup_{r>0}\frac{1}{(2r+1)}\sum_{n\in I_r}|a(m-n)|$$

We define Hardy-Littlewood maximal operator as follows

$$M a(m) = \sup_{m\in I}\frac{1}{|I|}\sum_{n\in I}|a(n)|$$

where the supremum is taken over all intervals containing m.

We define dyadic Hardy-Littlewood maximal operator as follows:

$$M_d a(m) = \sup_{m \in I} \frac{1}{|I|} \sum_{k \in I} |a(k)|$$

where supremum is taken over all dyadic intervals containing m.

4. Relations between Maximal operators

In the following lemmas, we give relations between maximal operators. For the proofs of the following lemmas, refer [1]. These relations will be used when we prove the weighted inequalities for maximal ergodic operators.

Lemma 4.1. *Given a sequence $\{a(m) : m \in \mathbb{Z}\}$, the following relation holds:*

$$M'a(m) \leq Ma(m) \leq 3M'a(m)$$

Lemma 4.2. *If $\boldsymbol{a} = \{a(k) : k \in \mathbb{Z}\}$ is a non-negative sequence with $\boldsymbol{a} \in \ell_1$, then*

$$|\{m \in \mathbb{Z} : M'a(m) > 4\lambda\}| \leq 3|\{m \in \mathbb{Z} : M_d a(m) > \lambda\}|$$

In the following lemma, we see that in the norm of BMO($\mathbb{Z}$) space, we can replace the average a_I of $\{a(n)\}$ by a constant b. The proof is similar to the proof in continuous version [15]. The second inequality follows from $||a| - |b|| \leq |a| - |b|$.

Lemma 4.3. *Consider a non-negative sequence $\boldsymbol{a} = \{a(k) : k \in \mathbb{Z}\}$. Then the following are valid.*

1. $\frac{1}{2} \|a\|_\star \leq \sup_{m \in I} \inf_{b \in \mathbb{Z}} \frac{1}{|I|} |a(m) - b| \leq \|a\|_\star$
2. $M^{\#}(|a|)(i) \leq M^{\#} a(i), i \in \mathbb{Z}$

5. Norm in Variable Sequence Spaces

Definition 5.1. Given a bounded sequence $\{p(n) : n \in \mathbb{Z}\}$ which takes values in $[1, \infty)$, define $\ell^{p(\cdot)}(\mathbb{Z})$ to be set of all sequences $\{a(n) : n \in \mathbb{Z}\}$ such that for some $\lambda > 0$, $\sum_{k \in \mathbb{Z}} (\frac{|a(k)|}{\lambda})^{p(k)} < \infty$.

We define modular functional for variable sequences spaces associated with $p(\cdot)$ as

$$\rho_{p(\cdot)}(a) = \sum_{k \in \mathbb{Z}} |a(k)|^{p(k)}$$

Further for a given sequence $\{a(k) : k \in \mathbb{Z}\}$ in $\ell^{p(\cdot)}(\mathbb{Z})$, we define

$$\|a\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}(\frac{a}{\lambda}) \leq 1 \right\}$$

$\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$ is a norm in $\ell^{p(\cdot)}(\mathbb{Z})$ [16].

6. WEIGHTS

Definition 6.1. For a fixed p, $1 < p < \infty$, we say that a non-negative sequence $\{w(n) : n \in \mathbb{Z}\}$ belongs to class A_p if there is a constant C such that, for all intervals I in $\mathbb{Z}$, we have

$$\left(\frac{1}{|I|}\sum_{k\in I} w(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} w(k)^{-\frac{1}{p-1}}\right)^{p-1} \leq C$$

Infimum of all such constants C is called A_p constant. We say that $\{w(m) : m \in \mathbb{Z}\}$ belongs to class A_1 if there a constant C such that, for all intervals I in $\mathbb{Z}$,

$$\frac{1}{|I|}\sum_{k\in I} w(k) \leq Cw(m)$$

for all $m \in I$. Infimum of all such constants C is called A_1 constant.
Let $1 \leq p < \infty$ and $\{w(n) : n \in \mathbb{Z}\} \in A_p(\mathbb{Z})$. We say that a sequence $\{a(n) : n \in \mathbb{Z}\}$ is in $\ell^p_w(\mathbb{Z})$ if

$$\sum_{n\in\mathbb{Z}} |a(n)|^p w(n) < \infty$$

We define norm in $\ell^p_w(\mathbb{Z})$ by

$$\|a\|_{\ell^p_w(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |a(k)|^p w(k)\right)^{\frac{1}{p}}$$

For a subset A of $\mathbb{Z}$, $w(A)$ always denotes $\sum_{k\in A} w(k)$.
For a given sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_w(\mathbb{Z}), 1 \leq p < \infty$, the weighted weak type (p,p) inequality for a non-negative weight sequence $\{w(n) : n \in \mathbb{Z}\}$ is as follows:

$$w(\{m \in \mathbb{Z} : Ma(m) > \lambda\}) \leq \frac{C}{\lambda^p}\sum_{m\in\mathbb{Z}} |a(m)|^p w(m) \tag{A4}$$

Definition 6.2. Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. Suppose $1 < p < \infty$ and $w : X \to \mathbb{R}$ be a non-negative integrable function. The function w is said to satisfy ergodic A_p condition,

$$esssup_{x\in X} \sup_{N\geq 1}\left(\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x)\right)\left(\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x)^{\frac{-1}{p-1}}\right)^{p-1} \leq C$$

The function w is said to satisfy ergodic A_1 condition,

$$esssup_{x\in X} \sup_{N\geq 1}\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x) \leq Cw(U^m x)$$

for $m = -N, -N+1, \ldots, N$

7. SINGULAR OPERATORS

Definition 7.1. A sequence $\{\phi(n)\}$ is said to be a singular kernel if there exist constants C_1 and $C_2 > 0$ such that

If $\phi = \{\phi(n)\}$ is a singular kernel and $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z}), 1 \leq p < \infty$, define
(S1): $\sum_{n=-N}^{N} \phi(n)$ converges as $N \to \infty$.

(S2): $\phi(0) = 0$ and $|\phi(n)| \leq \frac{C_1}{|n|}, n \neq 0$
(S3): $|\phi(n+1) - \phi(n)| \leq \frac{C_2}{n^2}, \quad n \neq 0.$

Define
$$T_\phi a(n) = (\phi \star a)(n) = \sum_{k \in \mathbb{Z}} \phi(n-k)a(k)$$
Since S2 implies that $\phi \in \ell^r$ for all $1 < r \leq \infty$, the above convolution is defined. The operator T_ϕ defined above is called discrete singular operator. For definition and results on singular operators on sequence spaces, we refer [20].
If ϕ is a singular kernel, then the truncations $\phi_N, N \geq 1$ are defined as follows.
$$\phi_N(n) = \begin{cases} \phi(n) & if \quad |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$
These truncations satisfy
$$(S3)': \quad \sup_n \sum_{|k|>2|n|} |\phi_N(k-n) - \phi_N(k)| \leq C_2$$
where C_2 does not depend on N. We remark that the $\{\phi_N\}$ need not satisfy (S3) uniformly in N (take $\phi(n) = \frac{1}{n}$ as an example)

The maximal singular operator corresponding to this singular operator is defined as
$$T^\star_\phi a(n) = \sup_N |\sum_{k=-N}^{N} \phi(k)a(n-k)|$$

8. Commutator on weighted sequence spaces $\ell^p_w(\mathbb{Z})$

Let $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$ and $f \in L^p(\mathbb{R}), 1 \leq p < \infty$.

Definition 8.1. We define commutator of singular operator as the operator of pointwise multiplication by a sequence $b = \{b(n) : n \in \mathbb{Z}\}$ and a singular operator T_ϕ on $\ell^p(\mathbb{Z})$. More precisely, we consider the operators given
$$([b, T_\phi]a)(n) = b(n)T_\phi a(n) - T_\phi(ba)(n) = \sum_{k=-\infty}^{\infty} \phi(k)[b(n) - b(n-k)]a(n-k).$$
and its maximal version $T^\star_\phi$ on $\ell^p(\mathbb{Z})$ which is defined as
$$([b, T_\phi]^\star a)(n) = \sup_N \left| \sum_{k=-N}^{N} \phi(k)[b(n) - b(n-k)]a(n-k) \right|.$$

Definition 8.2. We define commutator of maximal ergodic singular operator $\tilde{T}_\phi$
$$[b, \tilde{T}_\phi]^\star f(x) = \sup_{N \geq 1} \left| \sum_{k=-N}^{N} \phi(k)[b(x) - b(U^{-k}x)]f(U^{-k}x)) \right|.$$
and its truncated version corresponding to commutator of maximal ergodic singular operator $\tilde{T}^\star_\phi$ as follows: For $J \geq 1$
$$[b, \tilde{T}_\phi]^\star_J f(x) = \sup_{N \leq J} \left| \sum_{k=-N}^{N} [b(x) - b(U^{-k}x)]f(U^{-k}x)\phi(k) \right|.$$

Definition 8.3. We define discrete Hilbert transform and maximal discrete Hilbert transform for a sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$ as follows:

$$Ha(m) = \sum_{n\in\mathbb{Z}} \left|\frac{a(n)}{m-n}\right|.$$

$$H^\star a(n) = \sup_N \left|\sum_{k=-N}^{N} \frac{a(n-k)}{k}\right|, a \in \ell^p, \quad 1 \le p < \infty.$$

We define maximal ergodic Hilbert transform and truncated maximal ergodic Hilbert transform for a function $f \in L^p(\mathbb{R}), 1 \le p < \infty$ as follows:

$$\tilde{H}^\star f(x) = \sup_N \left|\sum_{k=-N}^{N} \frac{f(U^{-k}x)}{k}\right|.$$

$$\tilde{H}_J f(x) = \sup_{1\le n\le J} \left|\sum_{k=-n}^{n} \frac{f(U^{-k}x)}{k}\right|.$$

9. SOME RESULTS ON BMO$(\mathbb{Z})$

In order to prove the boundedness of commutator on weighted sequence spaces, we require the properties of BMO($\mathbb{Z}$) which we state and prove in the following lemmas.

One of the most important results about BMO is the John-Nirenberg inequality. As a consequence we get a family of equivalent norms on BMO($\mathbb{Z}$).

Lemma 9.1. *Let $b \in BMO(\mathbb{Z})$. Then there exists consstants $C_1, C_2 > 0$ such that, for every finite interval I in $\mathbb{Z}$ and $\lambda > 0$,*

$$\frac{|\{n \in I : |b(n) - b_I| > \lambda\}|}{|I|} \le C_1 e^{\frac{-C_2\lambda}{\|b\|_\star}}.$$

Proof. The key to the proof of the theorem is the Calderón-Zygmund decomposition restricted to an interval in $\mathbb{Z}$. Proof is same as the result in case of $\mathbb{R}$. For details refer [15]. ∎

John-Nirenberg theorem has an interesting corollary, namely, the reverse H$\ddot{o}$lder's inequality.

Corollary 9.2. *Let $b \in BMO(\mathbb{Z})$. Then for every finite $p > 1$,*

$$\sup_I \left(\frac{1}{|I|}\sum_{k\in I}|b(k) - b_I|^p\right)^{\frac{1}{p}} \le C_p \|b\|_\star.$$

Proof.

$$\begin{aligned}
&\left(\frac{1}{|I|}\sum_{k\in I}|b(k) - b_I|^p\right) \\
&= p\int_0^\infty \frac{\lambda^{p-1}|\{n \in I : |b(n) - b_I| > \lambda\}|}{|I|} d\lambda \\
&\le C_1 p \int_0^\infty \lambda^{p-1} e^{-C_2\frac{\lambda}{\|b\|_\star}} d\lambda \\
&\le \frac{C_1}{C_2^p} p \|b\|_\star^p \int_0^\infty \lambda^{p-1} e^{-\lambda} d\lambda = \frac{C_1}{C_2^p} p\Gamma_p \|b\|_\star^p = C_p \|b\|_\star^p.
\end{aligned}$$

∎

Lemma 9.3. *Let $b \in BMO(\mathbb{Z})$ and I, J be two finite intervals in $\mathbb{Z}, I \subset J$.*
(a) If $|J| \leq 2|I|$, then
$$|b_I - b_J| \leq 2 \|b\|_\star .$$
(b) If $|J| > 2|I|$, then
$$|b_I - b_J| \leq 2 \log[\frac{|J|}{|I|}] \|b\|_\star .$$

Proof.
(a)
$$|b_I - b_J| \leq \left| \frac{1}{|I|} \sum_{k \in I} [b(k) - b_J] \right| \leq \frac{2}{|J|} \sum_{k \in J} |b(k) - b_J| \leq 2 \|b\|_\star .$$

(b) Let $I = I_1 \subset I_2 \subset \dots I_n = J$. where $I_1, \dots I_n$ are intervals in $\mathbb{Z}$ such that $|I_{K+1}| \leq 2|I_k|$ and where $n \leq C \log\left(\frac{|J|}{|I|}\right)$. Repeated applications of (a) yields (b). ∎

Lemma 9.4. *Let $b \in BMO(\mathbb{Z})$, I any interval in $\mathbb{Z}$ and n_0 the centre of I. Then for each $r > 1$, there exists a constant C_r such that*
$$\left(\sum_{n \notin 3I} \frac{|b(n) - b_I|^r}{|n_0 - n|^r} \right)^{\frac{1}{r}} \leq \frac{C_r \|b\|_\star}{|I|^{\frac{1}{r'}}},$$
where $\frac{1}{r} + \frac{1}{r'} = 1$.

Proof. Recall that $3I = 2LI \cup 2RI$. For $k = 1, 2 \dots$, let $I_k = 3^k I$ and $J_k = I_{k+1} \setminus I_k$. Now
$$\left(\sum_{n \notin 3I} \frac{|b(n) - b_I|^r}{|n_0 - n|^r} \right)^{\frac{1}{r}} = \left(\sum_{k=1}^{\infty} \sum_{n \in J_k} \left(\frac{|b(n) - b_I|^r}{|n_0 - n|^r} \right)^{\frac{1}{r}} \right)$$
$$\leq \left(\sum_{k=0}^{\infty} \sum_{n \in J_k} \frac{|b(n) - b_I|^r}{2^{kr} |I|^r} \right)^{\frac{1}{r}}$$
$$\leq \left(\sum_{k=0}^{\infty} \sum_{n \in J_k} \frac{|b(n) - b_{I_{k+1}}|^r}{2^{kr} |I|^r} \right)^{\frac{1}{r}} + \left(\sum_{k=0}^{\infty} \sum_{n \in J_k} \frac{|b_I - b_{I_{k+1}}|^r}{2^{kr} |I|^r} \right)^{\frac{1}{r}}$$
$$= A_1 + A_2.$$

Then using Corollary 9.2, we have
$$A_1 \leq \left(\sum_{k=1}^{\infty} \frac{4}{2^{(r-1)k|I|^{r-1}}} \frac{1}{|I_{k+1}|} \sum_{n \in I_{k+1}} |b(n) - b_{I_{k+1}}|^r \right)^{\frac{1}{r}}$$
$$\leq \left(\sum_{k=1}^{\infty} \frac{4}{2^{(r-1)k|I|^{r-1}}} \|b\|_\star^r \right)^{\frac{1}{r}} \leq \frac{C}{|I|^{\frac{1}{r'}}} \|b\|_\star .$$

To estimate A_2, we use (b) part of Lemma 9.3
$$A_2 \leq \left(\sum_{k=1}^{\infty} \sum_{n \in I_{k+1} \setminus I_k} \frac{[\log(\frac{|I_{k+1}|}{|I|})]^r \|b\|_\star^r}{2^{rk} |I|^r} \right)^{\frac{1}{r}}$$
$$\leq C \left(\sum_{k=1}^{\infty} \frac{2^{k+2} |I| (\log 2^{k+1})^r \|b\|_\star^r}{2^{rk} |I|^r} \right)^{\frac{1}{r}}$$

$$\leq C \frac{\|b\|_\star 4^{\frac{1}{r}}}{|I|^{\frac{1}{r'}}} \left(\sum_{k=1}^{\infty} \frac{(k+1)^r}{2^{(r-1)k}} \right)^{\frac{1}{r}} \leq \frac{C \|b\|_\star}{|I|^{\frac{1}{r'}}}.$$

∎

10. WEIGHTED SHARP MAXIMAL SEQUENCE THEOREM

The proof of weighted sharp maximal sequence theorem uses Calderón-Zygmund decomposition theorem for sequences, relations between maximal operators and weighted good-Lambda estimate which were proved in Chapter 4.

Theorem 10.1 (Weighted sharp maximal sequence theorem). *Let $1 \leq p < \infty$ and $w \in A_p(\mathbb{Z})$. Then there exists a constant $C_{p,w} > 0$ such that*

$$\|Ma\|_{p,w} \leq C_{p,w} \left\|a^\#\right\|_{p,w}, \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

11. WEIGHTED MAXIMAL COMMUTATOR THEOREM

In this section we prove weighted strong type inequalities for the discrete maximal commutator. the strong type inequalities without weights for the discrete maximal commutators can be found in [21].

Definition 11.1. For a sequence $b = \{b(n)\} \in$ **BMO**$(\mathbb{Z})$, define maximal commutator of singular integral operator as follows

$$[b, T_\phi]^\star a(n) = \sup_N \Big| \sum_{k=-N}^{N} \phi(k)[b(n) - b(n-k)]a(n-k) \Big|.$$

We want to prove that the maximal commutator is bounded on $\ell^p_w(\mathbb{Z})$, where $1 < p < \infty$ which we state it as the following theorem.

Theorem 11.1 (Weighted maximal commutator theorem). *Let $1 < p < \infty$, $b \in BMO(\mathbb{Z})$. Then there exists a constant $C_P > 0$ such that*

$$\|[b, T_\phi]^\star a\|_{\ell^p_w(\mathbb{Z})} \leq C_p \|a\|_{\ell^p_w(\mathbb{Z})}.$$

Condition $(S3)$ in the definition of singular kernel plays a crucial role in this proof. Let ϕ_N denote truncation of ϕ which is defined as follows

$$\phi_N(k) = \begin{cases} \phi(k) & \text{if} \quad |k| \leq N, \\ 0 & \text{if} |k| > n. \end{cases} \tag{11.1}$$

The proof of the maximal commutator inequalities would have been simpler if the ϕ_N's satisfied $(S3)$ uniformly in N. However this is not true even for $\phi(n) = \frac{1}{n}$. To overcome this difficulty we dominate $[b, T_\phi]^\star$ by a sum of two operators $[b, T_\nu]^\star$ and $[b, T_{|\psi|}]^\star$, whose truncated kernels ν_N, ψ_N satisfy $S3$ uniformly in N. Then we prove the boundedness of the corresponding maximal operators on $\ell^p_w(\mathbb{Z})$. We define the kernels $\{\nu\}, \{\psi\}$ and their truncation$\{\nu_N\}, \{\psi_N\}$ as follows.

Definition 11.2. Consider the differentiable function ν and ψ defined on $(0, \infty)$ by

$$\nu(t) = \begin{cases} 1 & \text{if} \quad 0 < t \leq \frac{1}{2} \\ \frac{1}{2}[1 - \cos 2\pi t] & \text{if} \quad \frac{1}{2} < t \leq 1 \\ 0 & \text{if} \quad t > 1 \end{cases} \tag{11.2}$$

$$\psi(t) = \begin{cases} 0 & \text{if} \quad 0 < t \leq \frac{1}{2} \\ \frac{1}{2}[1 + \cos 2\pi t] & \text{if} \quad \frac{1}{2} < t \leq 1 \\ 1 & \text{if} \quad 1 < t < 2 \\ \frac{1}{2}[1 - \cos \frac{\pi t}{2}] & \text{if} \quad 2 \leq t \leq 4 \\ 0 & \text{if} \quad t > 4 \end{cases} \tag{11.3}$$

Observe that

$$|\chi_{[0,1)}(t) - \nu(t)| \leq \psi(t), \quad t \in (0, \infty). \tag{11.4}$$

For $j \in Z$, let

$$\nu_N(j) = \phi(j)\nu(\frac{|j|}{N})$$
$$\psi_N(j) = \phi(j)\psi(\frac{|j|}{N}).$$

Using the kernels $\{\nu_N\}$ and $\{\psi_N\}$, we define the operators $[b, T_\nu]^\star$ and $[b, T_{|\psi|}]^\star$ as

$$[b, T_\nu]^\star a(n) = \sup_{N \geq 1} \left| \sum_{j=-\infty}^{\infty} [b(n) - b(j)]\nu_N(n-j)a(j) \right|$$
$$[b, T_{|\psi|}]^\star a(n), = \sup_{N \geq 1} \sum_{j=-\infty}^{\infty} |[b(n) - b(j)]||\psi_N(n-j)||a(j)|.$$

Because of inequality 11.4, we can prove the following lemma.

Lemma 11.2. *For each $n \in \mathbb{Z}$*

$$[b, T_\phi]^\star a(n) \leq [b, T_\nu]^\star a(n) + [b, T_{|\psi|}]^\star a(n).$$

As we mentioned earlier in the following lemma, we prove both the truncated kernels $\{\nu_N\}$, $\{\psi_N\}$ satisfy $(S3)$ uniformly in N.

Lemma 11.3. *There exists a constant $C > 0$ such that*

$$|\nu_N(n-j) - \nu_N(n)| \leq \frac{C|j|}{(n-j)^2} \quad \textit{for} \quad |n| > 2|j| \quad \textit{and} \quad \forall N \geq 1. \tag{11.3[A]}$$

$$|\psi_N(n-j) - \psi_N(n)| \leq \frac{C|j|}{(n-j)^2} \quad \textit{for} \quad |n| > 2|j| \quad \textit{and} \quad \forall N \geq 1. \tag{11.3[B]}$$

Proof. We will prove first of the inequalities $11.3[A]$. The proof for second inequality $11.3[A]$ is similar.
Consider the kernel $\{\nu_N\}$. Let $|n| > 2|j|$. Then as in [2] we can show that

$$|\nu_N(n-j) - \nu_N(n)| = \left| \phi(n-j)\nu(\frac{|n-j|}{N}) - \phi(n)\nu(\frac{|n|}{N}) \right|$$
$$\leq |\phi(n-j) - \phi(n)||\nu(\frac{|n|}{N})| + |\phi(n-j)||\nu(\frac{|n|}{N}) - \nu(\frac{|n-j|}{N})|$$
$$\leq C\frac{|j|}{(n-j)^2} + C\frac{1}{|n-j|}\left|\nu(\frac{|n|}{N}) - \nu(\frac{|n-j|}{N})\right|.$$

Since supp $\nu \subseteq (0, 1]$ and $|n| > 2|j|$,

$$|\nu(\frac{|n-j|}{N}) - \nu(\frac{|n|}{N})| = 0 \quad \text{if} \quad \frac{|n-j|}{N} > 2.$$

If $\frac{|n-j|}{N} \leq 2$, applying the mean value theorem, we get

$$\left|\nu(\frac{|n-j|}{N}) - \nu(\frac{|n|}{N})\right| \leq \frac{|j|}{N}\nu'(t_0).$$

where t_0 is a point between $\frac{|n-j|}{N}$ and $\frac{|n|}{N}$. But $|\nu'(t)| \leq \pi, \forall t \in (0, \infty)$. Therefore,

$$|\nu(\frac{|n-j|}{N}) - \nu(\frac{|n|}{N})| \leq \frac{|j|\pi}{N} \leq \frac{2\pi|j|}{|n-j|}.$$

Hence, the kernels $\{\nu_N\}$ satisfy condition S3 uniformly. ∎

For proving the boundedness of the operators $[b, T_\nu]^\star$ and $[b, T_{|\psi|}]^\star$ on $\ell^p_w(\mathbb{Z})$, we need to consider the maximal operators $T^\star_\nu$ and $T^\star_{|\psi|}$ defined as:

$$T^\star_\nu = \sup_{N\geq 1} |\sum_{k=-\infty}^{\infty} \nu_N(n-k)a(k)|$$

$$T^\star_{|\psi|} = \sup_{N\geq 1} |\sum_{k=-\infty}^{\infty} \psi_N(n-k)a(k)|.$$

Lemma 11.4. *Let $1 < p < \infty$. Then there exists a constant $C_p > 0$ such that*

$$\|T^\star_\nu a\|_{\ell^p_w(\mathbb{Z})} \leq c_p \|a\|_{\ell^p_w(\mathbb{Z})} \quad \forall a \in \ell^p_w(\mathbb{Z})$$
$$\left\|T^\star_{|\psi|} a\right\|_{\ell^p_w(\mathbb{Z})} \leq c_p \|a\|_{\ell^p_w(\mathbb{Z})} \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

Proof. For a non negative real number α. let $[\alpha]$ denote the greatest integer less than or equal to α. Then

$$\begin{aligned}
&|\sum_{|n-j|\leq N} \phi(n-j)\nu(\frac{|n-j|}{N})a(j)| \\
&\leq |\sum_{|n-j|\leq[N/2]} \phi(n-j)a(j)| + \sum_{N\geq|n-j|>[N/2]} |\phi(n-j)||a(j)| \\
&\leq |\sum_{|n-j|\leq[N/2]} \phi(n-j)a(j)| + C_2 \sum_{N\geq|n-j|>[N/2]} \frac{|a(j)|}{|n-j|}. \\
&\leq C[T^\star_\phi a(n) + Ma(n)]
\end{aligned}$$

where Ma is the Hardy-Littlewood maximal sequence of $\{a(n) : n \in \mathbb{Z}\}$. Therefore,

$$T^\star_\nu a(n) \leq C[T^\star_\phi a(n) + Ma(n)].$$

Since we already prove that M and $T^\star_\phi$ are bounded on $\ell^p_w(\mathbb{Z})$ in Chapter4 and Chapter5 respectively , it follows that $T^\star_\nu$ is also bounded on $\ell^p_w(\mathbb{Z})$.
For the proof of second inequality, fix N and consider

$$\begin{aligned}
&\sum_{|n-j|\leq 4N} |\phi(n-j)|\psi(\frac{|n-j|}{N})|a(j)| \\
&\leq \sum_{4N\geq|n-j|>N/2} |\phi(n-j)||a(j)| \leq C_2 \sum_{4N\geq|n-j|>N/2} \frac{|a(j)|}{|n-j|} \\
&\leq \frac{C}{8N+1} \sum_{|n-j|\leq 4N} |a(j)| \leq CMa(n).
\end{aligned}$$

Therefore, $T^{\star}_{|\psi|}a(n) \leq CMa(n)$. It follows that $T^{\star}_{|\psi|}$ is bounded on $\ell^p_w(\mathbb{Z})$. ∎

Theorem 11.5. *Let $1 < p < \infty$ and $B \in BMO(\mathbb{Z})$. Then there exists a constant $C_p > 0$ such that*

$$\|[b,T_\nu]^\star a\|_{\ell^p_w(\mathbb{Z})} \leq c_p \|a\|_{\ell^p_w(\mathbb{Z})} \quad \forall a \in \ell^p_w(\mathbb{Z}).$$
$$\left\|[b,T_{|\psi|}]^\star a\right\|_{\ell^p_w(\mathbb{Z})} \leq c_p \|a\|_{\ell^p_w(\mathbb{Z})} \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

Proof. For J=1,2,3... define

$$V_J a(n) = \sup_{N \leq J} \left| \sum_{j=n-N}^{n+N} [b(n) - b(j)]\nu_N(n-j)a(j) \right|.$$
$$W_J a(n) = \sup_{N \leq J} \sum_{j=n-4N}^{n+4N} |b(n) - b(j)||\psi_N(n-j)||a(j)|.$$

Then

$$[b,T_\nu]^\star a(n) = \sup_J V_J a(n).$$
$$[b,T_{|\psi|}]^\star a(n) = \sup_J W_J a(n).$$

We will prove that

$$\|V_J a\|_{\ell^p_w(\mathbb{Z})} \leq C \|a\|_{\ell^p_w(\mathbb{Z})}.$$
$$\|W_J a\|_{\ell^p_w(\mathbb{Z})} \leq C \|a\|_{\ell^p_w(\mathbb{Z})}.$$

For the proof of above inequalities, we first obtain estimates for the corresponding sharp maximal sequences. Then we will prove Theorem 11.5 using weighted sharp maximal sequence theorem i.e Theorem 10.1. These estimates are proved in the following lemma. ∎

Lemma 11.6. *Let $r > 1$ and $\{a(n) : n \in \mathbb{Z}\}$ be a sequence. then there exist constants C and C_1 such that*

$$\sup_J (V_J a)^{\#}(n) \leq C \|b\|_\star [M(T^\star_\nu a)^r(n)]^{\frac{1}{r}} + [M(a)^r(n)]^{\frac{1}{r}} \tag{11.5}$$

$$\sup_J (W_J a)^{\#}(n) \leq C_1 \|b\|_\star [M(T^\star_{|\psi|} a)^r(n)]^{\frac{1}{r}} + [M(a)^r(n)]^{\frac{1}{r}}. \tag{11.6}$$

Proof. Fix $J \geq 1$ and $n \in \mathbb{Z}$. Then if I is an interval containing n, put

$$C_I = \sup_{N \leq J} |\sum_{i=-\infty}^{\infty} [b(i) - b_I]\nu_N(j_o - i)a\chi_{Z\setminus 3I}(i)|.$$

where j_o is centre of I. Then, for $j \in I$

$$\begin{aligned}
&|V_J a(j) - C_I| = \\
&\left| \sup_{N \leq J} |\sum_{i=-\infty}^{\infty} [b(j) - b(i)]\nu_N(j-i)a(i)| - \sup_{N \leq J} |\sum_{i=-\infty}^{\infty} [b_I - b(i)]\nu_N(j_o - i)a\chi_{Z\setminus 3I}(i)| \right| \\
&\leq \sup_{N \leq J} \left| \sum_{i=-\infty}^{\infty} [b(j) - b(i)]\nu_N(j-i)a(i) - \sum_{i=-\infty}^{\infty} [b_I - b(i)]\nu_N(j_o - i)a\chi_{Z\setminus 3I}(i) \right| \\
&\leq \sup_{N \leq J} \left| \sum_{i=-\infty}^{\infty} [b(j) - b_I]\nu_N(j-i)a(i) \right|
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{N\le J}\left|\sum_{i=-\infty}^{\infty}[b(i)-b_I]\nu_N(j_o-i)a\chi_{3I}(i)\right| \\
&+ \sup_{N\le J}\left|\sum_{i=-\infty}^{\infty}[b(i)-b_I][\nu_N(j-i)-\nu_N(j_o-i)]a\chi_{Z\setminus 3I}(i)\right| \\
&= A_1(j)+A_2(j)+A_3(j).
\end{aligned}
$$

For the first term, we have, with $\frac{1}{r}+\frac{1}{r'}=1$

$$
\begin{aligned}
\frac{1}{|I|}\sum_{j\in I}A_1(j) &\le \frac{1}{|I|}\sum_{j\in I}|b(j)-b_I|T_\nu^\star a(j) \\
&\le \left(\frac{1}{|I|}\sum_{j\in I}|b(j)-b_I|^{r'}\right)^{\frac{1}{r'}}\left(\frac{1}{|I|}\sum_{j\in I}|T_\nu^\star a(j)|^r\right)^{\frac{1}{r}} \\
&\le \|b\|_\star\,[M(T_\nu^\star a)^r(n)]^{\frac{1}{r}}.
\end{aligned}
$$

Now consider

$$
\begin{aligned}
\frac{1}{|I|}\sum_{j\in I}A_2(j) &\le \frac{1}{|I|}\sum_{j\in I}|T_\nu^\star[(b-b_I)a\chi_{3I}](j)| \\
&\le \left\{\frac{1}{|I|}\sum_{j\in I}|T_\nu^\star[(b-b_I)a\chi_{2I}](j)|^s\right\}^{\frac{1}{s}},
\end{aligned}
$$

where $s>1$. We can further replace the above summation over I by a summation over $\mathbb{Z}$. Then using the boundedness of $T_\nu^\star$ on l^s, we get

$$
\begin{aligned}
\frac{1}{|I|}\sum_{j\in I}A_2(j) &\le C\left\{\frac{1}{|I|}\sum_{j\in 3I}|b(j)-b_I|^s|a(j)|^s\right\}^{\frac{1}{s}} \\
&\le C\left\{\frac{1}{|3I|}\sum_{j\in 3I}|b(j)-b_I|^{sq}\right\}^{\frac{1}{sq}}\left\{\frac{1}{|3I|}\sum_{j\in 2I}|a(j)|^{sq'}\right\}^{\frac{1}{sq'}},
\end{aligned}
$$

where $q>1$ and $\frac{1}{q}+\frac{1}{q'}=1$
Now,

$$
\left\{\frac{1}{|3I|}\sum_{j\in 3I}|b(j)-b_I|^{sq}\right\}^{\frac{1}{sq}} \le C\left(\frac{1}{|3I|}\sum_{j\in 3I}|b(j)-b_{2I}|^{sq}\right)^{\frac{1}{sq}}+|b_{2I}-b_I| \le C\,\|b\|_\star.
$$

Therefore,

$$
\frac{1}{|I|}\sum_{j\in I}A_2(j) \le c\,\|b\|_\star\,[M(|a|^r)(n)]^{\frac{1}{r}}.
$$

provided we choose s and q' so that $sq'=r$. It remains to estimate $A_3(j), j\in I$.

$$
A_3(j) \le \sup_{N\le J}\sum_{i=-\infty}^{\infty}|b(i)-b_I||\nu_N(j-i)-\nu_N(j_0-i)||a\chi_{\mathbb{Z}\setminus 2I}(i)|.
$$

But since $i \notin 3I$ and $j \in I$, $|j_0 - i| > 2|j - j_0|$
Therefore,
$$|\nu_N(j-i) - \nu_N(j_0 - i)| \leq \frac{C|j-j_0|}{(j-i)^2} \leq \frac{C|j-j_0|}{(j_o-i)^2}.$$
Therefore,
$$A_3(j) \leq C \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i) - b_I| \frac{|j-j_0|}{(j_0-i)^2} |a\chi_{\mathbb{Z}\setminus 3I}(i)|$$
$$\leq C(|I|)\left(\sum_{i\notin 3I} \frac{|b(i)-b_I|^{r'}}{|j_0-i|^{r'}}\right)^{\frac{1}{r'}} \left(\sum_{i\notin 3I} \frac{|a(i)|^r}{|j_0-i|^r}\right)^{\frac{1}{r}}.$$
Now by Lemma 9.4
$$\left(\sum_{i\notin 3I} \frac{|b(i)-b_I|^{r'}}{|j_0-i|^{r'}}\right)^{\frac{1}{r'}} \leq \frac{C\,\|b\|_\star}{(|I|)^{\frac{1}{r}}}.$$
Now let $I_k = 3^k I$, then by standard techniques, we get
$$\left(\sum_{i\notin 3I} \frac{|a(i)|^r}{|j_0-i|^r}\right)^{\frac{1}{r}}. \leq \frac{C[M(|a|^r)(n)]^{\frac{1}{r}}}{|I|^{\frac{1}{r'}}}.$$
Therefore,
$$A_3(j) \leq C\,\|b\|_\star\,[M(|a|^r)(n)]^{1/r}.$$
and so,
$$\frac{1}{|I|} \sum_{j\in I} A_3(j) \leq C\,\|b\|_\star\,[M(|a|^r)(n)]^{1/r}.$$

For the proof of second inequality in Lemma 11.6, we proceed as follows. For $n \in \mathbb{Z}$ and any interval I containing n, we choose
$$C_I = \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i) - b_I||\psi_N(j_0-i)||a\chi_{\mathbb{Z}\setminus 3I}(i)|.$$
where j_0 is centre of I. Then
$$|W_j a(j) - C_I| =$$
$$\Bigg| \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(j)-b(i)||\psi_N(j-i)||a(i)|$$
$$- \sup_{N\leq j} \sum_{i=-\infty}^{\infty} |b(i)-b_I||\psi_N(j_0-i)||a\chi_{\mathbb{Z}\setminus 3I}(i)|\Bigg|$$
$$\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} \left| |b(j)-b(i)||\psi_N(j-i)||a(i)| - \Big(|b(i)-b_I||\psi_N(j_0-i)||a\chi_{Z\setminus 3I}(i)|\Big) \right|$$
$$\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} \Bigg| \Bigg[\Bigg| \Big((b(j)-b_I)|\psi_N(j-i)||a(i)| + (b_I - b(i))|\psi_N(j-i)||a\chi_{3I}(i)| \Big)$$
$$+ (b_I - b(i))|\psi_N(j-i)||a\chi_{Z\setminus 3I}(i)| \Bigg|\Bigg] - \Big(|b_I - b(i)||\psi_N(j_0-i)||a\chi_{Z\setminus 3I}| \Big) \Bigg|.$$

Now

$$\Big||x+y|-|z|\Big| \leq |x| + \Big||y|-|z|\Big|, \forall x,y,z \in \mathbb{C}. \tag{11.7}$$

We have

$$\begin{aligned}
&|W_j a(j) - C_I| \\
&\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} \Big||b(j)-b_I||\psi_N(j-i)||a(i)| + (b_I - b(i))|\psi_N(j-i)||a\chi_{3I}(i)|\Big| \\
&+ \Big||b(i)-b_I||\psi_N(j-i)||a\chi_{Z\setminus 3I(i)}| - |b(i)-b_I||\psi_N(j_0-i)||a\chi_{\mathbb{Z}\setminus 3I(i)}|\Big| \\
&\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(j)-b_I||\psi_N(j-i)||a(i)| \\
&+ \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i)-b_I||\psi_N(j-i)||a\chi_{3I}(i)| \\
&+ \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i)-b_I||\psi_N(j-i)-\psi_N(j_0-i)||a\chi_{\mathbb{Z}\setminus 3I(i)}| \\
&= B_1(j) + B_2(j) + B_3(j).
\end{aligned}$$

The estimates for each of these terms are obtained exactly as in the previous case by replacing ν by ψ . This concludes proof of Lemma 11.6. ∎

Here we prove the boundedness of sharp maximal sequences $\{(V_J a)^{\#}\}$, $\{(W_J a)^{\#}\}$ on $\ell^p_w(\mathbb{Z})$.

Theorem 11.7. *For $1<p<\infty$,*

$$\left\|(V_J a)^{\#}\right\|_{\ell^p_w(\mathbb{Z})} \leq C \left\|a\right\|_{\ell^p_w(\mathbb{Z})}, \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

$$\left\|(W_J a)^{\#}\right\|_{\ell^p_w(\mathbb{Z})} \leq C_1 \left\|a\right\|_{\ell^p_w(\mathbb{Z})}, \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

where C, C_1 are independent of J

Proof.

$$\begin{aligned}
&\left\|(V_J a)^{\#}\right\|_{\ell^p_w(\mathbb{Z})} \\
&= \left\{\sum_{n=-\infty}^{\infty} |(V_J a)^{\#}(n)|^p w(n)\right\}^{\frac{1}{p}} \\
&\leq C \left\{\sum_{n=-\infty}^{\infty} M(T^\star_\nu a)^r(n) w(n)]^{\frac{p}{r}}\right\}^{\frac{1}{p}} + \left\{\sum_{n=-\infty}^{\infty} [M(|a|^r(n)]^{\frac{p}{r}} w(n)\right\}^{\frac{1}{p}} \\
&\leq C \left\{\sum_{n=-\infty}^{\infty} [T^\star_\nu a(n)]^p w(n)\right\}^{\frac{1}{p}} + \left\{\sum_{n=-\infty}^{\infty} [a(n)]^p w(n)\right\}^{\frac{1}{p}} \\
&\leq C \left\|a\right\|_{\ell^p_w(\mathbb{Z})}.
\end{aligned}$$

By a similar argument

$$\left\|(W_J a)^\star\right\|_{\ell^p_w(\mathbb{Z})} \leq C \left\|a\right\|_{\ell^p_w(\mathbb{Z})}.$$

∎

It remains to prove $V_Ja, W_Ja \in \ell^p_w(\mathbb{Z})$. Then the boundedness of sequences $\{(V_Ja)\}, \{(W_Ja)\}$ on $\ell^p_w(\mathbb{Z})$ hold by the weighted sharp maximal sequence theorem i.e Theorem 10.1 as follows

$$\|(V_Ja)\|_p \leq \|M(V_Ja)\|_p \leq C_p \left\|(V_Ja)^{\#}\right\|_p, \quad \forall \quad a \in \ell^p_w(\mathbb{Z}).$$

$$\|(V_Ja)\|_p \leq \|M(V_Ja)\|_p \leq C_p \left\|(V_Ja)^{\#}\right\|_p, \quad \forall \quad \in \ell^p_w(\mathbb{Z}).$$

The $\ell^p_w(\mathbb{Z})$ norms of $\{(V_Ja)\}, \{(W_Ja)\}$(may depend on J).
Alternatively, we claim that

$$V_Ja(n) \leq C_J \|b\|_\star (M(|a|^r)(n))^{1/r}, \quad 1 < r < \infty.$$

and

$$W_Ja(n) \leq C'_J \|b\|_\star T^\star_{|\psi|}a(n).$$

We have

$$\begin{aligned} V_Ja(n) &= \sup_{N \leq J} |\sum_{i=n-N}^{n+N} [b(n) - b(i)]\nu_N(n-i)a(i)| \\ &= \sup_{N \leq J} |\sum_{i=-\infty}^{\infty} [b(n) - b(i)]\nu_N(n-i)a\chi_{I_J}(i)|. \end{aligned}$$

where $I_J = [n - J, n + J]$. We estimate this exactly as we estimated the term A_2 in Lemma 11.6 and we have

$$\begin{aligned} V_Ja(n) &\leq (2J+1)\left\{C\|b\|_\star + (\log J)\|b\|_\star\right\}\left\{M(|a|^r)(n)\right\}^{1/r} \\ &\leq C_J \|b\|_\star \left\{M(|a|^r)(n)\right\}^{1/r}. \end{aligned}$$

Therefore, choosing $r < p$ we see that $V_Ja \in \ell^p_w(\mathbb{Z})$ for $a \in \ell^p_w(\mathbb{Z})$.

Next let $n \in \mathbb{Z}$ and $I_{4J} = [n - 4J, n + 4J]$. Then for $i \in I_{4J}$

$$|b(n) - b(i)| \leq |b(n) - b_{I_{4J}}| + |b_{I_{4J}} - b(i)| \leq 2(8J+1)\|b\|_\star.$$

Therefore,

$$W_Ja(n) = \sup_{N \leq J} \sum_{i=n-4N}^{n+4N} |b(n) - b(i)||\psi_N(n-i)||a(i)| \leq C_J \|b\|_\star T^\star_{|\psi|}a(n).$$

So $W_Ja \in \ell^p_w(\mathbb{Z}), \forall a \in \ell^p_w(\mathbb{Z})$.
Hence we conclude the boundedness of operators V_J, W_J on $\ell^p_w(\mathbb{Z})$. Since the constants obtained in the inequalities stated in Theorem 11.7 are independent of J, boundedness of $[b, T_\nu]^\star$, $[b, T_{|\psi|}]^\star$ on $\ell^p_w(\mathbb{Z}), 1 < p < \infty$ follow immediately using weighted sharp maximal sequence theorem.

Corollary 11.8. *Let $1 < p < \infty$. If $b \in BMO(\mathbb{Z})$, then the commutator of the singular operator $[b, T_\phi]a$ exists for every $a \in \ell^p_w(\mathbb{Z})$.*

Proof. Note that finite sequences are dense in $\ell^p_w(\mathbb{Z})$ [23] and $[b, T_\phi]a$ exists for every finite sequence $\{a(n) : n \in \mathbb{Z}\}$. Since $[b, T_\phi]^\star$ is bounded on $\ell^p_w(\mathbb{Z})$, the proof follows. ∎

12. MAXIMAL COMMUTATOR OF SINGULAR OPERATOR ON VARIABLE SEQUENCE SPACES $\ell^{\ell^{p(\cdot)}}(\mathbb{Z})$

In this section, we prove strong type inequality for the maximal commutator of singular operator on $\ell^{p(\cdot)}(\mathbb{Z}), 1 < p_- \leq p_+ < \infty$, $1 \leq p < \infty$, using Rubio de Francia extrapolation method given in [16].

Theorem 12.1. *Given a sequence $\{a(n) : n \in \mathbb{Z}\}$, suppose $p(\cdot) \in \mathcal{S}$ such that $p_- > 1$. Let $[b, T_\phi]^\star$ be a maximal commutator of singular operator.*
Then,

$$\|[b, T_\phi]^\star a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq C \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

Proof. Take p_0 such that $1 < p_0 \leq p_- \leq p_+ < \infty$. Therefore by Lemma [11]

$$\begin{aligned}
&\|([b, T_\phi]^\star a)\|^{p_0}_{\ell^{p(\cdot)}(\mathbb{Z})} = \|([b, T_\phi]^\star a)^{p_0}\|_{\ell^{\frac{p(\cdot)}{p_0}}}(\mathbb{Z}) \\
&= \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |[b, T_\phi]^\star a(k)|^{p_0} |h(k)| \\
&\leq \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |[b, T_\phi]^\star a(k)|^{p_0} Rh(k) \\
&\leq C \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |a(k)|^{p_0} Rh(k) \\
&\leq C \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \||a|^{p_0}\|_{\ell^{\frac{p(\cdot)}{p_0}}(\mathbb{Z})} \|Rh\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} \\
&= C \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \||a|\|^{p_0}_{\ell^{p(\cdot)}(\mathbb{Z})} \|Rh\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} \\
&\leq 2C\|a\|^{p_0}_{\ell^{p(\cdot)}(\mathbb{Z})}.
\end{aligned}$$

∎

13. MAXIMAL ERGODIC COMMUTATOR OF SINGULAR OPERATOR

Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. We define the commutator of the truncated maximal ergodic singular operator as follows:

$$[b, \tilde{T}_\phi]^\star_J f(x) = \sup_{N \leq J} \left| \sum_{k=-N}^{N} \phi(k)[b(x) - b(U^{-k}x)] f(U^{-k}x) \right|.$$

Definition 13.1 (BMO(X))**.** For a probability space $(X, \mathcal{B}, \mu)$ and an invertible measure preserving transformation on X, the space BMO(X) is defined as the space of those functions $b \in L^1(X)$ satisfying

$$esssup_{x\in X}\left(\sup_{N\geq 1} \frac{1}{2N+1} \sum_{k=-N}^{N} |b(U^k x) - \frac{1}{2N+1} \sum_{j=-N}^{N} b(U^j x)|\right) = \|b\|_\star < \infty.$$

Now, we prove the strong type inequality for the maximal ergodic commutator of singular operator on $L^p_w(X, \mathcal{B}, \mu)$ spaces.

Theorem 13.1. *Let $(X, \mathcal{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. If w is an ergodic A_p weight, $1 < p < \infty$ and $b \in BMO(X)$, then there exists a constant $C_p > 0$ such that*

$$\left\|[b, \tilde{T}_\phi]^\star f\right\|_{L^p_w(X)} \leq C_p \|f\|_{L^p_w(X)} \quad \forall f \in L^p_w(X, \mathcal{B}, \mu).$$

Proof. Observe that if $b \in BMO(X)$, then $b \in L^1(X)$ and for a.e the sequence b_x given by $b_x(n) = b(U^n x)$ is in BMO($\mathbb{Z}$), with $\|b_x\|_\star \leq C$, where C is independent of x. For $J \geq 1$, let

$$[b, \tilde{T}_\phi]^\star_J f(x) = \sup_{N \leq J} \left| \sum_{k=-N}^{N} [b(x) - b(U^{-k}x)] f(U^{-k}x)\phi(k) \right|.$$

We will prove that

$$\left\| [b, \tilde{T}_\phi]^\star_J f \right\|_{L^p_w(X)} \leq C \|f\|_{L^p_w(X)} \quad \forall f \in L^p_w(X, \mathcal{B}, \mu),$$

where the constant C is independent of f and J. Then the theorem will follow by monotone convergence theorem. For a.e $x \in X$ let

$$[b_x, T_\phi]^\star_J a(n) = \sup_{N \leq J} \left| \sum_{k=-N}^{N} [b_x(n) - b_x(n-k)] a(n-k)\phi(k) \right|.$$

It is easy to see that $[b_x, T_\phi]^\star_J$ is sub linear. Let L be any integer greater than or equal to J. Observe that if $\{a(n) : n \in \mathbb{Z}\}$ is a sequence such that $a(n) = 0$ for $|n| \leq L$ then

$$[b, T_\phi]^\star_J a(n) = 0 \quad \text{for} \quad |n| \leq L - J.$$

For a.e $x \in X$, let $f_x(n) = f(U^n x)$ and $w_x(n) = w(U^n x)$. For $k \in \mathbb{Z}_+$, define

$$f_x^K(n) = \begin{cases} f_x(n) & \text{if} \quad |n| \leq K, \\ 0 & \text{if} \quad |n| > K. \end{cases}$$

Since $[b, T_\phi]^\star_J$ is sub linear, for $K, L \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we have

$$[b_x, T_\phi]^\star_J f_x(n) \leq [b_x, T_\phi]^\star_J f_x^{K+L}(n) + [b, T_\phi]^\star_J (f_x - f_x^{K+L})(n).$$

We can choose L large enough so that

$$[[b_x, T_\phi]^\star_J (f_x - f_x^{K+L})](n) = 0 \quad \text{if} \quad |n| \leq K.$$

Note that L depends only on J and not on K. Therefore,

$$[b_x, T_\phi]^\star_J f_x(n) \leq [b_x, T_\phi]^\star_J f_x^{K+L}(n)$$

for $|n| \leq K$.

Also, for a.e $x \in X$and $j \in \mathbb{Z}$, we have

$$[b, \tilde{T}_\phi]^\star_J f(U^j x) = \sup_{N \leq J} |\sum_{k=-N}^{N} [b(U^j x) - b(U^{j-k})] f(U^{j-k})\phi(k)|$$

$$= \sup_{N \leq J} |\sum_{k=-N}^{N} [b_x(j) - b_x(j-k)] f_x(j-k)\phi(k)|$$

$$= [b_x, T_\phi]^\star_J f_x(j).$$

Then

$$\int_X ([b, \tilde{T}_\phi]^\star_J f(x))^p w(x) d\mu = \frac{1}{2K+1} \sum_{j=-K}^{K} \int_X ([b, \tilde{T}_\phi]^\star_J f(U^j x))^p w(U^j x) d\mu$$

$$
\begin{aligned}
&= \frac{1}{2K+1}\sum_{j=-K}^{K}\int_X ([b_x,T_\phi]_J^\star f_x(j))^p w_x(j)d\mu \\
&\leq \frac{1}{2K+1}\sum_{j=-K}^{K}\int_X ([b_x,T_\phi]_J^\star f_x^{K+L}(j))^p w_x(j)d\mu \\
&\leq \frac{C}{2K+1}\int_X \sum_{j=-(K+L)}^{K+L} |f_x(j)|^p w_x(j)d\mu \\
&= \frac{C}{2K+1}\sum_{j=-(K+L)}^{K+L}\int_X |f(U^j x)|^p w(U^j x)d\mu \\
&\leq \frac{C[2(K+L)+1]\,\|f\|^p_{L^p_w(X)}}{2K+1}.
\end{aligned}
$$

Choosing K sufficiently large, we get

$$
\left\|[b,\tilde{T}_\phi]_J^\star f\right\|_{L^p_w(X)} \leq C_p\,\|f\|_{L^p_w(X)}\,.
$$

∎

Remark 13.1. Using the boundedness of maximal ergodic commutator of singular operator and Rubio de Francia method, we can prove that the maximal ergodic commutator of singular operator is bounded on variable $L^{p(\cdot)}(X,\mathcal{B},\mu)$ spaces. But Rubio de Francia method assumes ergodic maximal operator is bounded on the variable $L^{p(\cdot)}(X,\mathcal{B},\mu)$ spaces. With this assumption we can prove the boundedness of maximal ergodic commutator singular operator to variable $L^{p(\cdot)}(X,\mathcal{B},\mu)$ spaces.

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