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MASS TRANSPORTATION APPROACH FOR PARABOLIC p-BIHARMONIC EQUATIONS

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ABSTRACT. In this paper, we propose a mass transportation method to solving a parabolic p-biharmonic equations, which generalized the Cahn-Hilliard (**CH**) equations in $\mathbb{R}^N$, $N \in \mathbb{N}^*$. By using a time-step optimal approximation in the appropriate Wasserstein space, we define an approximate weak solution which converges to the exact solution of the problem. We also show that the solution under certain conditions may be unique. Therefore, we study the asymptotic behavior of the solution of the parabolic p-biharmonic problem.

Key words and phrases: Lebesgue and Sobolev spaces, p-Laplacian operator, Cahn-Hilliard equation, Optimal transportation method, q-Wasserstein metric.

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1. INTRODUCTION

The Cahn-Hilliard **(CH)** equation in its original formulation, proposed in [8, 9, 3] describes the dynamics of phase separation in binary alloys. It has been used also as a phenomenological model in several different areas, from the description of multicomponent polymeric systems in [26], and lithium-ion batteries in [33], to the modeling of nanoporosity during dealloying in [4], or inpainting of binary images in [31], and even to the formation of Saturn rings in [22]. Recently, **CH** type equations have also been employed to describe pattern formation in biological systems (see, for instance, [22, 24]) and diffuse interface tumor growth models, [29, 17]. In particular, a **CH** equation with degenerate mobility, obtained from the application of mixture theory to solid tumors, is described in [32]. The Cahn-Hilliard equation is indeed a fundamental equation and an essential building block in the phase field theory for moving interface problems (cf. [28]), it Adaptive methods for the Cahn-Hilliard equation is often combined with other fundamental equations of mathematical physics such as the Navier-Stokes equation (cf. [12, 19, 25] and the references therein) to be used as diffuse interface models for describing various interface dynamics, such as flow of two-phase fluids, from various applications.

In [2], Alain Miranville studies the Cahn-Hilliard equation, as well as some of its variants. Such variants have applications in biology and image inpainting. A Wasserstein approach to the numerical solution of the one-dimensional has been analysed in [13] and a non-local version in a two-component incompressible and immiscible mixture with linear mobilities has been studied in [10]. These authors have showed that time-discrete approximations by means of the incremental minimizing movement scheme converge to a weak solution in the limit. In the paper [18], an optimal control problem for a two-dimensional Cahn-Hilliard-Darcy system with mass sources that arises in the modeling of tumor growth has been analysed.

In this paper, we propose an approach based on optimal transportation, to study existence and uniqueness of solution for a class of non-linear parabolic biharmonic equations in the probability space under the Neumann boundary condition, say the problem (1.1)-(1.3):

$$\frac{\partial \rho}{\partial t} = -div_x\left(\rho|\nabla_x(\Delta_x(\rho)-\psi^{'}(\rho))|^{p-2}\nabla_x(\Delta_x(\rho)-\psi^{'}(\rho))\right) \quad \text{in} \quad [0,+\infty)\times\Omega, \tag{1.1}$$

$$\rho(0,x) = \rho_0(x) \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\rho\nabla_x(\rho)\cdot\nu = \rho|\nabla_x(\Delta_x(\rho)-\psi^{'}(\rho))|^{p-2}\nabla_x(\Delta_x(\rho)-\psi^{'}(\rho))\cdot\nu = 0 \quad \text{on} \quad [0,+\infty)\times\partial\Omega. \tag{1.3}$$

where $p > 1$ is a constant and $\psi : [0,+\infty) \to \mathbb{R}$ is a convex function of class C^2, and $\Omega \subset \mathbb{R}^N$, a bounded domain with smooth boundary $\partial\Omega$. Here, the initial datum $\rho_0 : \Omega \to (0,+\infty)$ is a probability density function. Of course, depending on the features of ψ and p, equations (1.1)-(1.3) occurs in the modeling of the evolution of a broad range of physical and biological phenomena having non-homogeneous properties such as, the interaction of particles, the flow of electrorheological fluids, fluids with temperature-depending viscosity, flow in porous non-homogeneous and anisotropic media and image processing.

In a recent work, some authors established the existence and the uniqueness of weak solution of (1.1)-(1.3) for different values of N and $p = 2$, see [8, 9, 3].

Optimal transportation method on the space of measures have demonstrated to be a valuable new approach in time-step approximation of nonlinear diffusion problems since the pioneer works of Otto [20] and Jordan-Kinderleher-Otto [21]. Today, a very broad fields on mathematics research such as, Partial Differential Equations, Fluids mechanics , Shape optimization to quote just a few, have been impacted by optimal transportation method. One can see for instance the works in ([1],[5], [6], [7], [15], [16],[20], [21], [30]) . In [20], Jordan, Kinderlehrer and Otto have studied existence of solutions of the heat equation:

$$\frac{\partial \rho(t,x)}{\partial t} = \Delta_x\rho(t,x), \quad \text{in} \quad [0,+\infty)\times\mathbb{R}^N. \tag{1.4}$$

For their purpose, they use a descent algorithm in the probability space endowed with the 2- Wasserstein distance W_2 to construct the approximate solutions of (1.4). In [1], M. Agueh used a variational approach similar as in [20] to prove existence of solutions for the p-parabolic equation:

$$(1.5)\quad \begin{cases} \frac{\partial \rho(t,x)}{\partial t} &= div_x\left(\rho(t,x)|\nabla_x\psi^{'}(\rho(t,x))|^{p-2}\nabla_x\psi^{'}(\rho(t,x))\right) \quad \text{in} \quad [0,+\infty)\times\Omega, \\ \rho(0,x) &= \rho_0(x) \quad \text{in} \quad \Omega, \\ & \rho(t,x)|\nabla_x\psi^{'}(\rho(t,x))|^{p-2}\nabla_x\psi^{'}(\rho(t,x))\cdot\nu = 0 \quad \text{on} \quad [0,+\infty)\times\partial\Omega, \end{cases}$$

with p a constant, and $p > 1$.
Our purpose is to investigate at the light of some previous works of [1], [20], the case of non-homogeneous equations induces by a p-biharmonic operator, using the optimal transportation approach. From the best of our knowledge, our approach contrasts with other treatments in the literature for the class of equations under consideration, which generalizes the work [13, 10].

For the sake of completeness, we recall below some tools related to our approach and of interest for this work. Thus, let's consider the following Monge problem

$$(1.6)\qquad (M): \inf_{T_{\#}\rho_1=\rho_2}\int_\Omega |T(x)-x|^q\rho dx,$$

where ρ_1, ρ_2 are two probability density on Ω and $q = \frac{p}{p-1}$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$. The condition $T_{\#}\rho_1 = \rho_2$ say that: For all continuous function $\phi : \Omega \to \mathbb{R}$, we have

$$\int_\Omega \phi(x)\rho_2 dx = \int_\Omega \phi(T(x))\rho_1 dx.$$

The Monge problem (1.6) can be associated to the Kantorovich problem

$$(1.7)\qquad (K): \inf_\gamma\left\{\int_{\Omega\times\Omega}|x-y|^q d\gamma, \quad \gamma\in\Pi(\rho_1,\rho_2)\right\},$$

which admits a solution γ_0.
Here $\Pi(\rho_1, \rho_2)$ denote the set of all probability measures on $\Omega\times\Omega$ whose marginals are ρ_1 and ρ_2. Both the Monge and Kantorovich's formulation play a central role in our approach of the time-step approximation of solutions of the problem (1.1).
Indeed, we fix $h > 0$ to be a time step and assume that ρ_0 is a probability density on Ω. Define ρ_k, $k \in \mathbb{N}^*$ as a solution of the variational problem

$$(1.8)\qquad (P_k): \inf_{\rho\in P(\Omega)}\left\{I(\rho) := E(\rho) + \frac{1}{qh^{q-1}}W_q^q(\rho,\rho_{k-1})\right\},$$

where

$$(1.9)\qquad E(\rho) = \int_\Omega\left(\psi(\rho)+\frac{1}{2}|\nabla_x(\rho)|^2\right)dx$$

and W_q is the q Wasserstein metric defined by

$$(1.10)\qquad W_q^q(\rho,\rho_{k-1}) := \inf_{\gamma\in\Pi(\rho,\rho_{k-1})}\int_{\Omega\times\Omega}|x-y|^q d\gamma.$$

Here $\Pi(\rho, \rho_{k-1})$ is the set of all probability measures on $\Omega\times\Omega$ whose marginals are ρ and ρ_{k-1}.
We prove in section (3) that the sequence $(\rho_k)_k$, satisfies the equation

$$(1.11)\qquad \frac{\rho_k(x)-\rho_{k-1}(x)}{h} + div_x\left\{\rho_k|\nabla_x(\Delta_x(\rho_k)-\psi^{'}(\rho_k))|^{p-2}\nabla_x(\Delta_x(\rho_k)-\psi^{'}(\rho_k))\right\} = o(h),$$

weakly, where $o(h)$ tends to 0 when h tends to 0. Accordingly equation (1.11) shows that the sequence $(\rho_k)_k$ is a time discretization of (1.1)-(1.3).

We define ρ^h as it follows

$$(1.12)\qquad \begin{cases} \rho^h(t,x) &= \rho_k(x) \quad \text{if} \quad (t,x)\in[hk,h(k+1))\times\Omega, \\ \rho^h(t,x) &= \rho_0(x) \quad \text{if} \quad (t,x)\in\{0\}\times\Omega, \end{cases}$$

and we show that the sequence $(\rho^h)_h$ converges weakly to $\rho = \rho(t,x)$ which solves the parabolic equations (1.1)-(1.3) weakly. Moreover, we use the transportation method to investigate the existence of local vanishing property of our problem.

This paper is organized as it follows: section 2 is devoted to the preliminary tools useful throughout the paper and in section 3, we establish the existence and the uniqueness of solution for the variational problem (P_k) and next prove that the Euler-Lagrange equation (1.11) is satisfies. In section 4, we establish our convergence results and in section 5 the existence and the uniqueness results for (1.1)-(1.3) are stated. We study in section 6 the asymptotic behavior of the solution of the parabolic bi-harmonic problem. We offer our conclusion and the further works in section 7.

2. Preliminaries

2.1. **Main assumptions.** Throughout this work, we will assume the following:

(ψ_1) $\psi : [0,+\infty) \to \mathbb{R}$ is convex function such that $\psi(0) = 0$,
(ψ_2) $\psi \in C^2((0,+\infty))$,
(ψ_3) $t \longmapsto t^N\psi(t^{-N})$ is convex and decreasing,
(H_{ρ_0}) : ρ_0 is a probability density on Ω such that,

$$E(\rho_0) := \int_\Omega \left(\psi(\rho_0) + \frac{|\nabla_x(\rho_0)|^2}{2}\right) dx < +\infty. \tag{2.1}$$

2.2. **Lebesgue-Sobolev spaces.** We recall in this section some definitions and fundamental properties of the Lebesgue and Sobolev space.

Definition 2.1. Let ρ be a probability measure on Ω, and $p > 1$ a constant. We denote by $L^p_\rho(\Omega)$ the Lebesgue space defined by :

$$L^p_\rho(\Omega) := \left\{u : \Omega \to \mathbb{R}; \int_\Omega |u(x)|^p \rho(x)dx < +\infty\right\}, \tag{2.2}$$

with the norm

$$\|u\|_{L^p_\rho(\Omega)} = \left(\int_\Omega |u(x)|^p\rho(x)dx\right)^{\frac{1}{p}}, \tag{2.3}$$

for all $u \in L^p_\rho(\Omega)$.
We denote by $W^{1,p}_\rho(\Omega)$ the Sobolev space defined by

$$W^{1,p}_\rho(\Omega) := \left\{u \in L^p_\rho(\Omega), \quad |\nabla u| \in L^p_\rho(\Omega)\right\} \tag{2.4}$$

equipped with the norm

$$\|u\|_{W^{1,p}_\rho(\Omega)} := \|u\|_{L^p_\rho(\Omega)} + \|\nabla u\|_{L^p_\rho(\Omega)}. \tag{2.5}$$

It is well known that $L^p_\rho(\Omega)$ and $W^{1,p}_\rho(\Omega)$ are Banach spaces respectively with the norms (2.3) and (2.5) .
We denote by q the conjugate of p which is defined by

$$q = \frac{p}{p-1}.$$

Proposition 2.1. *(Hölder inequality). Let $\rho \in P(\Omega)$ be a probability density and $p > 1$, $q > 1$ two constants such that $\frac{1}{p} + \frac{1}{q} = 1$.*
If $u \in L^p_\rho(\Omega)$ and $v \in L^q_\rho(\Omega)$, then:

$$\int_\Omega |u(x)v(x)|\rho(x)dx \le \|u\|_{L^p_\rho(\Omega)}\|v\|_{L^q_\rho(\Omega)}.$$

Furthermore, if p_1, p_2, p_3 *are such that* $\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}$, *we have*

$$\|uv\|_{L^{p_1}_\rho(\Omega)} \leq 2\|u\|_{L^{p_2}_\rho(\Omega)}\|v\|_{L^{p_3}_\rho(\Omega)},$$

for $u \in L^{p_2}_\rho(\Omega)$ *and* $v \in L^{p_3}_\rho(\Omega)$.

Proposition 2.2. *Let* $\rho \in P(\Omega)$ *be a probability density and* p_1, p_2 *two constants such that* $p_1 \leq p_2$. *Then, we have the following continuous injection:*

$$L^{p_2}_\rho(\Omega) \hookrightarrow L^{p_1}_\rho(\Omega). \tag{2.6}$$

Furthermore,

$$\|u\|_{L^{p_1}_\rho(\Omega)} \leq 2\|u\|_{L^{p_2}_\rho(\Omega)}$$

Theorem 2.3. *Assume that* $p > 1$. *Then the Banach spaces* $L^p_\rho(\Omega)$ *and* $W^{1,p}_\rho(\Omega)$ *are separable, reflexive and uniformly convex.*

2.3. **Mass transportation theory.** In this section, $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $P(\Omega)$ denote the set of all probability density on Ω.

Definition 2.2. Let $\rho_1, \rho_2 \in P(\Omega)$ and γ a probability measure on $\Omega \times \Omega$.
We said that γ have ρ_1 and ρ_2 as its marginals, if one of the following equivalent condition holds:

(i) For all Borel set $A \subset \Omega$,

$$\gamma(A \times \Omega) = \rho_1(A), \quad \text{and} \quad \gamma(\Omega \times A) = \rho_2(A).$$

(ii) For $(\phi_1, \phi_2) \in L^1_{\rho_1}(\Omega) \times L^1_{\rho_2}(\Omega)$,

$$\int_{\Omega\times\Omega} [\phi_1(x) + \phi_2(y)]d\gamma(x,y) = \int_\Omega \phi_1(x)\rho_1 dx + \int_\Omega \phi_2(x)\rho_2(y)dy.$$

We denote by $\Pi(\rho_1, \rho_2)$, the set of all probability measures satisfying (i) or (ii).

Definition 2.3. Let $\rho_1, \rho_2 \in P(\Omega)$. A borel map $T : \Omega \longrightarrow \Omega$ is said to push ρ_1 forward to ρ_2, if

(i) For all Borel set $A \subset \Omega$,

$$\rho_2(A) = \rho_1(T^{-1}(A)).$$

(ii) For $\phi_1 \in L^1_{\rho_1}(\Omega)$,

$$\int_\Omega \phi_1(y)\rho_2(y)dy = \int_\Omega \phi_1(T(x))\rho_1(x)dx.$$

When (i) or (ii) holds, we write that $\rho_2 = T_\#\rho_1$ and we said that T pushes ρ_1 forward to ρ_2.

Proposition 2.4. *(see* [1]*) Let* $c : \mathbb{R}^N \to [0, +\infty)$ *be strictly convex and* $\rho_1, \rho_2 \in P(\Omega)$. *Then,*

(i) *There is a function* $v : \overline{\Omega} \to \mathbb{R}$ *such that* $T := id_{\overline{\Omega}} - \nabla c^*(\nabla u)$ *pushes* ρ_1 *forward to* ρ_2, *where* c^* *is the Legendre transform of* c *and* $u(x) = \inf_{x\in\overline{\Omega}}\{c(x-y) - v(y)\}$, *for all* $x \in \overline{\Omega}$.

(ii) $T := id_{\overline{\Omega}} - \nabla c^*(\nabla u)$ *is the unique minimizer of the Monge problem*

$$(M) : \inf_T \left\{ \int_\Omega c(T(x) - x)\rho_1 dx, \quad T_\#\rho_1 = \rho_2 \right\}. \tag{2.7}$$

(iii) *The probability measure* $\gamma_T := (id_\Omega \times T)_\#\rho_1$ *defined by*

$$\gamma_T(B) := \rho_1(\{x \in \Omega, \quad (x, T(x)) \in B\}),$$

for all Borel set $B \subset \Omega \times \Omega$ *is the unique solution of Kantorovich problem*

$$(K) : \inf_\gamma \left\{ \int_{\Omega\times\Omega} c(x-y)d\gamma, \quad \gamma \in \Pi(\rho_1, \rho_2) \right\}. \tag{2.8}$$

(iv) *If* $c(x) = |x|^q$ *, with* $q > 1$*. The Monge cost*

$$W_q(\rho_1, \rho_2) := \left[\inf_T \left(\int_\Omega |T(x) - x|^q \rho_1 dx, \quad T_\# \rho_1 = \rho_2\right)\right]^{\frac{1}{q}}$$

is the q-Wasserstein metric.

3. EULER LAGRANGE EQUATION OF THE PROBLEM (P_k)

Here, we establish the existence and uniqueness of the solution of problem (P_k) and show that the sequence $(\rho_k)_k$ is a time discretization of (1.1)-(1.3).

Proposition 3.1. *Assume that hypotheses* (H_{ρ_0}), (ψ_1), (ψ_2) and (ψ_3) are fulfilled. Then, the problem

$$(P_1) : \inf_{\rho \in P(\Omega)} \left\{ I(\rho) := E(\rho) + \frac{1}{qh^{q-1}} W_q^q(\rho, \rho_0) \right\} \tag{3.1}$$

admits a unique solution ρ_1 and $E(\rho_1) < +\infty$.

Proof. Let denote l the infimum of I over $P(\Omega)$. Show that l is finite.
If $\rho = \rho_0$, then $I(\rho_0) = E(\rho_0)$. Then, by using hypothesis (H_{ρ_0}), we deduce that $I(\rho_0)$ is finite.
Let ρ is an probability density on Ω . Since ψ is convex, then by Jessen's inequality we obtain:

$$\int_\Omega \psi(\rho) dy \geq |\Omega| \psi\left(\frac{1}{|\Omega|}\right). \tag{3.2}$$

Therefore $\int_\Omega \frac{|\nabla \rho|^2}{2} dx \geq 0$ and $W_q^q(\rho, \rho_0) \geq 0$. Consequently

$$I(\rho) \geq |\Omega| \psi\left(\frac{1}{|\Omega|}\right). \tag{3.3}$$

We conclude that l is finite.
Let $(\rho_n)_n$ be a minimizing sequence of (P_1) in $P(\Omega)$.
Then the sequence $(I(\rho_n))_n$ is bounded in $\mathbb{R}$. Thus, there exist a constant $K \geq 0$ such that $I(\rho_n) \leq K$ for all $n \in \mathbb{N}$. Consequently

$$\int_\Omega \frac{|\nabla \rho_n|^2}{2} dx \leq K - \int_\Omega \psi(\rho_n) dx - \frac{1}{qh^{q-1}} W_q^q(\rho_n, \rho_0). \tag{3.4}$$

Since $W_q^q(\rho_n, \rho_0) \geq 0$, then we use Jessen's inequality (3.2) to obtain

$$\int_\Omega \frac{|\nabla \rho_n|^2}{2} dx \leq K - |\Omega| \psi\left(\frac{1}{|\Omega|}\right). \tag{3.5}$$

Consequently, the sequence $(\rho_n)_n$ is bounded in $H^1(\Omega)$. Thus, (ρ_n) converge strongly to some ρ_1 in $L^2(\Omega)$, (up to a subsequence) and ρ_1 is a probability density on Ω.

Since ψ is C^1, we have

$$\liminf \int_\Omega \psi(\rho_n) dx = \int_\Omega \psi(\rho_1) dx. \tag{3.6}$$

Therefore, since $\rho_n \to \rho$ strongly in $L^2(\Omega)$, then

$$\liminf \int_\Omega \frac{|\nabla \rho_n|^2}{2} dx \geq \int_\Omega \frac{|\nabla \rho_1|^2}{2} dx. \tag{3.7}$$

Let γ_n be a solution of Kantorovich problem

$$W_q^q(\rho_n, \rho_0) = \inf_{\gamma \in \Pi(\rho_n, \rho_0)} \int_{\Omega \times \Omega} |x - y|^q d\gamma. \tag{3.8}$$

Note that $P(\Omega \times \Omega)$ is tight, then $(\gamma_n)_n$ converges narrowly to a probability measure γ_1 in $P(\Omega \times \Omega)$, (up to a subsequence), and $\gamma_1 \in \Pi(\rho_1, \rho_0)$.
Then we obtain that

$$\liminf \int_{\Omega\times\Omega} |x-y|^q d\gamma_n \geq \int_{\Omega\times\Omega} |x-y|^q d\gamma_1. \tag{3.9}$$

Noting that $W_q^q(\rho_n, \rho_0) = \int_{\Omega\times\Omega} |x-y|^q d\gamma_n$, and $\int_{\Omega\times\Omega} |x-y|^q d\gamma_1 \geq W_q^q(\rho_1, \rho_0)$. We conclude that

$$\liminf W_q^q(\rho_n, \rho_0) \geq W_q^q(\rho_1, \rho_0). \tag{3.10}$$

From (3.9), (3.10), (3.7) and (3.6), we have

$$\liminf I(\rho_n) \geq I(\rho_1). \tag{3.11}$$

Then $I(\rho_1) = \inf_{\rho\in P(\Omega)} I(\rho)$. Consequently ρ_1 is a solution of the problem (P_1) and $E(\rho_1) < +\infty$. We obtain uniqueness of ρ_1 by using the convexity of $\rho \longmapsto E(\rho)$ and the strict convexity of the map $\rho \longmapsto W_q^q(\rho, \rho_0)$. ∎

By induction, we obtain existence and uniqueness of the sequence $(\rho_k)_k$ such that ρ_k is a unique solution of the problem (P_k).

Theorem 3.2. *Assume that hypotheses* (H_{ρ_0}), (ψ_1),(ψ_2) and (ψ_3) hold. Then, the Kantorovich problem

$$(K): \inf_{\gamma\in\Pi(\rho_k,\rho_{k-1})} \left\{ \int_{\Omega\times\Omega} |x-y|^q d\gamma \right\} \tag{3.12}$$

admits a unique solution γ_k, and

$$supp(\gamma_k) \subset \left\{ (x,y): \quad y = x - h|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]|^{p-2}\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)] \right\}.$$

Proof. Since the cost function $(x,y) \mapsto c(x,y) = |x-y|^p$ is convex, then the Kantorovich problem (3.12) admit a unique solution γ_k.
Let $\phi \in C_c^\infty(\Omega, \Omega)$ be a test function, and consider the flow map $(T_\varepsilon)_{\varepsilon\in\mathbb{R}}$ in $C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$, such that

$$\begin{cases} \frac{\partial T_\varepsilon}{\partial \varepsilon} &= \phi \circ T_\varepsilon \\ T_0 &= id \end{cases}. \tag{3.13}$$

Define: $\rho_\varepsilon = T_{\varepsilon\#}\rho_k$.
The function ρ_ε is a probability density on Ω and satisfy

$$\frac{\partial \rho_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = -div_x(\rho_k \phi), \quad \text{see [1]}. \tag{3.14}$$

Consequently, by using (3.14), we obtain

$$\frac{d}{d\varepsilon}\left[\int_\Omega \psi(\rho_\varepsilon(x))dx\right]|_{\varepsilon=0} = \int_\Omega < \nabla_x(\psi^{'}(\rho_k(x))), \phi(x) > \rho_k(x)dx, \tag{3.15}$$

and

$$\frac{d}{d\varepsilon}\left[\int_\Omega \frac{|\nabla_x(\rho_\varepsilon)|^2}{2}dx\right]|_{\varepsilon=0} = -\int_\Omega < \nabla_x(\Delta_x(\rho_k)), \phi(x) > \rho_k(x)dx. \tag{3.16}$$

Let γ^ε be a probability measure on $\Omega \times \Omega$ defined by

$$\int_{\Omega\times\Omega} \Phi(x,y)d\gamma^\varepsilon(x,y) = \int_{\Omega\times\Omega} \Phi(T_\varepsilon(x),y)d\gamma_k(x,y), \tag{3.17}$$

for all $\Phi \in C_b^0(\Omega \times \Omega)$. Then $\gamma^\varepsilon \in \Pi(\rho_\varepsilon, \rho_{k-1})$.
By using the definition of γ^ε, we have

$$\frac{d}{d\varepsilon}\left[\int_{\Omega\times\Omega} |x-y|^q d\gamma_\varepsilon\right]|_{\varepsilon=0} = q\int_{\Omega\times\Omega} < |x-y|^{q-2}(x-y), \phi(x) > d\gamma_k(x,y). \tag{3.18}$$

The solution ρ_k of the problem (P_k) satisfies

$$\frac{d}{d\varepsilon}\left[\int_\Omega(\psi(\rho_\varepsilon)+\frac{|\nabla_x(\rho_\varepsilon)|^2}{2})dx+\frac{1}{qh^{q-1}}W_q^q(\rho_\varepsilon,\rho_{k-1})\right]|_{\varepsilon=0}=0. \tag{3.19}$$

Note that γ^ε is admissible for (P_k), then

$$W_q^q(\rho_\varepsilon,\rho_{k-1})\le\int_{\Omega\times\Omega}|x-y|^q d\gamma_\varepsilon. \tag{3.20}$$

By using the inequality (3.20), we obtain

$$I(\rho_\varepsilon):=E(\rho_\varepsilon)+\frac{1}{qh^{q-1}}W_q^q(\rho_\varepsilon,\rho_{k-1})\le E(\rho_\varepsilon)+\frac{1}{qh^{q-1}}\int_{\Omega\times\Omega}|x-y|^q d\gamma_\varepsilon. \tag{3.21}$$

So, for $\varepsilon>0$, we have

$$\begin{aligned}\frac{I(\rho_\varepsilon)-I(\rho_k)}{\varepsilon}&:=\frac{E(\rho_\varepsilon)-E(\rho_k)}{\varepsilon}+\frac{W_q^q(\rho_\varepsilon,\rho_k)-W_q^q(\rho_k,\rho_{k-1})}{qh^{q-1}\varepsilon}\le\\ &\frac{E(\rho_\varepsilon)-E(\rho_k)}{\varepsilon}+\frac{\int_{\Omega\times\Omega}|x-y|^q d\gamma_\varepsilon-\int_{\Omega\times\Omega}|x-y|^q d\gamma_k}{qh^{q-1}\varepsilon}.\end{aligned} \tag{3.22}$$

We use (3.16), (3.18) , (3.19) and (3.22) and we tend ε to 0,

$$D_1(k)+D_2(k)\ge 0, \tag{3.23}$$

with,

$$\begin{aligned}D_1(k)&:=\int_\Omega<\nabla_x[-\Delta_x(\rho_k)+\psi'(\rho_k)],\phi(x)>\rho_k dx\quad\text{and}\\ D_2(k)&:=\frac{1}{h^{q-1}}\int_{\Omega\times\Omega}<|x-y|^{q-2}(x-y),\phi(x)>d\gamma_k(x,y).\end{aligned}$$

Changing ϕ by $-\phi$ in (3.23), we obtain the desired equality

$$D_1(k)+D_2(k)=0. \tag{3.24}$$

Finally, we obtain

$$y=x-h|\nabla_x[\Delta_x(\rho_k)-\psi'(\rho_k)]|^{p-2}\nabla_x[\Delta_x(\rho_k)-\psi'(\rho_k)]-\gamma_k\quad a.e. \tag{3.25}$$

Now, let show that $(\rho_k)_k$ is a time discretization of the parabolic equation(1.1).
Let $\Phi\in C_c^\infty(\Omega,\mathbb{R})$ be a test function. We define $T_k:\Omega\to\Omega$ by

$$T_k(x)=x-h|\nabla_x[\Delta_x(\rho_k)-\psi'(\rho_k)]|^{p-2}\nabla_x[\Delta_x(\rho_k)-\psi'(\rho_k)]. \tag{3.26}$$

We have $T_k\#\rho_k=\rho_{k-1}$, then

$$\int_\Omega(\rho_k(x)-\rho_{k-1}(x))\Phi(x)dx=\int_{\Omega\times\Omega}(\Phi(x)-\Phi(T_k(x)))\rho_k(x)dx. \tag{3.27}$$

Using Taylor's formula

$$\Phi(T_k(x))=\Phi(x)+(T_k(x)-x)\cdot\nabla_x\Phi(x)+(T_k(x)-x)^\tau\nabla_x^2\Phi(x+\theta(T_k(x)-x))\cdot(T_k(x)-x), \tag{3.28}$$

with $\theta\in[0,1]$ and $(T_k(x)-x)^\tau$ is the transpose of $T_k(x)-x$.
We use (3.28) and (3.26) in (3.27), then

$$\begin{aligned}\int_\Omega(\rho_k-\rho_{k-1})\Phi(x)dx=h\int_\Omega<|\nabla_x[\Delta_x(\rho_k)-\psi'(\rho_k)]|^{p-2}\nabla_x[\Delta_x(\rho_k)-\psi'(\rho_k)],\nabla_x\Phi(x)>dx+\\ \frac{1}{2}\int_\Omega<(x-T_k(x))^\tau,\nabla_x^2\Phi(x+\theta V_k)(x-T_k(x))>\rho_k dx.\end{aligned} \tag{3.29}$$

Thus, by using Neumann boundary condition, we obtain

$$\int_\Omega (\rho_k - \rho_{k-1})\Phi(x)dx = -h\int_\Omega div_x\left(|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]|^{p-2}\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]\right)\Phi(x)dx + \frac{1}{2}\int_\Omega < (x - T_k(x))^\tau, \nabla_x^2\Phi(x+\theta V_k)(x - T_k(x)) > \rho_k(x)dx. \tag{3.30}$$

In (3.30), $V_k := |\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]|^{p-2}\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]$.
Define $A_k(\Phi) = \int_\Omega < (x - T_k(x))^\tau, \nabla_x^2\Phi(x+\theta V_k)(x - T_k(x)) > \rho_k dx$ and show that $A_k(\Phi)$ tends to 0 when h tends to 0.
We have

$$|A_k(\Phi)| \le \sup_{x\in\Omega}|\nabla_x^2\Phi(x)|\int_\Omega |T_k(x) - x|^2\rho_k(x)dx. \tag{3.31}$$

Since ρ_k is the solution of (P_k), then $I(\rho_k) \le I(\rho_{k-1})$. Consequently

$$E(\rho_k) - E(\rho_{k-1}) \ge \frac{1}{qh^{q-1}}W_q^q(\rho_k, \rho_{k-1}). \tag{3.32}$$

Therefore, since γ_k is the solution of (3.12), then

$$\begin{aligned} W_q^q(\rho_k, \rho_{k-1}) &= \int_{\Omega\times\Omega}|x-y|^q d\gamma_k \\ &= h^q\int_\Omega \left|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]\right|^p \rho_k dx. \end{aligned} \tag{3.33}$$

From (3.32) and (3.33), we obtain that

$$E(\rho_k) - E(\rho_{k-1}) \ge \frac{h}{q}\int_\Omega \left|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]\right|^p \rho_k dx. \tag{3.34}$$

Taking the sum over $k = 1, \dots, \frac{T}{h}$ in (3.34), we get

$$E(\rho_0) - E(\rho_{\frac{T}{h}}) \ge \frac{h}{q}\sum_{k=1}^{\frac{T}{h}}\int_\Omega \left|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]\right|^p \rho_k dx \ge \frac{h}{q}\int_\Omega \left|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]\right|^p \rho_k dx. \tag{3.35}$$

We use Jensen's inequality in (3.35) and obtain:

$$E(\rho_0) - |\Omega|\psi\left(\frac{1}{|\Omega|}\right) \ge \frac{h}{q}\int_\Omega \left|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]\right|^p \rho_k dx \tag{3.36}$$

So, by using inequalities (3.36) and (3.33), we have

$$\int_{\Omega\times\Omega}|x-y|^q d\gamma_k \le qh^{q-1}\left[E(\rho_0) - |\Omega|\psi\left(\frac{1}{|\Omega|}\right)\right]. \tag{3.37}$$

• If $q \le 2$, then

$$\int_{\Omega\times\Omega}|x-y|^2 d\gamma_k \le [\text{diam}(\Omega)]^{2-q}\int_{\Omega\times\Omega}|x-y|^q d\gamma_k. \tag{3.38}$$

• If $q \ge 2$, then

$$\int_{\Omega\times\Omega}|x-y|^2 d\gamma_k \le \left(\int_{\Omega\times\Omega}|x-y|^q d\gamma_k\right)^{\frac{2}{q}}. \tag{3.39}$$

Consequently, by using (3.37) we obtain

$$\begin{aligned} &\int_{\Omega\times\Omega}|x-y|^2 d\gamma_k \le qh^{q-1}[\text{diam}(\Omega)]^{2-q}\left[E(\rho_0) - |\Omega|\psi(\frac{1}{|\Omega|})\right], \quad \text{if } q \le 2 \\ &\int_{\Omega\times\Omega}|x-y|^2 d\gamma_k \le q^{\frac{2}{q}}h^{\frac{2}{p}}\left[E(\rho_0) - |\Omega|\psi(\frac{1}{|\Omega|})\right]^{\frac{2}{q}}, \quad \text{if } q \ge 2 \end{aligned} \tag{3.40}$$

Finally, we deduce that

$$(3.41)\qquad |A(\Phi)| \le qh^{q-1}[\text{diam}(\Omega)]^{2-q}\sup_{x\in\Omega}|\nabla_x^2\Phi(x)|\left[E(\rho_0)-|\Omega|\psi(\frac{1}{|\Omega|})\right],\quad \text{if } q\le 2$$

$$|A(\Phi)| \le q^{\frac{2}{q}}h^{\frac{2}{p}}\sup_{x\in\Omega}|\nabla_x^2\Phi(x)|\left[E(\rho_0)-|\Omega|\psi(\frac{1}{|\Omega|})\right]^{\frac{2}{q}},\quad \text{if } q\ge 2$$

The inequality (3.41) proves that $A_k(\Phi)$ tends to 0 when h tends to 0. Hence, the sequence $(\rho_k)_k$ satisfies the Euler-Lagrange equation (1.11) . ∎

Next, let's show that the sequence $(\rho^h)_h$ converges weakly (up to a subsequence) to a function $\rho = \rho(t,x)$ which solves the parabolic equations (1.1).

4. Convergence results

In this section, we assume that the initial datum ρ_0 is a probability density which satisfies

$$\int_\Omega \left(\frac{|\nabla(\rho_0)|^2}{2}+\psi(\rho_0)\right)dx<\infty.$$

Using the previous results, we prove that the sequence $(\nabla_x(\rho^h)_h$ is bounded in $L^2([0,T]\times\Omega)$. Then, we deduce that ρ^h converges strongly to ρ in $L^2([0,T]\times\Omega)$.
Finally, we use the strong convergence of $(\rho^h)_h$ to ρ, to prove the weak convergence of the nonlinear term

$$(4.1)\qquad \{div_x\{\rho^h|\nabla_x[\Delta_x(\rho^h)-\psi(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h)-\psi(\rho^h)]\}\}_h$$

to

$$(4.2)\qquad div_x\{\rho|\nabla_x[\Delta_x(\rho)-\psi(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho)-\psi(\rho)]\}.$$

Theorem 4.1. *Assume that* ρ_0 *satisfy* $m\le\rho_0\le M$ *and hypothesis* (H_{ρ_0}), (ψ_1), (ψ_2) and (ψ_3) are fulfilled. Then,

i) The sequence $(\rho^h)_h$ converge strongly to some ρ in $L^2([0,T]\times\Omega)$.
ii) The sequence $(|\nabla_x(\Delta_x(\rho^h)-\psi(\rho^h))|^{p-2}\nabla_x(\Delta_x(\rho^h)-\psi(\rho^h)))_h$ converge weakly to some σ in $[L^q([0,T]\times\Omega)]^N$.
iii) If $t\longmapsto u(t)$ is a positive test function whose support is in $[-T,T]$ for $0<T<\infty$. Then

$$(4.3)\qquad \lim_{h\to 0}\int_{\Omega_T}\left|\nabla_x(\Delta_x(\rho^h)-\psi(\rho^h))\right|^p\rho^h u(t)dtdx = L_1$$

where

$$(4.4)\qquad L_1 := \int_{\Omega_T} <\sigma,\nabla_x(\Delta_x(\rho)-\psi(\rho))>\rho(t,x)u(t)dtdx,$$

with,

$$\Omega_T := [0,T]\times\Omega.$$

Furthermore, $div_x\{\rho^h|\nabla_x(\Delta_x(\rho^h)-\psi(\rho^h))|^{p-2}\nabla_x(\Delta_x(\rho^h)-\psi(\rho^h))\}_h$ converges weakly to $div_x(\rho\sigma)$ in $[C_c^\infty(\mathbb{R}\times\Omega)]'$, and $div_x(\rho\sigma)=div_x[\rho|\nabla_x(\Delta_x(\rho)-\psi(\rho))|^{p-2}\nabla_x(\Delta_x(\rho)-\psi(\rho))]$ weakly.

Proof.

i) Since ρ_k minimize I over $P(\Omega)$, then

$$(4.5)\qquad \int_\Omega\left(\frac{|\nabla_x(\rho_k)|^2}{2}+\psi(\rho_k)\right)dx+\frac{1}{qh^{q-1}}W_q^q(\rho_k,\rho_{k-1})\le\int_\Omega\left(\frac{|\nabla_x(\rho_{k-1})|^2}{2}+\psi(\rho_{k-1})\right)dx.$$

Thus, since $W_q^q(\rho_k,\rho_{k-1})\ge 0$, we have

$$(4.6)\qquad \int_\Omega\frac{|\nabla_x(\rho_j)|^2}{2}dx-\int_\Omega\frac{|\nabla_x(\rho_{j-1})|^2}{2}dx\le\int_\Omega\psi(\rho_{j-1})dx-\int_\Omega\psi(\rho_j)dx.$$

By taking the sum for $j = 1, \dots, k$ in (4.6), we obtain that

$$\int_\Omega \frac{|\nabla_x(\rho_k)|^2}{2}dx - \int_\Omega \frac{|\nabla_x(\rho_0)|^2}{2}dx \leq \int_\Omega \psi(\rho_0)dx - \int_\Omega \psi(\rho_k)dx. \tag{4.7}$$

Consequently, by using definition of ρ^h and the Jessen's inequality in previous relation, we have

$$\int_\Omega \frac{|\nabla_x(\rho^h)|^2}{2}dx \leq \int_\Omega \left(\frac{|\nabla_x(\rho_0)|^2}{2} + \psi(\rho_0)\right)dx - |\Omega|\psi\left(\frac{1}{|\Omega|}\right). \tag{4.8}$$

Consequently, the sequence $(\rho^h(t,.))_h$ is bounded in $H^1(\Omega)$. We deduce that $(\rho^h(t,.))_h$ converge strongly to some $\rho(t,.)$ in $L^2([\Omega)$, for all $t \geq 0$.
Since $m \leq \rho_0 \leq M$, then $m \leq \rho^h \leq M$ (see the maximum principle in [1]). Consequently, we use dominate convergence theorem and we deduce that the sequence $(\rho^h)_h$ converge strongly to ρ in $L^2([0,T] \times \Omega)$.

ii) We use (3.36), then

$$E(\rho_0) - |\Omega|\psi\left(\frac{1}{|\Omega|}\right) \geq \frac{h}{q}\int_\Omega \left|\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]\right|^p \rho_k dx. \tag{4.9}$$

We integrate (4.11) on $[0,T]$ and obtain that

$$\int_{[0,T]\times\Omega} \left|\nabla_x\left[\Delta_x(\rho^h) - \psi^{'}(\rho^h)\right]\right|^p \rho^h dtdx \leq qT\left[E(\rho_0) - |\Omega|\psi\left(\frac{1}{|\Omega|}\right)\right] \tag{4.10}$$

By using the maximum principle $m \leq \rho^h \leq M$, we conclude that

$$\int_{[0,T]\times\Omega} \left|\nabla_x\left[\Delta_x(\rho^h) - \psi^{'}(\rho^h)\right]\right|^p dtdx \leq \frac{qT}{m}\left[E(\rho_0) - |\Omega|\psi\left(\frac{1}{|\Omega|}\right)\right]. \tag{4.11}$$

Thus, the sequence $(\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)])_h$ is bounded in $[L^p([0,T] \times \Omega)]^N$ and $(|\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)])_h$ is bounded in $[L^q([0,T] \times \Omega)]^N$.
Consequently, $(|\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)])_h$ converge weakly to some σ in $[L^q([0,T] \times \Omega)]^N$.

iii) The proof of (4.3) will be derived from the three following lemmas:

∎

Lemma 4.2. *For* $0 < T < +\infty$, *we have*

$$\begin{aligned} &\int_{\Omega_T} < \sigma, \nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)] > \rho u(t)dtdx \\ &\leq \liminf_{h\to 0}\int_{\Omega_T} |\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)]|^p \rho^h u(t)dtdx, \end{aligned} \tag{4.12}$$

with $\Omega_T := [0,T] \times \Omega$.

Proof. We set

$$U(t,x) = |\nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)]|^{p-2}\nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)] \quad \text{and} \tag{4.13}$$

$$V(t,x) = \nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)] - \nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)]. \tag{4.14}$$

Since u is positive and $v \longmapsto |v|^{p-2}v$ is monotone, we have

$$\int_{\Omega_T} A(t,x)u(t)dtdx \geq 0, \tag{4.15}$$

where,

$$A(t,x) :=< \sigma^h - U(t,x), V(t,x) > \rho^h$$

with

$$\sigma^h = |\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)].$$

By the previous inequality, we obtain

$$\begin{aligned}(4.16)\qquad &\int_{\Omega_T} < \sigma^h, \nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)] > \rho^h u(t)dtdx \\ &\geq \int_{\Omega_T} < \sigma^h, \nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)] > \rho^h u(t)dtdx + \\ &\int_{\Omega_T} < U(t,x), V(t,x) > \rho^h u(t)dtdx.\end{aligned}$$

Then, using the strong convergence of ρ^h to ρ, the weak convergence of $(\sigma^h)_h$ to σ and the weak convergence of $(\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)])_h$ to $\nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)]$, we have

$$(4.17)\qquad \lim_{h\to 0} \int_{\Omega_T} < \sigma^h, \nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)] > \rho^h u(t)dtdx = L_2,$$

where

$$L_2 := \int_{[0,T]\times\Omega} < \sigma, \nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)] > \rho u(t)dtdx.$$

Also

$$(4.18)\qquad \lim_{h\to 0} \int_{\Omega_T} < U(t,x), V(t,x) > \rho^h u(t)dtdx = 0.$$

By tending h to 0 in (4.16) and using (4.17) and (4.18), we obtain the proof of lemma (4.2). ∎

Lemma 4.3. *For $0 < T < +\infty$, we have*

$$\begin{aligned}&\limsup_{h\to 0} \int_{\Omega_T} \left|\nabla_x(\Delta_x(\rho^h) - \psi^{'}(\rho^h))\right|^p \rho^h u(t)dtdx \leq \\ (4.19)\qquad &\int_\Omega \left[\rho_0\psi^{'}(\rho_0) - \psi^*(\psi^{'}(\rho_0))\right] u(0)dx + \int_{[0,T]\times\Omega} \left[\rho\psi^{'}(\rho) - \psi^*(\psi^{'}(\rho))\right] u^{'}(t)dtdx \\ &+ \int_{[0,T]\times\Omega} \frac{|\nabla_x(\rho)|^2}{2} u^{'}(t)dtdx + \int_\Omega \frac{|\nabla_x(\rho_0)|^2}{2} u(0)dx.\end{aligned}$$

Proof. Since ρ_k minimize I over $P(\Omega)$, we obtain energy-inequality

$$(4.20)\qquad I(\rho_{k-1}) - I(\rho_k) \geq \frac{1}{qh^{q-1}} W_q^q(\rho_k, \rho_{k-1}).$$

Consequently, using the expression of I and (3.26), we obtain

$$(4.21)\qquad \int_\Omega \left[\frac{|\nabla(\rho_{k-1})|^2}{2} + \psi(\rho_{k-1})\right] dx - \int_\Omega \left[\frac{|\nabla(\rho_k)|^2}{2} + \psi(\rho_k)\right] dx \geq K_1,$$

where,

$$(4.22)\qquad K_1 := \frac{h}{q} \int_\Omega |\nabla_x(\Delta(\rho_k) - \psi(\rho_k))|^p \rho_k dx.$$

Multiplying the previous inequality by $u \geq 0$, we obtain after integration

$$(4.23)\qquad Z_1(h) + Z_2(h) \geq \int_{[0,T]\times\Omega} |\nabla_x[\Delta_x(\rho^\tau) - \psi^{'}(\rho^\tau)]|^p u(t)\rho^\tau dtdx,$$

where

$$(4.24)\qquad Z_1(h) := \sum_{k=1}^{\frac{T}{\tau}} \int_{(k-1)\tau}^{k\tau} \int_\Omega \left[\frac{\psi(\rho_{k-1}) - \psi(\rho_k)}{\tau}\right] u(t)dtdx \quad \text{and}$$

$$(4.25)\qquad Z_2(h) := \sum_{k=1}^{\frac{T}{\tau}} \int_{(k-1)\tau}^{k\tau} \int_\Omega \left[\frac{|\nabla_x(\rho_{k-1})|^2 - |\nabla_x(\rho_k)|^2}{2\tau}\right] u(t)dtdx,$$

with $\tau = \frac{h}{q}$, and ρ^τ is a approximate solution defined by

$$\rho^\tau(t,x) = \rho_k(x), \quad \text{if } (t,x) \in [\tau k, \tau(k+1)). \tag{4.26}$$

Notice that

$$Z_1(h) = \int_{[0,T]\times\Omega} \psi(\rho^\tau)\left[\frac{u(t)-u(t-\tau)}{\tau}\right] dtdx + \frac{1}{\tau}\int_0^\tau \int_\Omega \psi(\rho^\tau)u(t-\tau)dtdx. \tag{4.27}$$

We tend h to 0 in (4.27), and obtain

$$\lim_{h\to 0} Z_1(h) = \int_{[0,T]\times\Omega} \psi(\rho)u^{'}(t)dtdx + \int_\Omega \psi(\rho_0)u(0)dx. \tag{4.28}$$

Therefore

$$\begin{aligned} Z_2(h) = &\int_{[0,T]\times\Omega} \frac{|\nabla_x(\rho^\tau)|^2}{2}\left[\frac{u(t)-u(t-\tau)}{\tau}\right] dtdx \\ + &\frac{1}{\tau}\int_0^\tau \int_\Omega \frac{|\nabla_x(\rho^\tau)|^2}{2} u(t-\tau)dtdx, \end{aligned} \tag{4.29}$$

and

$$\lim_{h\to 0} Z_2(h) = \int_{[0,T]\times\Omega} \frac{|\nabla_x(\rho)|^2}{2}u^{'}(t)dtdx + \int_\Omega \frac{|\nabla_x(\rho_0)|^2}{2}u(0)dx. \tag{4.30}$$

We use (4.23) and (4.28), (4.29), (4.30) and obtain

$$\limsup_{h\to 0} \int_{[0,T]\times\Omega} |\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)]|^p \rho^h u(t)dtdx \quad \le \quad L_3 + L_4 + L_5,$$

where,

$$\begin{aligned} L_3 &:= \left[\int_{[0,T]\times\Omega} \psi(\rho)u^{'}(t)dtdx + \int_\Omega \psi(\rho_0)u(0)dx\right] \\ L_4 &:= \int_{[0,T]\times\Omega} \frac{|\nabla_x(\rho)|^2}{2}u^{'}(t)dtdx \quad \text{and} \\ L_5 &:= \int_\Omega \frac{|\nabla_x(\rho_0)|^2}{2}u(0)dx. \end{aligned}$$

From the definition of ψ^*, we have $\psi^*(a) \ge ab - \psi(b)$ for all $a, b > 0$ and we obtain the equality if $a = \psi^{'}(b)$. Then, using $\psi(\rho_0) = \rho_0\psi^{'}(\rho_0) - \psi^*(\psi^{'}(\rho_0))$ and $\psi(\rho) = \rho\psi^{'}(\rho) - \psi^*(\psi^{'}(\rho))$ in (4.31), we obtain (4.19). ∎

Lemma 4.4. *For $0 < T < \infty$, we have*

$$\int_{[0,T]\times\Omega} < \sigma, \nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)] > \rho u(t)dtdx \ge J_1 + J_2 + J_3,$$

with,

$$\begin{aligned} J_1 &:= \int_\Omega \left[\rho_0\psi^{'}(\rho_0) - \psi^*(\psi^{'}(\rho_0))\right] u(0)dx \\ J_2 &:= \int_{[0,T]\times\Omega} \left[\rho\psi^{'}(\rho) - \psi^*(\psi^{'}(\rho))\right] u^{'}(t)dtdx \quad \textit{and} \\ J_3 &:= \int_{[0,T]\times\Omega} \frac{|\nabla_x(\rho)|^2}{2}u^{'}(t)dtdx + \int_\Omega \frac{|\nabla_x(\rho_0)|^2}{2}u(0)dx. \end{aligned}$$

Proof. Define $\Psi(t,x) = [\Delta_x(\rho) - \psi^{'}(\rho(t,x))]u(t)$, $\Psi(t,.) \in W^{1,p}(\Omega)$.
Approximating Ψ by $C_c^{\infty}(\Omega)$ functions and using (3.30), we have

$$\int_{\Omega} \frac{\rho_k - \rho_{k-1}}{h}\Psi(t,x)dx = \int_{\Omega} G(k,t,x)\rho_k u(t)dtdx + 0(h), \tag{4.31}$$

with

$$G(k,t,x) :=< |\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)]|^{p-2}\nabla_x[\Delta_x(\rho_k) - \psi^{'}(\rho_k)], \nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)] >, \tag{4.32}$$

where $0(h)$ tends to 0 when h tends to 0.
By using the definition of ρ^h, we obtain after integration

$$\sum_{k=1}^{\frac{T}{h}} \int_{(k-1)h}^{kh} \int_{\Omega} \frac{\rho_k - \rho_{k-1}}{h}\Psi(t,x)dtdx = \int_{[0,T]\times\Omega} G(k,t,x)\rho^h u(t)dtdx + 0(h).$$

Noting that

$$\sum_{k=1}^{\frac{T}{h}} \int_{(k-1)h}^{kh} \int_{\Omega} \frac{\rho_k - \rho_{k-1}}{h}\Psi(t,x)dtdx = A(h) + B(h), \tag{4.33}$$

where

$$\begin{aligned} A(h) &= \sum_{k=1}^{\frac{T}{h}} \int_{(k-1)h}^{kh} \int_{\Omega} \frac{\rho_k - \rho_{k-1}}{h}\Delta_x(\rho)u(t)dtdx \\ &= -\frac{1}{h}\int_0^h \int_{\Omega} \rho_0 \Delta(\rho)u(t)dtdx \\ &- \int_{[0,T]\times\Omega} \rho^h(t)\Delta_x(\rho(t+h))\left[\frac{u(t+h)-u(t)}{h}\right]dtdx \\ &- \int_{[0,T]\times\Omega} \rho^h(t,x)u(t)\left[\frac{\Delta_x(\rho(t+h)) - \Delta_x(\rho)}{h}\right]dtdx, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} B(h) &= -\sum_{k=1}^{\frac{T}{h}} \int_{(k-1)h}^{kh} \int_{\Omega} \frac{\rho_k - \rho_{k-1}}{h}\psi^{'}(\rho)u(t)dtdx \\ &= \frac{1}{h}\int_0^h \int_{\Omega} \rho_0 \psi^{'}(\rho)u(t)dtdx \\ &+ \int_{[0,T]\times\Omega} \rho^h(t)\psi^{'}(\rho(t+h))\left[\frac{u(t+h)-u(t)}{h}\right]dtdx \\ &+ \int_{[0,T]\times\Omega} \rho^h(t,x)u(t)\left[\frac{\psi^{'}(\rho(t+h)) - \psi^{'}(\rho)}{h}\right]dtdx. \end{aligned} \tag{4.35}$$

Consequently, we tend h to 0 in 4.34 and we obtain that

$$\begin{aligned} \lim_{h\to 0} A(h) &= -\int_{\Omega} \rho_0 \Delta_x(\rho_0)u(0)dx \\ &- \int_{[0,T]\times\Omega} \rho\Delta_x(\rho)u^{'}(t)dtdx \\ &- \int_{[0,T]\times\Omega} \rho u(t)\Delta_x(\frac{\partial\rho(t)}{\partial t})dtdx. \end{aligned}$$

By using boundary condition $\rho\nabla_x(\rho)\cdot\nu = 0 \quad \text{on} \quad \partial\Omega$, we deduce that

$$
\begin{aligned}
\lim_{h\to 0} A(h) &= \int_\Omega |\nabla_x(\rho_0)|^2 u(0)dx \\
&+ \int_{[0,T]\times\Omega} \frac{|\nabla_x(\rho)|^2}{2} u'(t)dtdx \\
&\geq \int_\Omega \frac{|\nabla_x(\rho_0)|^2}{2} u(0)dx + \int_{[0,T]\times\Omega} \frac{|\nabla_x(\rho)|^2}{2} u'(t)dtdx.
\end{aligned}
\tag{4.36}
$$

We rewrite $B(h)$ as follow

$$
\begin{aligned}
B(h) &= \frac{1}{h}\int_0^h \int_\Omega \rho_0 \psi'(\rho)u(t)dtdx \\
&+ \int_{[0,T]\times\Omega} \rho^h(t)\psi'(\rho(t+h)) \left[\frac{u(t+h)-u(t)}{h}\right] dtdx \\
&+ \int_{[0,T]\times\Omega} \rho(t+h,x)u(t) \left[\frac{\psi'(\rho(t+h))-\psi'(\rho)}{h}\right] dtdx \\
&+ \int_{[0,T]\times\Omega} (\rho^h(t,x)-\rho(t+h,x)) \left[\frac{\psi'(\rho(t+h))-\psi'(\rho)}{h}\right] u(t)dtdx.
\end{aligned}
\tag{4.37}
$$

Since the Legendre transform ψ^* of ψ is convex, then

$$
\psi^*(\psi'(\rho)) - \psi^*(\psi'(\rho(t+h,x))) \geq \rho(t+h,x)[\psi'(\rho) - \psi'(\rho(t+h,x))]. \tag{4.38}
$$

Consequently

$$
\int_{[0,T]\times\Omega} \rho(t+h,x) \left[\frac{\psi'(\rho(t+h))-\psi'(\rho)}{h}\right] u(t)dtdx \geq K(h), \tag{4.39}
$$

where

$$
K(h) := \int_{[0,T]\times\Omega} \left[\frac{\psi^*(\psi'(\rho(t+h,x))) - \psi^*(\psi'(\rho))}{h}\right] u(t)dtdx. \tag{4.40}
$$

From (4.39), we have

$$
K(h) = K_1(h) + K_2(h) + K_3(h),
$$

where

$$
\begin{aligned}
K_1(h) &:= -\frac{1}{h}\int_0^h \int_\Omega \psi^*(\psi'(\rho))u(t)dtdx \\
K_2(h) &:= -\int_{[h,T+h]\times\Omega} \psi^*(\psi'(\rho)) \left[\frac{u(t)-u(t-h)}{h}\right] dtdx \\
K_3(h) &:= \frac{1}{h}\int_T^{T+h} \psi^*(\psi'(\rho))u(t)dtdx.
\end{aligned}
$$

We combine (4.37) and (4.41). Then, $B(h)$ is as follow

$$\begin{aligned} B(h) \;\geq\; & \frac{1}{h}\int_0^h\int_\Omega \rho_0\psi^{'}(\rho)u(t)dtdx \\ &+ \int_{[0,T]\times\Omega}\rho^h(t)\psi^{'}(\rho(t+h))\left[\frac{u(t+h)-u(t)}{h}\right]dtdx \\ &- \int_{[h,T+h]\times\Omega}\psi^*(\psi^{'}(\rho))\left[\frac{u(t)-u(t-h)}{h}\right]dtdx \\ &- \frac{1}{h}\int_0^h\int_\Omega \psi^*(\psi^{'}(\rho))u(t)dtdx \\ &+ \frac{1}{h}\int_T^{T+h}\psi^*(\psi^{'}(\rho))u(t) \\ &+ \int_{[0,T]\times\Omega}(\rho^h(t,x)-\rho(t+h,x))\left[\frac{\psi^{'}(\rho(t+h))-\psi^{'}(\rho)}{h}\right]u(t)dtdx. \end{aligned} \tag{4.41}$$

Since $(\rho^h)_h$ converges strongly to ρ, then

$$\lim_{h\to 0}\int_{[0,T]\times\Omega}(\rho^h(t,x)-\rho(t+h,x))\left[\frac{\psi^{'}(\rho(t+h))-\psi^{'}(\rho)}{h}\right]u(t)dtdx = 0. \tag{4.42}$$

We tend h to 0 in (4.41), and using (4.42), we have

$$\begin{aligned} \lim_{h\to 0} B(h) \geq & \int_\Omega\left[\rho_0\psi^{'}(\rho_0)-\psi^*(\psi^{'}(\rho_0))\right]u(0)dx + \\ & \int_{[0,T]\times\Omega}\left[\rho\psi^{'}(\rho)-\psi^*(\psi^{'}(\rho))\right]u^{'}dtdx. \end{aligned} \tag{4.43}$$

Finally, we combine relation (4.43),(4.36) and (4.33) and we reach(4.31).
To get the proof of (4.3), we use the results in the three previous lemmas .

Now, let show that

$$\left\{div_x\left(\rho^h|\nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)]\right)\right\}_h$$

converges to
$div_x(\rho\sigma) = div_x\left(\rho|\nabla_x[\Delta_x(\rho)-\psi^{'}(\rho)]|^{p-2}\nabla_x[\Delta_x(\rho)-\psi^{'}(\rho)]\right)$ in $[C_c^\infty([0,T]\times\Omega)]^{'}$.

Let $\varepsilon > 0$ be small and $\phi \in C_c^\infty(\Omega)$ be a test function. Define $\Psi_\varepsilon(t,x) = \Delta_x(\rho) - \psi^{'}(\rho) - \varepsilon\phi(x)$. $\Psi_\varepsilon \in W^{1,p}([0,T]\times\Omega)$.
We use the fact that $v \longmapsto v|v|^{p-2}$ is monotone to derive

$$\int_{[0,T]\times\Omega} < \sigma^h - |\nabla_x(\Psi_\varepsilon)|^{p-2}\nabla_x(\Psi_\varepsilon), \nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)] - \nabla_x\Psi_\varepsilon > \rho^h u(t)dtdx \geq 0, \tag{4.44}$$

where σ^h is defined above. Thus

$$\begin{aligned} & \int_{[0,T]\times\Omega}|\nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)]|^p\rho^h u(t)dtdx \\ & -\int_{[0,T]\times\Omega} < \sigma^h, \nabla_x\Psi_\varepsilon > \rho^h u(t)dtdx \\ & -\int_{[0,T]\times\Omega} < |\nabla_x(\Psi_\varepsilon)|^{p-2}\nabla_x(\Psi_\varepsilon), \nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)] - \nabla_x\Psi_\varepsilon > u(t)dtdx \geq 0. \end{aligned} \tag{4.45}$$

We tend h to 0 in the previous inequality, and we use (4.3), to get

$$\int_{[0,T]\times\Omega} < \sigma, \nabla_x \left[\Delta_x(\rho) - \psi^{'}(\rho)\right] > \rho u(t)dtdx - \int_{[0,T]\times\Omega} < \sigma, \nabla_x \Psi_\varepsilon > \rho u(t)dtdx - \int_{[0,T]\times\Omega} < |\nabla_x(\Psi_\varepsilon)|^{p-2}\nabla_x(\Psi_\varepsilon), \nabla_x[\Delta(\rho) - \psi^{'}(\rho)] - \nabla_x\Psi_\varepsilon > \rho u(t)dtdx \geq 0. \tag{4.46}$$

By using definition of Ψ_ε, the previous inequality becomes

$$\int_{[0,T]\times\Omega} < \sigma, \nabla_x\phi(x) > \rho u(t)dtdx \geq K_4(h), \tag{4.47}$$

with

$$K_4(h) := \int_{[0,T]\times\Omega} < |\nabla_x(\Psi_\varepsilon)|^{p-2}\nabla_x(\Psi_\varepsilon), \nabla_x\phi(x). > \rho u(t)dtdx. \tag{4.48}$$

We tend ε to 0 , and we have

$$\int_{[0,T]\times\Omega} < \sigma, \nabla_x\phi(x) > \rho u(t)dtdx \geq \int_{[0,T]\times\Omega} < |\nabla_x(\Delta_x(\rho) - \psi^{'}(\rho))|^{p-2}\nabla_x(\Delta_x(\rho) - \psi^{'}(\rho)), \nabla_x\phi(x) > \rho u(t)dtdx. \tag{4.49}$$

Replacing ϕ by $-\phi$ in the previous inequality, we obtain the equality:

$$\int_{[0,T]\times\Omega} < \sigma, \nabla_x\phi(x) > \rho u(t)dtdx = \int_{[0,T]\times\Omega} < |\nabla_x(\Delta_x(\rho) - \psi^{'}(\rho))|^{p-2}\nabla_x(\Delta_x(\rho) - \psi^{'}(\rho)), \nabla_x\phi(x) > \rho u(t)dtdx. \tag{4.50}$$

Finally, we deduce that the sequence

$$\left\{div_x\left(\rho^h|\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h) - \psi^{'}(\rho^h)]\right)\right\}_h$$

converges to
$div_x(\rho\sigma) = div_x\left(\rho|\nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)]|^{p-2}\nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)]\right)$ in $[C_c^\infty([0,T]\times\Omega)]'$. ∎

5. EXISTENCE AND UNIQUENESS OF SOLUTION

In this section, we show the existence and thank to additional assumption the uniqueness of weak solutions of the parabolic biharmonic equation (1.1)-(1.3).

Theorem 5.1. *Assume that hypothesis* (H_{ρ_0}), (ψ_1), (ψ_2) and (ψ_3) are fulfilled. Then , the sequence $(\rho^h)_h$ converges strongly to a positive function $\rho(t,x)$ and $\rho \in L^\infty([0,\infty[\times\Omega)$. Also ρ is a weak solution of the equation (1.1). That is, for all $\phi(t,x) \in C_c^\infty([0,\infty[\times\Omega)$, $supp\phi(.,x) \subset [-T,T]$, for $0 < T < \infty$, we have:

$$\int_{[0,T]\times\Omega} \rho\left[\frac{\partial\phi(t,x)}{\partial t} + < |\nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)]|^{p-2}\nabla_x[\Delta_x(\rho) - \psi^{'}(\rho)], \nabla_x\phi(t,x) >\right] dtdx = Y_0, \tag{5.1}$$

with

$$Y_0 := -\int_\Omega \rho_0\phi(0,x)dx.$$

Proof. Using (3.27):

$$\sum_{k=1}^{\frac{T}{h}}\int_{[(k-1)h,kh]\times\Omega}\frac{\rho_k-\rho_{k-1}}{h}\phi(t,x)dtdx =$$
$$\int_{[0,T]\times\Omega} < |\nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)], \nabla_x\phi(t,x) > \rho^h dtdx + 0(h),$$

where $0(h)$ tends to 0 when h tends to 0.
Note that:

$$\sum_{k=1}^{\frac{T}{h}}\int_{[(k-1)h,kh]\times\Omega}\frac{\rho_k-\rho_{k-1}}{h}\phi(t,x)dtdx = \int_{[0,T]\times\Omega}\rho^h\left[\frac{\phi(t-h,x)-\phi(t,x)}{h}\right]dtdx - \frac{1}{h}\int_0^h\int_\Omega \rho^h\phi(t-h,x)dtdx. \tag{5.2}$$

Replacing the previous relation in (5.2), we have:

$$\int_{[0,T]\times\Omega}\rho^h\left[\frac{\phi(t-h,x)-\phi(t,x)}{h}\right]dtdx - \frac{1}{h}\int_0^h\int_\Omega \rho^h\phi(t-h)dtdx - \int_{[0,T]\times\Omega} < |\nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)]|^{p-2}\nabla_x[\Delta_x(\rho^h)-\psi^{'}(\rho^h)], \nabla_x\phi(t,x) > \rho^h dtdx = 0(h). \tag{5.3}$$

We tend h to 0 in (5.3) and use theorem (4.1) to obtain:

$$\int_{[0,T]\times\Omega}\rho\frac{\partial\phi(t,x)}{\partial t}dtdx + \int_{[0,T]\times\Omega} < |\nabla_x[\Delta_x(\rho)-\psi^{'}(\rho)]|^{p-2}\nabla_x[\Delta_x(\rho)-\psi^{'}(\rho)], \nabla_x\phi(t,x) > \rho dtdx = -\int_\Omega \rho_0\phi(0,x)dx.$$

We conclude that ρ is a weak solution of the parabolic equation (1.1)-(1.3). ∎

Theorem 5.2. *Assume that hypothesis* (H_{ρ_0}),(ψ_1), (ψ_2) and (ψ_3) are fulfilled. Let ρ^1 and ρ^2 be two weak solutions of (1.1)-(1.3) satisfying $\frac{\partial\rho^i}{\partial t}\in L^1(\Omega)$, for $i=1,2$, with initial datum $\rho^1(0,.)$ and $\rho^2(0,.)$ respectively satisfying $m\le\rho^1(0,.),\rho^2(0,.)\le M$. Then,

$$\int_\Omega[\rho^1(T,x)-\rho^2(T,x)]^+dx \le 0, \tag{5.4}$$

for all $T\ge 0$.

Proof. Define $\theta_\delta:\mathbb{R}\to[0,1]$, by:

$$\theta_\delta(s) = \begin{cases} 0 & \text{if} \quad s\le 0 \\ \frac{s}{\delta} & \text{if} \quad 0\le s\le\delta \\ 1 & \text{if} \quad s\ge\delta. \end{cases}, \tag{5.5}$$

By using definition of the weak solution, we have:

$$\int_{[0,T]\times\Omega}\phi\frac{\partial}{\partial t}(\rho^1(t,x)-\rho^2(t,x))dtdx = \int_{[0,T]\times\Omega} < \rho^1A(\rho^1)-\rho^2A(\rho^2), \nabla_x\phi > dtdx, \tag{5.6}$$

where

$$A(\rho) = |\nabla_x[\Delta_x(\rho)-\psi^{'}(\rho)]|^{p-2}\nabla_x[\Delta_x(\rho)-\psi^{'}(\rho)].$$

We use $\theta_\delta([-\Delta(\rho^1)+\psi^{'}(\rho^1)]-[-\Delta(\rho^2)+\psi^{'}(\rho^2))$ in 5.6; we have:

$$\int_{\Omega_T}\theta_\delta([-\Delta_x(\rho^1)+\psi^{'}(\rho^1)]-[-\Delta_x(\rho^2)+\psi^{'}(\rho^2)])\frac{\partial}{\partial t}(\rho^1(t,x)-\rho^2(t,x))dtdx =$$

$$\int_{\Omega_T}<\rho^1A(\rho^1)-\rho^2A(\rho^2),\nabla_x(\theta_\delta([-\Delta_x(\rho^1)+\psi^{'}(\rho^1)]-[-\Delta_x(\rho^2)+\psi^{'}(\rho^2)])>dtdx$$

$$=-\frac{1}{\delta}\int_{\Omega_{T,\delta}}<-A(\rho^1)+A(\rho^2),\nabla_x[-\Delta_x(\rho^1)+\psi^{'}(\rho^1)]-\nabla_x[-\Delta_x(\rho^2)+\psi^{'}(\rho^2)]>\rho^2dtdx+$$

$$\frac{1}{\delta}\int_{\Omega_{T,\delta}}<(\rho^1-\rho^2)A(\rho^1),\nabla_x[-\Delta_x(\rho^1)+\psi^{'}(\rho^1)]-\nabla_x[-\Delta_x(\rho^2)+\psi^{'}(\rho^2)]>\rho^2dtdx,$$

where

$$\Omega_{T,\delta}:=\Omega_T\cap\{0<-\Delta_x(\rho^1)+\Delta_x(\rho^2)+\psi^{'}(\rho^1)-\psi^{'}(\rho^2)\le\delta\}$$

and $\Omega_T:=[0,T]\times\Omega$.
Since $v\mapsto v|v|^{p-2}$ is monotone, we have

$$-\frac{1}{\delta}\int_{\Omega_{T,\delta}}<-A(\rho^1)+A(\rho^2),\nabla_x[-\Delta_x(\rho^1)+\psi^{'}(\rho^1)]-\nabla_x[-\Delta_x(\rho^2)+\psi^{'}(\rho^2)]>\rho^2dtdx\le 0.$$

If $\delta\to 0^+$, then $|\Omega_{T,\delta}|\to 0$ and
$\theta_\delta(-\Delta_x(\rho^1)+\Delta_x(\rho^2)+\psi^{'}(\rho^1)-\psi^{'}(\rho^2))\to sign^+(-\Delta_x(\rho^1)+\Delta_x(\rho^2)+\psi^{'}(\rho^1)-\psi^{'}(\rho^2)) = sign^+(\rho^1-\rho^2)$; with $sign(s)=\frac{s}{|s|}$ for all $s\in\mathbb{R}^*$. Then,

$$\int_{[0,T]\times\Omega}\frac{\partial(\rho^1-\rho^2)^+}{\partial t}=\int_{[0,T]\times\Omega}sign^+(\rho^1-\rho^2)\frac{\partial}{\partial t}(\rho^1-\rho^2)\le 0. \tag{5.7}$$

This implies

$$\int_\Omega(\rho^1(T,x)-\rho^2(T,x))^+dx\le 0 \tag{5.8}$$

for all $T\ge 0$. Then the solution of equation (1.1) is unique. ∎

6. ASYMPTOTIC BEHAVIOR

In this section, we study the asymptotic behavior of the solution of the parabolic bi-harmonic equation (1.1)-(1.3). We establish the regularity of the solution in this lemma.

Lemma 6.1. *Assume that hypothesis* (H_{ρ_0}), (ψ_1), (ψ_2) and (ψ_3) hold. Let ρ be a solution of parabolic p-biharmonic equation (1.1). Then, there exist a constant $\lambda>0$ such that

$$\int_\Omega A_\rho(\phi_1)\rho dx-\int_\Omega A_\rho(\phi_2)\rho dx \ \ge\ \int_\Omega<\nabla_x(A_\rho)(\phi_2),\phi_1(x)-\phi_2(x)>\rho dx \tag{6.1}$$

$$+\ \lambda\int_\Omega|\phi_1(x)-\phi_2(x)|^q\rho dx. \tag{6.2}$$

for all $\phi_1,\phi_2\in[L^q(\Omega)]^N$, and where

$$A_\rho:=-\Delta_x(\rho)+\psi^{'}(\rho).$$

Proof. Since ρ is a solution of the equation (1.1), then there exist a sequence $(\rho_k)_k$ defined in (1.8) which converge to ρ. Therefore, the optimal map whose push ρ_k forward to ρ_{k-1} is defined by

$$T_k(x)=x+h|\nabla_x[-\Delta_x(\rho_k)+\psi^{'}(\rho_k)]|^{p-2}\nabla_x[-\Delta_x(\rho_k)+\psi^{'}(\rho_k)]. \tag{6.3}$$

Consequently

$$\nabla_x[-\Delta_x(\rho_k)+\psi^{'}(\rho_k)]=\left|\frac{T_k(x)-x}{h}\right|^{q-2}\left(\frac{T_k(x)-x}{h}\right). \tag{6.4}$$

In [1], it is know that the map T_k is differentiable and $\nabla_x T_k = id - (p-1)|\nabla_x u_k|^{p-2} D^2 u_k$, where u_k is a semi-concave function. We deduce that $\nabla_x[-\Delta_x(\rho_k) + \psi^{'}(\rho_k)]$ is differentiable and

$$D^2[-\Delta_x(\rho_k) + \psi^{'}(\rho_k)] = -(q-1)(p-1)h^{3-p-q}|T_k(x) - x|^{q-2}|\nabla u_k|^{p-2} D^2 u_k. \tag{6.5}$$

Since u_k is semi-concave, then $-D^2 u_k$ is diagonalizable with non-negative eigenvalues (see [1]). Consequently

$$< D^2[-\Delta_x(\rho_k) + \psi^{'}(\rho_k)]z, z > \geq \lambda |z|^q, \tag{6.6}$$

for all $z \in \mathbb{R}^N$ and for some $\lambda > 0$.
From (6.6), we obtain that

$$A_{\rho_k}(z_1) - A_{\rho_k}(z_2) \geq < \nabla_x A_{\rho_k}(z_2), z_1 - z_2 > + \lambda |z_1 - z_2|^q. \tag{6.7}$$

By using, the fact that A_{ρ_k} converge weakly to A_ρ, we obtain (6.7). ∎

Theorem 6.2. *Assume that hypothesis* (H_{ρ_0}), (ψ_1), (ψ_2) and (ψ_3) *are fulfilled. Let* ρ *be a solution of parabolic* p*-biharmonic equation (1.1). Then*

$$[E(\rho(t,.)) - E(\rho_\infty)] \leq e^{-p(\lambda q)^p t}[E(\rho_0) - E(\rho_\infty)] \tag{6.8}$$

and

$$W_q^q(\rho(t,.), \rho_\infty) \leq \frac{1}{\lambda} e^{-p(\lambda q)^p t}[E(\rho_0) - E(\rho_\infty)], \tag{6.9}$$

where $\lambda > 0$ *is a constant and* ρ_∞ *is a probability density on* Ω *whose satisfy*

$$\rho_\infty \nabla_x[-\Delta_x(\rho_\infty) + \psi^{'}(\rho_\infty] = 0 \quad \text{in } \Omega. \tag{6.10}$$

Proof. Let ρ_1 and ρ_2 two probability density on Ω and $T : \Omega \to \Omega$ the map whose push ρ_1 forward to ρ_2 in the Monge-Kantorovich problem

$$(M) : \inf_{T_{\#}\rho_1 = \rho_2} \int_\Omega \frac{|T(x) - x|^q}{q} \rho_1 dx. \tag{6.11}$$

Since ψ is convex, then

$$\int_\Omega \psi(\rho_2)dx - \int_\Omega \psi(\rho_1)dx \geq \int_\Omega < \psi^{'}(\rho_1), \rho_2 - \rho_1 > dx. \tag{6.12}$$

Therefore,

$$\int_\Omega \frac{|\nabla_x \rho_2|^2}{2} dx - \int_\Omega \frac{|\nabla_x \rho_1|^2}{2} dx \geq \int_\Omega < \nabla_x \rho_1, \nabla_x(\rho_2) - \nabla_x(\rho_1) > dx \tag{6.13}$$

By using boundary condition $\rho \nabla_x \rho \cdot \nu = 0 \quad \text{on } \partial\Omega$, we obtain that,

$$\int_\Omega < \nabla_x \rho_1, \nabla_x(\rho_2) - \nabla_x(\rho_1) > dx = \int_\Omega (-\Delta_x \rho_1)\rho_2 dx - \int_\Omega (-\Delta_x \rho_1)\rho_1 dx. \tag{6.14}$$

Using (6.14), (6.12) and lemma (6.1), we obtain that

$$E(\rho_2) - E(\rho_1) \geq \int_\Omega < T(x) - x, \nabla_x[-\Delta_x \rho_1 + \psi^{'}(\rho_1)] > \rho_1 dx + \lambda \int_\Omega |T(x) - x|^q \rho_1 dx. \tag{6.15}$$

where $E(\rho) = \int_\Omega [\psi(\rho) + \frac{|\nabla_x \rho|^2}{2}]dx$. Noting that $\int_\Omega |T(x) - x|^q \rho_1 dx \geq W_q^q(\rho_1, \rho_2)$, where W_q is the q-Wasserstein metric. Consequently

$$E(\rho_2) - E(\rho_1) \geq \int_\Omega < T(x) - x, \nabla_x[-\Delta_x \rho_1 + \psi^{'}(\rho_1)] > \rho_1 dx + \lambda W_q^q(\rho_1, \rho_2). \tag{6.16}$$

If $\rho_1 = \rho_\infty$ satisfy $\rho_\infty \nabla_x[-\Delta_x \rho_\infty + \psi^{'}(\rho_\infty)] = 0$ and $\rho_2 = \rho$ then (6.16) becomes

$$W_q^q(\rho(t,.), \rho_\infty) \leq \frac{1}{\lambda}[E(\rho(t,.)) - E(\rho_\infty)]. \tag{6.17}$$

We use Young inequality in (6.16) and obtain

$$E(\rho_2) - E(\rho_1) \geq -\frac{1}{q\mu}\int_\Omega |T(x)-x|^q \rho_1 dx \tag{6.18}$$

$$- \frac{\mu^p}{p}\int_\Omega |\nabla_x[-\Delta_x \rho_1 + \psi'(\rho_1)|^p \rho_1 dx \tag{6.19}$$

$$+ \lambda \int_\Omega |T(x)-x|^q \rho_1 dx. \tag{6.20}$$

By using in the previous relation $\mu = \frac{1}{\lambda q}$, $\rho_1 = \rho$ and $\rho_2 = \rho_\infty$, we obtain

$$\begin{aligned} E(\rho_\infty) - E(\rho) &\geq -\lambda \int_\Omega |T(x)-x|^q \rho dx \\ &- \frac{1}{p(\lambda q)^p}\int_\Omega |\nabla_x[-\Delta_x \rho + \psi'(\rho)|^p \rho dx \\ &+ \lambda \int_\Omega |T(x)-x|^q \rho dx \\ &\geq -\frac{1}{p(\lambda q)^p}\int_\Omega |\nabla_x[-\Delta_x \rho + \psi'(\rho)]|^p \rho dx. \end{aligned} \tag{6.21}$$

By using $\frac{d}{dt}[E(\rho) - E(\rho_\infty)] = -\int_\Omega |\nabla_x[-\Delta_x \rho + \psi'(\rho)]|^p \rho dx$, and the previous inequality, we obtain that

$$\frac{d}{dt}[E(\rho) - E(\rho_\infty] \leq -p(\lambda q)^p [E(\rho) - E(\rho_\infty)]. \tag{6.22}$$

From (6.22), we deduce that

$$[E(\rho) - E(\rho_\infty)] \leq e^{-p(\lambda q)^p t}[E(\rho_0) - E(\rho_\infty)]. \tag{6.23}$$

Combining (6.23) and (6.17), we conclude that

$$W_q^q(\rho(t,.), \rho_\infty) \leq \frac{1}{\lambda} e^{-p(\lambda q)^p t}[E(\rho_0) - E(\rho_\infty)]. \tag{6.24}$$

∎

7. SUMMARY

In this work, we have developed a new approach based on optimal transportation, to study existence and uniqueness of solutions for a class of non-linear parabolic biharmonic equations in the probability space under the Neumann boundary condition. We established a regularity result to analyze the asymtotic behavior of the solution. In a forthcoming paper [23], we will discretize the problem using the finite element method. An a priori and a posteriori estimator will be performed to study the convergence of the method.

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