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ON THE OSCILLATORY BEHAVIOR OF SELF ADJOINT FRACTIONAL EXTENSIBLE BEAM EQUATIONS

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ABSTRACT. The main objective of this paper is to study the oscillatory behavior of the solutions of self adjoint fractional extensible beam equations by using integral average method. Some new sufficient conditions are established with various boundary conditions over a cylindrical domains. Examples illustrating the results are given.

Key words and phrases: Extensible beam equations; Self adjoint; Fractional; Oscillation.

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1. INTRODUCTION

The problem of oscillation and nonoscillation of beam equations has been investigated by many authors, see ([1], [4], [10], [11], [19], [22], [23]). In the papers, the authors have discussed the existence of arbitrarily large zeroes of solutions of beam equations with forcing term. In the present paper we have obtained sufficient conditions for solutions on the boundary domains with certain boundary value problems to have a zero. In fact we consider various boundary conditions such as hinged ends, sliding ends and hinged-sliding ends.

The beam equations were proposed by Woinowsky-krieger[21] as a imitation for the transverse deflection u(x,t) of an extensible beam of nature length L whose ends are held a fixed distance apart. The imitation has also been discussed by Eisley[5] and Burgreen[2]. Recently, Dickey[4] introduces the initial-boundary conditions for the beam equations, representing a vibrating string. These method is an adaptation in studying the oscillatory behavior of solutions of hyperbolic equations ([3], [9], [12], [20], [24]).

The theory of the fractional differential equations is an important tool in modeling real world phenomena. The derivative first appeared in the 17^{th} century in a more general form of the integer order differential equations, extending those equations to an arbitrary order. The definition is most frequently involoved in nonlocal that is Riemann-Liouville derivative and Caupto derivative ([6], [7], [8], [13], [14], [15], [16], [18]). They are used in physics, electrochemistry, electromagentic and control theory field.

In 1985, N. Yoshida[23] studied the forced oscillations of extensible beams which motivates this paper. To the authors' knowledge, there has been no previous work made on the oscillation of fractional beam equations. In this article we initiate the forced oscillation of self adjoint fractional extensible beam equations of the form,

$$\frac{\partial}{\partial t}(r(t)D^{\alpha}_{+,t}u(x,t)) + p\frac{\partial^4 u(x,t)}{\partial x^4} - \left(q + m\int_{\Omega}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$+ g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right) = f(x,t), \quad (x,t) \in \Omega\times\mathbb{R}_+ = G. \tag{1.1}$$

Where, Ω = (0, L), $\alpha \in$ (0, 1), $\mathbb{R}_+$ = (0, ∞). Then p is a non negative constant, q, m are constants. $u(x,t) \in C^{1+\alpha}(G,R^1) \cap C^4(G,R^1)$ and $D^{\alpha}_{+,t}$ is the Riemann-Liouville fractional derivative of order α of $u(x,t)$ with respect to t.
We assume the following conditions,
(A_1) r(t) is continuous and $\int_0^\infty \frac{1}{r(s)}ds = \infty$.
We define $Q(t,s) = \int_s^t \frac{1}{r(\eta)}d\eta$, $(t,s) \in (0,\infty)$.
(A_2) $g(x,t,E(t))$ is a real-valued continuous function in G $\times\,\mathbb{R}^1$.
(A_3) $E(t)g(x,t,E(t)) \geq 0$ for all $(x,t,E(t)) \in$ G $\times\,\mathbb{R}^1$.
(A_4) $g(x,t,-E(t)) = -g(x,t,E(t))$ for all $(x,t,E(t)) \in$ G $\times\,\mathbb{R}_+$.
(A_5) $F(t), K(t)$ are continous functions, $f(x,t) \in C(\overline{G},\mathbb{R}_+)$, $F(t) = \int_\Omega f(x,t)\psi(x)dx$, $K(t) = \int_\Omega f(x,t)dx$.
By a solution of the boundary value problem we mean a solution of the problem.
A function $u(x,t)$: G $\to \mathbb{R}^1$ is said to be oscillatory in G, if it has a zero in $\Omega\times(0,\infty)$ for any $t > 0$. otherwise it is nonoscillatory.
This paper is organized as follows, the preliminaries are given in section 2. In section 3, we discuss the forced oscillation problems with boundary conditions that are hinged, sliding and hinged-sliding ends. In section 4, we provide the suitable examples illustrate our main results.

2. PRELIMINARIES

We present the definition of the Riemann-Liouville derivatives and integrals which are given in this section. We have studied a lemma which is uesd in the sequel.

Definition: 2.1. [13] *The Riemann-Liouville fractional partial derivative of order* $0 < \alpha < 1$ *with respect to* t *of a function* $u(x,t)$ *is given by*

$$(D^{\alpha}_{+,t}u)(x,t) := \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\nu)^{-\alpha}u(x,\nu)d\nu$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, *where* Γ *is the gamma function.*

Definition: 2.2. [13] *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $x : \mathbb{R}_+ \to \mathbb{R}$ *on the half-axis* $\mathbb{R}_+$ *is given by*

$$(I^{\alpha}_{+}x)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1}x(\nu)d\nu \quad \textit{for} \quad t > 0$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Lemma 2.1. [16] *Let x be solution of (1.1) and*

$$E(t) := \int_0^t (t-\nu)^{-\alpha}x(\nu)d\nu \quad \textit{for} \quad \alpha \in (0,1) \quad \textit{and} \quad t > 0.$$

Then $E'(t) = \Gamma(1-\alpha)(D^{\alpha}_{+}x)(t)$.

3. MAIN RESULTS

In this section, we study the oscillation of 1.1 with hinged ends, sliding ends and hinged-sliding ends.

Oscillation of extensible beam with hinged. We treat the case, where the ends of the beam are hinged and satisfy the condition

$$u(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0. \tag{3.1}$$

In the following theorem, we reduce the multidimensional problems to one dimensional problem by using Jenson's inequality and integral averaging method.

Theorem 3.1. *Assume that m* $\geq$ *0, there exists a postive function*
ψ*(x)* $\in \mathbb{C}^4(\Omega)$, *such that*
1. $p\psi^4(x)$ - $q\psi''(x) \geq c\psi(x)$ *in* Ω *for constant c* $\geq$ *0,*
2. $\psi''(x) \geq$ *0 in* Ω, *and*
3. $\psi(0)$ = $\psi(L)$ = $\psi''(0)$ = $\psi''(L)$ = *0.*
Each solution of 1.1 satisfying the boundary condition 3.1 is oscillatory in $\Omega \times \mathbb{R}_+$, *if the inequality*

$$\frac{d}{dt}(r(t)D^{\alpha}_{+}U(t)) + cU(t) \leq \pm F(t) \tag{3.2}$$

is oscillatory at t = ∞.

Proof. Assume on the contrary that u is a nonoscillatory in $\Omega \times \mathbb{R}_+$. We consider $u(x,t) > 0$, multiplying 1.1 by $\psi(x)$ and integrating over Ω.

$$\int_{\Omega}\frac{\partial}{\partial t}(r(t)D^{\alpha}_{+,t}u(x,t))\psi(x)dx + p\int_{\Omega}\frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx - q\int_{\Omega}\frac{\partial^2 u(x,t)}{\partial x^2}\psi(x)dx$$

$$- m\int_{\Omega}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\int_{\Omega}\frac{\partial^2 u(x,t)}{\partial x^2}\psi(x)dx + \int_{\Omega} g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)\psi(x)dx$$

$$= \int_{\Omega} f(x,t)\psi(x)dx \tag{3.3}$$

Intgrating by parts and using (3.1),

$$\int_{\Omega}\frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx = \int_0^L \frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx = \int_{\Omega} u(x,t)\psi^4(x)dx, \tag{3.4}$$

$$\int_{\Omega}\frac{\partial^2 u(x,t)}{\partial x^2}\psi(x)dx = \int_0^L \frac{\partial^2 u(x,t)}{\partial x^2}\psi(x)dx = \int_{\Omega} u(x,t)\psi''(x)dx. \tag{3.5}$$

Also, we have Jenson's inequality and Lemma 2.1,

$$\int_{\Omega} g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)\psi(x)dx \geq g(x,t,E(t)) \tag{3.6}$$

Equations 3.4, 3.5 and 3.6 are substituted in Equation 3.3,

$$\frac{\partial}{\partial t}\left(r(t)\left(D^{\alpha}_{+,t}\int_{\Omega} u(x,t)\psi(x)dx\right)\right) + \int_{\Omega}(p\psi^4(x) - q\psi''(x))u(x,t)dx \leq \int_{\Omega} f(x,t)\psi(x)dx.$$

$$\frac{d}{dt}(r(t)D^{\alpha}_{+}U(t)) + cU(t) \leq \int_{\Omega} f(x,t)\psi(x)dx, \tag{3.7}$$

where $U(t) = \int_{\Omega} u(x,t)\psi(x)dx$. ie, $U(t) > 0$ is a solution of 3.2.
The case $u(x,t) < 0$ can be considered by the same method, we get

$$\frac{d}{dt}(r(t)D^{\alpha}_{+}U(t)) + cU(t) \leq -\int_{\Omega} f(x,t)\psi(x)dx. \tag{3.8}$$

From Equations 3.7 and 3.8, we get

$$\frac{d}{dt}(r(t)D^{\alpha}_{+}U(t)) + cU(t) \leq \pm F(t).$$

Hence the proof is complete. ∎

Corollary 3.2. *Assume that m ≥ 0 and $p\psi^4(x) + q\psi''(x) \geq 0$. Each solution of 1.1 satisfying 3.1 is oscillatory, if*

$$\liminf_{t\to\infty}\int_T^t Q(t,s)F(s)ds = -\infty, \tag{3.9}$$

$$\limsup_{t\to\infty}\int_T^t Q(t,s)F(s)ds = \infty \tag{3.10}$$

for all large T.

Proof. We see that $\psi(x) = sin\frac{\pi}{L}x$ satisfies the conditions of Theroem 3.1 with
c = p$(\frac{\pi}{L})^4$ +q$(\frac{\pi}{L})^2$. From 3.7, we have

$$\frac{d}{dt}(r(t)D^{\alpha}_{+}U(t)) \leq F(t), \quad t \geq T. \tag{3.11}$$

Integrating the Equation **??** twice from T to t, we get

$$E(t) \leq E(T) + r(T)E'(T)\int_T^t \frac{1}{r(s)}ds + \frac{1}{\Gamma(1-\alpha)}\int_T^t \frac{1}{r(s)}\left(\int_T^s F(\eta)d\eta\right)ds$$

$$= E(T) + r(T)E'(T)Q(t,T) + \frac{1}{\Gamma(1-\alpha)} \int_T^t Q(t,s)F(s)ds, \quad t \geq T,$$

where E(T) and r(T)$E'(T)$ are constant. We divide the above equations by Q(t,T) and let $t \to \infty$. In view of (3.9 and 3.10) we conclude that

$$\liminf_{t\to\infty} \frac{E(t)}{Q(t,T)} = -\infty,$$

$$\limsup_{t\to\infty} \frac{E(t)}{Q(t,T)} = \infty$$

which leads to a contradiction. ∎

Corollary 3.3. *Assume that m $\geq$ 0 and $p\psi^4(x) + q\psi''(x) \geq$ 0. Every solution of 1.1 satisfying 3.1 is oscillatory, if*

$$\int_t^{t+\frac{\pi}{w}} F(s)sinw(s-t)ds \tag{3.12}$$

is oscillatory in G, where w = ($p\psi^4(x) + q\psi''(x))^{\frac{1}{2}}$.

Proof. Let us to prove that Equation 3.2 is oscillatory. Assume on the contrary that there exists an eventually non negative solution of 3.2.
Multiplying k(t, s) = sin w(t-s) in 3.7, integrating over $(s, s+\frac{\pi}{w})$, we get

$$\int_s^{s+\frac{\pi}{w}} \frac{d}{dt}(r(t)D_+^\alpha U(t))k(t,s)dt + c\int_s^{s+\frac{\pi}{w}} U(t)k(t,s)dt \leq \int_s^{s+\frac{\pi}{w}} F(t)k(t,s)dt \tag{3.13}$$

$$\int_s^{s+\frac{\pi}{w}} \frac{d}{dt}(r(t)D_+^\alpha U(t))k(t,s)dt = \frac{w}{\Gamma(1-\alpha)}\left(r\left(s+\frac{\pi}{w}\right)E\left(s+\frac{\pi}{w}\right) - r(s)E(s)\right) +$$

$$\frac{1}{\Gamma(1-\alpha)}\int_s^{s+\frac{\pi}{w}} E(t)\frac{\partial}{\partial t}(r(t)k_t(t,s))dt \tag{3.14}$$

Equation 3.14 is substituted in 3.13, gives

$$\frac{w}{\Gamma(1-\alpha)}\left(r\left(s+\frac{\pi}{w}\right)E\left(s+\frac{\pi}{w}\right) - r(s)E(s)\right) + \frac{1}{\Gamma(1-\alpha)}\int_s^{s+\frac{\pi}{w}} E(t)\frac{\partial}{\partial t}\left(r(t)k_t(t,s)\right)dt$$
$$+ c\int_s^{s+\frac{\pi}{w}} U(t)k(t,s)dt \leq \int_s^{s+\frac{\pi}{w}} F(t)k(t,s)dt.$$

Therefore,

$$\frac{w}{\Gamma(1-\alpha)}\left(r\left(t+\frac{\pi}{w}\right)E\left(t+\frac{\pi}{w}\right) - r(t)E(t)\right) \leq \int_t^{t+\frac{\pi}{w}} F(s)sinw(s-t)ds. \tag{3.15}$$

The left side of 3.15 is non negative, but the right side of 3.15 oscillates, which is contradicts 3.2, as t tends to infinity. ∎

Corollary 3.4. *Assume that m $\geq$ 0 and $p\psi^4(x) + q\psi''(x) \geq$ 0. Then every solution u(x,t) of the fractional extensible beam equation*

$$\frac{\partial}{\partial t}(r(t)D_{+,t}^\alpha u(x,t)) + p\frac{\partial^4 u(x,t)}{\partial x^4} - \left(q + m\int_\Omega \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}$$
$$+ g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right) = 0$$

is oscillatory in G.

Oscillation of extensible beam with sliding. We deal the case of sliding with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = \frac{\partial^3 u}{\partial x^3}(0,t) = \frac{\partial^3 u}{\partial x^3}(L,t) = 0. \tag{3.16}$$

In the following theorem, we reduce the multidimensional problems to one dimensional problem.

Theorem 3.5. *Every solution of 1.1 satisfying the boundary condition 3.16 is oscillatory in G, if the inequality*

$$\frac{d}{dt}(r(t)D_{+}^{\alpha}U(t)) \leq \pm K(t) \tag{3.17}$$

is oscillatory.

Proof. Assume on the contrary that u is a nonoscillatory in G. We consider $u(x,t) > 0$ and integrating over Ω.

$$\int_\Omega \frac{\partial}{\partial t}(r(t)D_{+,t}^{\alpha}u(x,t))dx + p\int_\Omega \frac{\partial^4 u(x,t)}{\partial x^4}dx - m\int_\Omega \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi \int_\Omega \frac{\partial^2 u(x,t)}{\partial x^2}dx$$

$$-q\int_\Omega \frac{\partial^2 u(x,t)}{\partial x^2}dx + \int_\Omega g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx = \int_\Omega f(x,t)dx. \tag{3.18}$$

We know that,

$$\int_\Omega \frac{\partial^4 u(x,t)}{\partial x^4}dx = 0, \tag{3.19}$$

$$\int_\Omega \frac{\partial^2 u(x,t)}{\partial x^2}dx = 0 \tag{3.20}$$

and

$$\int_\Omega g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx \geq g(x,t,E(t)). \tag{3.21}$$

Equations 3.19, 3.20 and 3.21 are substituted in Equation 3.18, we get

$$\frac{d}{dt}(r(t)D_{+}^{\alpha}U(t)) + g(x,t,E(t)) \leq \int_\Omega f(x,t)dx. \tag{3.22}$$

where $U(t) = \int_\Omega u(x,t)\psi(x)dx$ and $\psi(x) = 1$.

$$\frac{d}{dt}(r(t)D_{+}^{\alpha}U(t)) \leq \int_0^t f(x,t)dx. \tag{3.23}$$

The case $u(x,t) < 0$ can be considered by the same method, we get

$$\frac{d}{dt}(r(t)D_{+}^{\alpha}U(t)) \leq -\int_0^t f(x,t)dx. \tag{3.24}$$

From Equations 3.23 and 3.24, we get

$$\frac{d}{dt}(r(t)D_{+}^{\alpha}U(t)) \leq \pm K(t).$$

Hence the proof is complete. ∎

Theorem 3.6. *Assume the condition*
g(x,t,E(t)) $\geq$ *h(t)J(E(t))* $\forall$ *(x,t,E(t))* $\in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$*, where h(t), J(E(t)) are continuous, positive and J(E(t)) is convex in* $\mathbb{R}_+$*.*
Each solution of 1.1 satisfying 3.16 is oscillatory in G, if the inequality

$$\frac{d}{dt}(r(t)D_+^\alpha U(t)) + h(t)J(E(t)) \leq \pm K(t) \tag{3.25}$$

is oscillatory at $t = \infty$*.*

Proof. By the Theroem 3.5, we are using the Equation 3.22

$$\frac{d}{dt}(r(t)D_+^\alpha U(t)) + g(x,t,E(t)) \leq \int_\Omega f(x,t)dx.$$

Using the condition, we get

$$\frac{d}{dt}(r(t)D_+^\alpha U(t)) + h(t)J(E(t)) \leq \pm K(t).$$

Hence the proof is complete. ∎

Corollary 3.7. *Every solution of 1.1 satisfying 3.16 is oscillatory in G, if*

$$\liminf_{t\to\infty} \int_T^t Q(t,s)K(s)ds = -\infty, \tag{3.26}$$

$$\limsup_{t\to\infty} \int_T^t Q(t,s)K(s)ds = \infty \tag{3.27}$$

for all large T.

Corollary 3.8. *Assume that* $g(x,t,E(t)) = c_0E(t)$ *(*c_0 *is a positive constant). Each solution of 1.1 satisfying 3.16 is oscillatory, if*

$$\int_t^{t+\frac{\pi}{\widetilde{w}}} K(s)sin\widetilde{w}(s-t)ds \tag{3.28}$$

is oscillatory in G, where $\widetilde{w} = (c_0)^{\frac{1}{2}}$*.*

Oscillation of extensible beam with hinged-sliding ends. We deals with the case of hinged-slinding ends

$$u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial u}{\partial x}(L,t) = \frac{\partial^3 u}{\partial x^3}(L,t) = 0. \tag{3.29}$$

Let $\Phi(x)$ is a non negative function, $\Phi(x) \in C^4(\Omega)$ which satisfies the boundary conditions

$$\Phi(0) = \Phi''(0) = \Phi'(L) = \Phi'''(L) = 0.$$

Then $\Phi(x) = sin\frac{\pi}{2L}x$. Hence, $sin\frac{\pi}{L}x$ is replaced by $sin\frac{\pi}{2L}x$ in Theorem 3.1.

Theorem 3.9. *Assume that q, m = 0, and* $\psi^4(x) \geq \epsilon\psi(x)$ *in* Ω *for some* $\epsilon \geq 0$*. Then there exists a solution of 1.1 satisfying the boundary condition 3.29 which is oscillatory in G, if the inequality*

$$\frac{d}{dt}(r(t)D_+^\alpha U(t)) + p\epsilon U(t) \leq \pm F(t) \tag{3.30}$$

is oscillatory.

Proof. Assume on the contrary that u is nonoscillatory in G. We consider $u(x,t) > 0$, multiplying 1.1 by $\psi(x)$ and integrating over Ω, we get

$$\int_\Omega \frac{\partial}{\partial t}(r(t)D^\alpha_{+,t}u(x,t))\psi(x)dx + p\int_\Omega \frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx$$

$$+\int_\Omega g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)\psi(x)dx = \int_\Omega f(x,t)\psi(x)dx \tag{3.31}$$

Taking into account that,

$$\int_\Omega g\left(x,t,\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)\psi(x)dx \geq g(x,t,E(t)), \tag{3.32}$$

Equation 3.32 are substituted in Equation 3.31, we get

$$\frac{d}{dt}\left(r(t)\left(D^\alpha_{+,t}\int_\Omega u(x,t)\psi(x)dx\right)\right) + p\epsilon\int_\Omega u(x,t)\psi(x)dx \leq \int_\Omega f(x,t)\psi(x)dx$$

$$\frac{d}{dt}(r(t)D^\alpha_+U(t)) + p\epsilon U(t) \leq \int_0^t f(x,t)\psi(x)dx. \tag{3.33}$$

Next, we consider $u(x,t) < 0$, similarly, we obtain

$$\frac{d}{dt}(r(t)D^\alpha_+U(t)) + p\epsilon U(t) \leq -\int_0^t f(x,t)\psi(x)dx. \tag{3.34}$$

From Equations 3.33 and 3.34, we have

$$\frac{d}{dt}(r(t)D^\alpha_+U(t)) + p\epsilon U(t) \leq \pm F(t).$$

Hence the proof is complete. ∎

Corollary 3.10. *Each solution of (1.1) satisfying (HSE) is oscillatory in G, if*

$$\liminf_{t\to\infty}\int_T^t Q(t,s)F(s)ds = -\infty, \tag{3.35}$$

$$\limsup_{t\to\infty}\int_T^t Q(t,s)F(s)ds = \infty \tag{3.36}$$

for all large T.

4. EXAMPLE

We provide a few examples to illustrate our results established in Section 3.

Example: 4.1. *We consider the fractional extensible beam equations*

$$\frac{\partial}{\partial t}(D^{\frac{1}{2}}_{+,t}u(x,t)) + \left(\frac{L}{\pi}\right)^4\frac{\partial^4 u(x,t)}{\partial x^4} - \left(\left(\frac{L}{\pi}\right)^2 + \left(\frac{L}{\pi}\right)^4\int_\Omega\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$+g\left(x,t,\int_0^t (t-s)^{-\frac{1}{2}}u(x,t)ds\right) = f(x,t), \qquad (x,t)\in G, \tag{4.1}$$

with boundary conditions 3.1

$$u(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0.$$

Here, $r(t) = 1$, $p(t) = \left(\frac{L}{\pi}\right)^4$, $q(t) = \left(\frac{L}{\pi}\right)^2$, $m(t) = \left(\frac{L}{\pi}\right)^4$, $g(x,t,E(t)) = E(t)$ *and*
$f(x,t) = \left(S_t(-\frac{3}{2},1) + \sqrt{\pi}S_t(\frac{1}{2},1) + (2 + \frac{L}{2}sin^2t)sint\right) sin\frac{\pi}{L}x$.
Therefore,

$$\int_T^t Q(t,s)F(s)ds$$
$$= \int_T^t Q(t,s)\left(\int_0^L \left(S_s(-\frac{3}{2},1) + \sqrt{\pi}S_s(\frac{1}{2},1) + (3 + \frac{L}{2}sin^2s)sins\right) sin\frac{\pi}{L}xsin\frac{\pi}{L}x\right) ds$$
$$= \frac{L}{2}\int_T^t Q(t,s)\left(S_s(-\frac{3}{2},1) + \sqrt{\pi}S_s(\frac{1}{2},1) + (3 + \frac{L}{2}sin^2s)sins\right) ds = \infty.$$

Thus all the conditions of Corollary 3.2 are satisfied. Hence, every solution of 4.1 and 3.1 is oscillatory in G. In fact, $u(x,t) = sintsin\frac{\pi}{L}x$ *is one such solution.*

Example: 4.2. *We consider the fractional extensible beam equations*

$$\frac{\partial}{\partial t}(D^{\frac{1}{2}}_{+,t}u(x,t)) + \left(\frac{L}{\pi}\right)^4 \frac{\partial^4 u(x,t)}{\partial x^4} - \left(-\left(\frac{2L}{\pi}\right)^2 + \left(\frac{2L^3}{\pi^4}\right)\int_\Omega \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}$$
$$+ g\left(x,t,\int_0^t (t-s)^{-\frac{1}{2}}u(x,t)ds\right) = f(x,t), \qquad (x,t) \in G, \tag{4.2}$$

with boundary conditions 3.1

$$u(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0.$$

Here, $r(t) = 1$, $p(t) = \left(\frac{L}{\pi}\right)^4$, $q(t) = -\left(\frac{2L}{\pi}\right)^2$, $m(t) = \left(\frac{2L^3}{\pi^4}\right)$, $g(x,t,E(t)) = E(t)$ *and*
$f(x,t) = \left(E_t(-\frac{3}{2},1) + \sqrt{\pi}E_t(\frac{1}{2},1) + (1 + e^{-2t})e^{-t}\right) sin\frac{\pi}{L}x$.
Therefore,

$$\int_T^t Q(t,s)F(s)ds$$
$$= \int_T^t Q(t,s)\left(\int_0^L \left(E_s(-\frac{3}{2},1) + \sqrt{\pi}E_s(\frac{1}{2},1) + (1 + e^{-2s})e^{-s}\right) sin\frac{\pi}{L}xsin\frac{\pi}{L}x\right) ds$$
$$\leq \frac{L}{2}\int_0^\infty Q(t,s)(3 + e^{-2s})e^{-s}ds < \infty.$$

The conditions of Corollary 3.2 are not satisfied. Infact, $u(x,t) = e^{-t}sin\frac{\pi}{L}x$ *is a nonoscillatory solution of 4.2.*

Example: 4.3. *We consider the fractional extensible beam equations*

$$\frac{\partial}{\partial t}(r(t)D^{\alpha}_{+,t}u(x,t)) + p\frac{\partial^4 u(x,t)}{\partial x^4} - \left(q + m\int_\Omega \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad (x,t) \in G, \tag{4.3}$$

with boundary conditions 3.1

$$u(0,t) = u(L,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0.$$

where $m \geq 0$ *and* $p\psi^4(x) + q\psi''(x) \geq 0$. *Corollary (3.4) implies that each solution of 4.3 satisfying 3.1 is oscillatory.*

An oscillatory solution of u(x, t) = $(sin(\frac{\pi}{L}x)V(t))$ *and r(t) = 1, here V(t) is a solution of the fractional Duffing's equation*

$$\frac{d}{dt}(D_+^\alpha(V(t))) + \left(p\left(\frac{\pi}{L}\right)^4 + q\left(\frac{\pi}{L}\right)^2\right)V(t) + \frac{mL}{2}\left(\frac{\pi}{L}\right)^4 V^3(t) = 0.$$

5. CONCLUSION

We have mainly focussed on deriving some new sufficient conditions for the forced oscillation of self adjoint fractional extensible beam equations with some boundary conditions. The results are essentially new and complements the previous existing literature in the classical case. We have also presented a few examples to illustrate our new results.

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