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# OPTIMAL CONDITIONS USING MULTI-VALUED $\mathbb{G}$-PREŠIĆ TYPE MAPPING

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ABSTRACT. In the present paper, some best proximity results have been presented using the concept of $\mathbb{G}$-Prešić type multi-valued mapping. These results are the extensions of Prešić's theorem in the non-self mapping. A suitable example has also been given. Here, some applications are presented in $\theta$-chainable space and ordered metric space.

*Key words and phrases:* Best proximity point; Multi-valued mappping; Fixed point.

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## 1. INTRODUCTION

In the year 1922, S. Banach[2] introduced the fixed point theory. This theory plays a significant role in non-linear analysis. Banach presented his famous Banach Contraction Principle by which he threw the light on the concept of fixed point. Afterthat several other mathematicians [7], [8], [4] extended and presented their ideas about this concept. In 1965, Prešić [10], [11] generalised the Banach's idea into product spaces and presented some results on fixed point. He proved the following:

**Theorem 1.1** ([10]). *Assume that* $(\mathbb{Y}, \Im)$ *is a complete metric space and* $\mathfrak{k} \geq 1$ *such that* $\mathfrak{k} \in \mathbf{N}$. *Suppose,* $\mathbb{F} : \mathbb{Y}^{\mathfrak{k}} \to \mathbb{Y}$ *be a mapping satisfying the following condition:*

$$\Im(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) < \sum_{i=1}^{\mathfrak{k}} \gamma_i \Im(\mathbf{u}_i, \mathbf{u}_{i+1})$$

*for each* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}, \mathbf{u}_{\mathfrak{k}+1} \in \mathbb{Y}$, *where* $\gamma_1, \gamma_2, \cdots, \gamma_{\mathfrak{k}}$ *are non-negative constants such that* $\sum_{i=1}^{\mathfrak{k}} \gamma_i < 1$. *Then, there exists a unique fixed point in* $\mathbb{Y}$. *Again, if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}$ *are some points in* $\mathbb{Y}$ *and for* $n \in \mathbf{N}$, $\mathbf{u}_{n+\mathfrak{k}} = \mathbb{F}(\mathbf{u}_n, \mathbf{u}_{n+1}, \cdots, \mathbf{u}_{n+\mathfrak{k}-1})$. *Then,* $\{\mathbf{u}_n\}$ *converges to the fixed point of* $\mathbb{F}$.

The work of Prešić can further be extended by several famous mathematicians [13], [14], [3], [6], [12] in different ways and different generalised spaces. In 1969, Nadlar [9] extended the concept of Banach's principle into multi-valued mapping. He used the Pompeiu-Hausdorff metric to present his result.
Suppose, $\mathcal{C}$ be a non-empty subset of a metric space $(\mathbb{Y}, \Im)$. Now, for $\mathfrak{p} \in \mathbb{Y}$,

$$\Im(\mathfrak{p}, \mathcal{C}) = inf\{\Im(\mathfrak{p}, \mathfrak{g}) : \mathfrak{g} \in \mathcal{C}\}$$

Assume that $CB(\mathbb{Y})$ be the set of all non-empty closed and bounded subsets of $\mathbb{Y}$. Now, for $\mathcal{C}, \mathcal{D} \in CB(\mathbb{Y})$,

$$\delta(\mathcal{C}, \mathcal{D}) = sup\{\Im(\mathfrak{p}, \mathcal{D}) : \mathfrak{p} \in \mathcal{C}\}$$
$$H(\mathcal{C}, \mathcal{D}) = max\{\delta(\mathcal{C}, \mathcal{D}), \delta(\mathcal{D}, \mathcal{C})\}$$

The metric $H$ is called Pompeiu-Hausdorff metric. Nadlar stated the following:

**Theorem 1.2** ([9]). *Suppose,* $(\mathbb{Y}, \Im)$ *be a complete metric space and there is a mapping* $\mathbb{F} : \mathbb{Y} \to CB(\mathbb{Y})$ *such that for all* $\mathfrak{p}, \mathfrak{g} \in \mathbb{Y}$,

$$H(\mathbb{F}(\mathfrak{p}), \mathbb{F}(\mathfrak{g})) \leq \varrho \Im(\mathfrak{p}, \mathfrak{g})$$

*where,* $\varrho \in [0, 1)$. *Then,* $\mathbb{F}$ *has a fixed point in* $\mathbb{Y}$.

In the year 2006, Eldred et al. [5] first revealed the concept of best proximity point. In 2019, Usman Ali et al. [1] presented their ideas on the Prešić-type single valued non-self mapping.
In the present paper, two best proximity results are shown using Pompeiu-Hausdorff metric where Prešić-type multivalued non-self mapping has been taken. Here, a suitable example has also been given in support of the theorem. Also, some consequences and application parts are given in $\theta$-chainable space and ordered metric space.

## 2. PRELIMINARIES

Suppose, $(\mathbb{Y}, \Im)$ be a metric space. Here, we consider a graph $\mathbb{G}$ such that $\mathbb{V}(\mathbb{G}) = \mathbb{Y}$ and $\mathbb{E}(\mathbb{G})$ be the set of all edges containing all loops. Here, we assume that $\mathbb{G}$ has no parallel edges. We can denote $\mathbb{G}$ as $(\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$.

Suppose, $\mathbb{C}$ and $\mathbb{D}$ are two non-empty subsets of a metric space $(\mathbb{Y}, \Im)$ and $\Delta$ denote the diagonal of the cartesian product $\mathbb{Y} \times \mathbb{Y}$. Here, we use the following notations:

$$\Im(\mathbb{C}, \mathbb{D}) = \{inf\ \Im(\mathbf{u}, \mathbf{v}) : \mathbf{u} \in \mathbb{C}, \mathbf{v} \in \mathbb{D}\}$$
$$\mathbb{C}_0 = \{\mathbf{u} \in \mathbb{C} : \Im(\mathbf{u}, \mathbf{v}) = \Im(\mathbb{C}, \mathbb{D}) \text{ for some } \mathbf{v} \in \mathbb{D}\}$$
$$\mathbb{D}_0 = \{\mathbf{v} \in \mathbb{D} : \Im(\mathbf{u}, \mathbf{v}) = \Im(\mathbb{C}, \mathbb{D}) \text{ for some } \mathbf{u} \in \mathbb{C}\}$$

Here, we give the following definition which is useful to our theorems.

**Definition 2.1. (Best Proximity Point)**: Suppose, $(\mathbb{Y}, \Im)$ be a metric space and $\mathbb{C}, \mathbb{D}$ be two non-empty subsets of $\mathbb{Y}$. An element $\mathbf{u}_* \in \mathbb{C}$ is said to be a best proximity point of the mapping $\mathbb{F} : \mathbb{C} \longrightarrow \mathbb{D}$ if $\Im(\mathbf{u}_*, \mathbb{F}(\mathbf{u}_*)) = \Im(\mathbb{C}, \mathbb{D})$.

**Definition 2.2. (P-Property)**: Let $(\mathbb{C}, \mathbb{D})$ be a pair of non-empty subsets of a metric space $(\mathbb{Y}, \Im)$ such that $\mathbb{C}_0$ is non-empty. Then, the pair $(\mathbb{C}, \mathbb{D})$ is said to have $P$-property iff $\Im(\mathbf{u}_1, \mathbf{v}_1) = \Im(\mathbf{u}_2, \mathbf{v}_2) = \Im(\mathbb{C}, \mathbb{D})$ implies that $\Im(\mathbf{u}_1, \mathbf{u}_2) = \Im(\mathbf{v}_1, \mathbf{v}_2)$ where $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{D}$.

## 3. MAIN RESULTS

**Definition 3.1.** Let, $\Xi, \Upsilon$ be the family of all functions $\varphi, \varpi : [0, \infty) \to [0, \infty)$ such that
i) $\varphi, \varpi$ are increasing.
ii) Both must attain continuity.
iii) $\varphi(0) = 0, \quad \varphi(\mathfrak{t}) < \mathfrak{t}$ for each $\mathfrak{t} \in [0, \infty)$.

**Definition 3.2.** Suppose, $\mathbb{C}$ and $\mathbb{D}$ are two non-empty closed subsets of a metric space $(\mathbb{Y}, \Im)$ which is complete such that $\mathbb{C}_0 \neq \emptyset$ and $\mathfrak{k} \geq 1$ such that $\mathfrak{k} \in \mathbf{N}$. Let, $\mathbb{F} : \mathbb{C}^{\mathfrak{k}} \to CB(\mathbb{D})$ be a mapping. Assume that for every path $\{\mathbf{u}_i\}_{i=1}^{\mathfrak{k}+1}$ of $\mathfrak{k} + 1$ vertices in $\mathbb{G}$, the following conditions are satisfied:
i) There exist non-negative constants $\gamma_i$s such that $\sum_{i=1}^{\mathfrak{k}} \gamma_i < 1$ and

$$\begin{aligned} H(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) \leq & \sum_{i=1}^{\mathfrak{k}} (\gamma_i \varphi(\Im(\mathbf{u}_i, \mathbf{u}_{i+1}))) \\ & - \varpi\big(max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}\big) \end{aligned}$$

ii) If $\mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})$ and $\mathbb{F}(\mathbf{u}_3, \mathbf{u}_4, \cdots, \mathbf{u}_{\mathfrak{k}+2}) \subseteq \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})$ are such that $\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) < max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}$, then $(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) \in E(\mathbb{G})$.

**Theorem 3.1.** *Let us assume that $\mathbb{C}$ and $\mathbb{D}$ are two non-empty closed subsets of a complete metric space $(\mathbb{Y}, \Im)$ such that $\mathbb{C}_0 \neq \emptyset$ and $\mathfrak{k} \geq 1$ such that $\mathfrak{k} \in \mathbf{N}$. Let, $\mathbb{F} : \mathbb{C}^{\mathfrak{k}} \to CB(\mathbb{D})$ be a mapping satisfying the above two conditions of the Definition(3.2). Suppose that the following assertions hold:*
*i) There exists a path $\{\mathbf{u}_i\}_{i=1}^{\mathfrak{k}+1}$ of $\mathfrak{k}+1$ vertices in $\mathbb{G}$ such that $\mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})$.*
*ii) $\mathbb{F}(\mathbb{C}_0^{\mathfrak{k}}) \subseteq \mathbb{D}_0$ and the pair $(\mathbb{C}, \mathbb{D})$ satisfies the property such that*

$$\begin{aligned} &\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D}) = \Im(\mathbf{u}_{\mathfrak{k}+2}, \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) \\ &\Rightarrow \Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) \leq H(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) \end{aligned}$$

*iii) There exist $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}) \in \mathbb{C}_0^{\mathfrak{k}}$ and $\mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}_0$ such that*

$$\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D})$$

*iv) $\mathbb{F}$ is continuous.*
*Then, $\mathbb{F}$ has a best proximity point in $\mathbb{C}^{\mathfrak{k}}$.*

*Proof.* From condition (iii), there exist $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}) \in \mathbb{C}_0^\mathfrak{k}$ and $\mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}_0$ such that

$$\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k})) = dist(\mathbb{C}, \mathbb{D}) \tag{3.1}$$

Since, $\mathbb{F}(\mathbb{C}_0^\mathfrak{k}) \subseteq \mathbb{D}_0$, there exist $(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \in \mathbb{C}_0^\mathfrak{k}$ and $\mathbf{u}_{\mathfrak{k}+2} \in \mathbb{C}_0$ such that

$$\Im(\mathbf{u}_{\mathfrak{k}+2}, \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) = dist(\mathbb{C}, \mathbb{D}) \tag{3.2}$$

Thus, continuing in this way, by mathematical induction, we get,

$$\Im(\mathbf{u}_{n+\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D}) \tag{3.3}$$

Again, since the pair $(\mathbb{C}, \mathbb{D})$ satisfies condition (ii), then we can write from equations $(3.1)$ and $(3.2)$,

$$\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) \leq H(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}))$$

Let, $\gamma = \sum_{i=1}^{\mathfrak{k}} \gamma_i < 1$. Suppose that there is a path $\{\mathbf{u}_i\}_{i=1}^{\mathfrak{k}+1}$ of $\mathfrak{k}+1$ vertices in $\mathbb{G}$ such that $\mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k})$.
Since, $\mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \in CB(\mathbb{D})$, there exists $\mathbb{F}(\mathbf{u}_3, \mathbf{u}_4, \cdots, \mathbf{u}_{\mathfrak{k}+2}) \subseteq \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})$ such that

$$\begin{aligned}
\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) &\leq H(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) \\
&\leq \sum_{i=1}^{\mathfrak{k}} (\gamma_i \varphi(\Im(\mathbf{u}_i, \mathbf{u}_{i+1}))) - \varpi\big(max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}\big) \\
&\leq \sum_{i=1}^{\mathfrak{k}} (\gamma_i \varphi(\Im(\mathbf{u}_i, \mathbf{u}_{i+1}))) \\
&< \sum_{i=1}^{\mathfrak{k}} (\gamma_i \Im(\mathbf{u}_i, \mathbf{u}_{i+1})) \\
&\leq \gamma\, max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\} \\
&< max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}
\end{aligned}$$

Hence, $(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) \in E(\mathbb{G})$
Similarly, as $\mathbb{F}(\mathbf{u}_3, \mathbf{u}_4, \cdots, \mathbf{u}_{\mathfrak{k}+2}) \in CB(\mathbb{D})$, there exists $\mathbb{F}(\mathbf{u}_4, \mathbf{u}_5, \cdots, \mathbf{u}_{\mathfrak{k}+3}) \subseteq \mathbb{F}(\mathbf{u}_3, \mathbf{u}_4, \cdots, \mathbf{u}_{\mathfrak{k}+2})$ such that

$$\begin{aligned}
\Im(\mathbf{u}_{\mathfrak{k}+2}, \mathbf{u}_{\mathfrak{k}+3}) &< \gamma\, max\{\Im(\mathbf{u}_{i+1}, \mathbf{u}_{i+2}) : i = 1, 2, \cdots, \mathfrak{k}\} \\
&< max\{\Im(\mathbf{u}_{i+1}, \mathbf{u}_{i+2}) : i = 1, 2, \cdots, \mathfrak{k}\}
\end{aligned}$$

So, $(\mathbf{u}_{\mathfrak{k}+2}, \mathbf{u}_{\mathfrak{k}+3}) \in E(\mathbb{G})$
Proceeding this way, as $\mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}}) \in CB(\mathbb{D})$, there exists $\mathbb{F}(\mathbf{u}_{n+2}, \mathbf{u}_{n+3}, \cdots, \mathbf{u}_{n+\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}})$ such that

$$\begin{aligned}
\Im(\mathbf{u}_{n+\mathfrak{k}}, \mathbf{u}_{n+\mathfrak{k}+1}) &< \gamma\, max\{\Im(\mathbf{u}_{i+n-1}, \mathbf{u}_{i+n}) : i = 1, 2, \cdots, \mathfrak{k}\} \\
&< max\{\Im(\mathbf{u}_{i+n-1}, \mathbf{u}_{i+n}) : i = 1, 2, \cdots, \mathfrak{k}\}
\end{aligned}$$

Hence, $(\mathbf{u}_{n+\mathfrak{k}}, \mathbf{u}_{n+\mathfrak{k}+1}) \in E(\mathbb{G})$ for all $n \in \mathbf{N}$
Now, we will prove that $\{\mathbf{u}_n\}$ is a Cauchy sequence.
Let,

$$\eta = max\Big\{\frac{\Im(\mathbf{u}_i, \mathbf{u}_{i+1})}{\zeta^i} : i = 1, 2, \cdots, \mathfrak{k}\Big\}$$

where, $\zeta = \gamma^{\frac{1}{\mathfrak{k}}}$
Now, by mathematical induction we have to prove that

$$\Im(\mathbf{u}_n, \mathbf{u}_{n+1}) \leq \eta\zeta^n \quad \forall n \in \mathbf{N} \tag{3.4}$$

Let, the $\mathfrak{k}$ inequalities be $\Im(\mathbf{u}_n, \mathbf{u}_{n+1}) \leq \eta\zeta^n, \Im(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) \leq \eta\zeta^{n+1}, \cdots, \Im(\mathbf{u}_{n+\mathfrak{k}-1}, \mathbf{u}_{n+\mathfrak{k}}) \leq \eta\zeta^{n+\mathfrak{k}-1}$
Now,

$$\begin{aligned}\Im(\mathbf{u}_{n+\mathfrak{k}}, \mathbf{u}_{n+\mathfrak{k}+1}) &< \gamma\, max\{\Im(\mathbf{u}_{i+n-1}, \mathbf{u}_{i+n}) : i = 1, 2, \cdots, \mathfrak{k}\}\\ &\leq \gamma\, max\{\eta\zeta^{i+n-1} : i = 1, 2, \cdots, \mathfrak{k}\}\\ &\leq \gamma\, \eta\zeta^n \quad [As \quad \zeta = \gamma^{1/\mathfrak{k}} < 1]\\ &= \eta\zeta^{n+\mathfrak{k}}\end{aligned}$$

Thus, the proof of $(3.4)$ is complete.
Now, for $m, n \in \mathbf{N}$ and $m > n$, using $(3.4)$ we get,

$$\begin{aligned}\Im(\mathbf{u}_n, \mathbf{u}_m) &\leq \Im(\mathbf{u}_n, \mathbf{u}_{n+1}) + \Im(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) + \cdots + \Im(\mathbf{u}_{m-1}, \mathbf{u}_m)\\ &< \eta\zeta^n + \eta\zeta^{n+1} + \cdots + \eta\zeta^{m-1}\end{aligned}$$

Since, $\zeta = \gamma^{1/\mathfrak{k}} < 1$, we conclude from the above inequality,

$$\lim_{m,n\to\infty} \Im(\mathbf{u}_n, \mathbf{u}_m) = 0$$

Hence, $\{\mathbf{u}_n\}$ is a Cauchy sequence.
Since, $(\mathbb{Y}, \Im)$ is complete and $\mathbb{C}$ is closed, so the sequence $\{\mathbf{u}_n\}$ converges to a point $\mathbf{u}_* \in \mathbb{C}$.
As, $\mathbb{F}$ is continuous,

$$\mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}}) \to \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*) \quad as \quad n \to \infty$$

The continuity of the metric implies that

$$dist(\mathbb{C}, \mathbb{D}) = \Im(\mathbf{u}_{n+\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}})) \to \Im(\mathbf{u}_*, \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*))$$

Hence,

$$\Im(\mathbf{u}_*, \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*)) = dist(\mathbb{C}, \mathbb{D})$$

Therefore, $\mathbb{F}$ has a best proximity point in $\mathbb{C}^{\mathfrak{k}}$. ∎

**Theorem 3.2.** *Supppse, $\mathbb{C}$ and $\mathbb{D}$ are two non-empty closed subsets of a complete metric space $(\mathbb{Y}, \Im)$ such that $\mathbb{C}_0 \neq \emptyset$ and $\mathfrak{k} \geq 1$ such that $\mathfrak{k} \in \mathbf{N}$. Let, $\mathbb{F} : \mathbb{C}^{\mathfrak{k}} \to CB(\mathbb{D})$ be a mapping satisfying the above two conditions of the Definition(3.2). Suppose that the following assertions hold:*
*i) There exists a path $\{\mathbf{u}_i\}_{i=1}^{\mathfrak{k}+1}$ of $\mathfrak{k}+1$ vertices in $\mathbb{G}$ such that $\mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})$.*
*ii) $\mathbb{F}(\mathbb{C}_0^{\mathfrak{k}}) \subseteq \mathbb{D}_0$ and the pair $(\mathbb{C}, \mathbb{D})$ satisfies the property such that*

$$\begin{aligned}&\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D}) = \Im(\mathbf{u}_{\mathfrak{k}+2}, \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}))\\ &\Rightarrow \Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) \leq H(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}))\end{aligned}$$

*iii) There exist $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}) \in \mathbb{C}_0^{\mathfrak{k}}$ and $\mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}_0$ such that*

$$\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D})$$

*iv) For any termwise connected sequence $\{\mathbf{u}_n\} \in \mathbb{C}$ if $\mathbf{u}_n \to \mathbf{u}_*$ and $\mathbb{F}(\mathbf{u}_{n+2}, \mathbf{u}_{n+3}, \cdots, \mathbf{u}_{n+\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}})$ for all $n \in \mathbf{N}$, then there exists a subsequence $\{\mathbf{u}_{n(r)}\}$ such that $(\mathbf{u}_{n(r)}, \mathbf{u}_*) \in E(\mathbb{G})$ for all $r \in \mathbf{N}$.*
*Then, $\mathbb{F}$ has a best proximity point in $\mathbb{C}^{\mathfrak{k}}$.*

*Proof.* From the proof of Theorem$(3.1)$, there exists a Cauchy sequence $\{\mathbf{u}_n\} \in \mathbb{C}$ such that

$$\Im(\mathbf{u}_{n+\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D}) \quad \forall \quad n \in \mathbf{N}$$

and $\mathbf{u}_n \to \mathbf{u}_*$ as $n \to \infty$ with $\mathbf{u}_* \in \mathbb{C}$.
From the condition (iv), there exists a subsequence $\{\mathbf{u}_{n(r)}\}$ of $\{\mathbf{u}_n\}$ such that $(\mathbf{u}_{n(r)}, \mathbf{u}_*) \in E(\mathbb{G})$. Since, for each $n \in \mathbf{N}$, we have $(\mathbf{u}_n, \mathbf{u}_{n+1}) \in E(\mathbb{G})$ and $\mathbb{F}(\mathbf{u}_{n+2}, \mathbf{u}_{n+3}, \cdots, \mathbf{u}_{n+\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+\mathfrak{k}})$, so for any $r \in \mathbf{N}$, we obtain,

$$\begin{aligned}
&\Im(\mathbf{u}_*, \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*))\\
&\leq \Im(\mathbf{u}_*, \mathbf{u}_{n(r)+\mathfrak{k}+1}) + \Im(\mathbf{u}_{n(r)+\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_{n(r)+1}, \mathbf{u}_{n(r)+2}, \cdots, \mathbf{u}_{n(r)+\mathfrak{k}}))\\
&+ H(\mathbb{F}(\mathbf{u}_{n(r)+1}, \mathbf{u}_{n(r)+2}, \cdots, \mathbf{u}_{n(r)+\mathfrak{k}}), \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*))\\
&= \Im(\mathbf{u}_*, \mathbf{u}_{n(r)+\mathfrak{k}+1}) + dist(\mathbb{C}, \mathbb{D})\\
&+ H(\mathbb{F}(\mathbf{u}_{n(r)+1}, \mathbf{u}_{n(r)+2}, \cdots, \mathbf{u}_{n(r)+\mathfrak{k}}), \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*))\\
&\leq \Im(\mathbf{u}_*, \mathbf{u}_{n(r)+\mathfrak{k}+1}) + dist(\mathbb{C}, \mathbb{D})\\
&+ H(\mathbb{F}(\mathbf{u}_{n(r)+1}, \mathbf{u}_{n(r)+2}, \cdots, \mathbf{u}_{n(r)+\mathfrak{k}}), \mathbb{F}(\mathbf{u}_{n(r)+2}, \mathbf{u}_{n(r)+3}, \cdots, \mathbf{u}_{n(r)+\mathfrak{k}}, \mathbf{u}_*))\\
&+ H(\mathbb{F}(\mathbf{u}_{n(r)+2}, \mathbf{u}_{n(r)+3}, \cdots, \mathbf{u}_{n(r)+\mathfrak{k}}, \mathbf{u}_*), \mathbb{F}(\mathbf{u}_{n(r)+3}, \mathbf{u}_{n(r)+4}, \cdots, \mathbf{u}_{n(r)+\mathfrak{k}}, \mathbf{u}_*, \mathbf{u}_*))\\
&+ \cdots + H(\mathbb{F}(\mathbf{u}_{n(r)+\mathfrak{k}}, \mathbf{u}_*, \cdots, \mathbf{u}_*), \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*))\\
&< \Im(\mathbf{u}_*, \mathbf{u}_{n(r)+\mathfrak{k}+1}) + dist(\mathbb{C}, \mathbb{D})\\
&+ \{\gamma_1 \Im(\mathbf{u}_{n(r)+1}, \mathbf{u}_{n(r)+2}) + \gamma_2 \Im(\mathbf{u}_{n(r)+2}, \mathbf{u}_{n(r)+3}) + \cdots + \gamma_{\mathfrak{k}} \Im(\mathbf{u}_{n(r)+\mathfrak{k}}, \mathbf{u}_*)\}\\
&+ \{\gamma_1 \Im(\mathbf{u}_{n(r)+2}, \mathbf{u}_{n(r)+3}) + \gamma_2 \Im(\mathbf{u}_{n(r)+3}, \mathbf{u}_{n(r)+4}) + \cdots + \gamma_{\mathfrak{k}-1} \Im(\mathbf{u}_{n(r)+\mathfrak{k}}, \mathbf{u}_*)\}\\
&+ \cdots + \gamma_1 \Im(\mathbf{u}_{n(r)+\mathfrak{k}}, \mathbf{u}_*)
\end{aligned}$$

Letting $r \to \infty$ in the above inequality, we get,

$$\Im(\mathbf{u}_*, \mathbb{F}(\mathbf{u}_*, \mathbf{u}_*, \cdots, \mathbf{u}_*)) = dist(\mathbb{C}, \mathbb{D})$$

Therefore, $\mathbb{F}$ has a best proximity point i.e. $\mathbf{u}_* \in \mathbb{C}^{\mathfrak{k}}$. ∎

## 4. Illustration

**Example 4.1.** *Let,* $\mathbb{Y} = \mathbf{R}$ *be a metric space endowed with the metric* $\Im(\mathbf{u}, \mathbf{v}) = |\mathbf{u} - \mathbf{v}|$ *for all* $\mathbf{u}, \mathbf{v} \in \mathbb{C}$*. Let,* $\mathbb{C} = [-1, -\frac{1}{2}]$ *and* $\mathbb{D} = [0, 1]$*. Now, we define a graph* $V(\mathbb{G}) = \mathbb{Y}$*,* $E(\mathbb{G}) = \Delta \cup \{(-1, -\frac{\mathbf{n}+1}{\mathbf{n}+2}), (-\frac{\mathbf{n}+1}{\mathbf{n}+2}, -\frac{\mathbf{n}}{\mathbf{n}+1}) : \mathbf{n} \in \mathbf{N}\}$*. Then,* $(\mathbb{Y}, \Im)$ *is a complete metric space. We define a mapping,* $\mathbb{F} : \mathbb{C} \times \mathbb{C} \to CB(\mathbb{D})$ *such that*

$$\mathbb{F}(\mathbf{a}, \mathbf{b}) = \begin{cases} \{0\} & \mathbf{a} = \mathbf{b} \in \mathbb{C} \\ \left[0, \frac{1}{\mathbf{n}+3}\right] & \mathbf{a} = -\frac{\mathbf{n}}{\mathbf{n}+1}, \mathbf{b} = -\frac{\mathbf{n}+1}{\mathbf{n}+2} \quad \mathbf{n} \in \mathbf{N} \\ \{1\} & otherwise \end{cases}$$

*Then,* $\mathbb{F}$ *satisfies the weak inequality used in Theorem*$(3.1)$ *with* $\gamma_1 = \frac{1}{3}$, $\gamma_2 = \frac{79}{120}$ *and* $\varphi(\mathfrak{t}) = \frac{99\mathfrak{t}}{100}$, $\varpi(\mathfrak{t}) = \frac{\mathfrak{t}}{1000}$ *for all* $\mathfrak{t} \in [0, \infty)$.
$\therefore$ *All the conditions of Theorem*$(3.1)$ *are satisfied and*

$$\Im(-\frac{1}{2}, \mathbb{F}(-\frac{1}{2}, -\frac{1}{2})) = dist(\mathbb{C}, \mathbb{D}) = \frac{1}{2}$$

*So, the best proximity point of* $\mathbb{F}$ *is* $-\frac{1}{2}$.

## 5. CONSEQUENCES

**Corollary 5.1.** *Let, $(\mathbb{C},\mathbb{D})$ be a pair of non-empty closed subsets of a complete metric space $(\mathbb{Y},\Im)$ such that $\mathbb{C}_0$ is non-empty and $\mathfrak{k}$ be a positive integer. Let, $\mathbb{F}:\mathbb{C}^{\mathfrak{k}}\to\mathbb{D}$ be a mapping such that*

$$\begin{aligned}\Im(\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}}),\mathbb{F}(\mathbf{u}_2,\mathbf{u}_3,\cdots,\mathbf{u}_{\mathfrak{k}+1}))\leq&\sum_{i=1}^{\mathfrak{k}}(\gamma_i\varphi(\Im(\mathbf{u}_i,\mathbf{u}_{i+1})))\\&-\varpi\big(max\{\Im(\mathbf{u}_i,\mathbf{u}_{i+1}):i=1,2,\cdots,\mathfrak{k}\}\big)\end{aligned}$$

*for all $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}+1}\in\mathbb{C}$, where $\gamma_i$ are non-negative constants such that $\sum_{i=1}^{\mathfrak{k}}\gamma_i<1$. Suppose the following assertions hold:*
*i) $\mathbb{F}(\mathbb{C}_0^{\mathfrak{k}})\subseteq\mathbb{D}_0$ and the pair $(\mathbb{C},\mathbb{D})$ satisfies the $P$-property.*
*ii) There exist $(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}})\in\mathbb{C}_0^{\mathfrak{k}}$ and $\mathbf{u}_{\mathfrak{k}+1}\in\mathbb{C}_0$ such that*

$$\Im(\mathbf{u}_{\mathfrak{k}+1},\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}}))=\Im(\mathbb{C},\mathbb{D})$$

*iii) $\mathbb{F}$ is continuous.*
*Then, $\mathbb{F}$ has a unique best proximity point in $\mathbb{C}^{\mathfrak{k}}$.*

**Corollary 5.2.** *Assume that $(\mathbb{Y},\Im)$ be a complete metric space such that $\mathfrak{k}$ be a positive integer. Suppose, $\mathbb{F}:\mathbb{Y}^{\mathfrak{k}}\to\mathbb{Y}$ be a continuous mapping such that*

$$\begin{aligned}\Im(\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}}),\mathbb{F}(\mathbf{u}_2,\mathbf{u}_3,\cdots,\mathbf{u}_{\mathfrak{k}+1}))\leq&\sum_{i=1}^{\mathfrak{k}}(\gamma_i\varphi(\Im(\mathbf{u}_i,\mathbf{u}_{i+1})))\\&-\varpi\big(max\{\Im(\mathbf{u}_i,\mathbf{u}_{i+1}):i=1,2,\cdots,\mathfrak{k}\}\big)\end{aligned}$$

*for all $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}+1}\in\mathbb{Y}$ where $\gamma_i$s are non-negative constants such that $\sum_{i=1}^{\mathfrak{k}}\gamma_i<1$. Suppose there exist $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}+1}\in\mathbb{Y}$ such that $\mathbf{u}_{\mathfrak{k}+1}=\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}})$. Thus, $\mathbb{F}$ has a unique fixed point in $\mathbb{Y}^{\mathfrak{k}}$.*

## 6. APPLICATION

We state the following theorem in $\theta$-chainable space [15].

**Theorem 6.1.** *Assume that $\mathbb{C}$ and $\mathbb{D}$ are two non-empty closed subsets of a complete $\theta$-chainable space $(\mathbb{Y},\Im)$ such that $\mathbb{C}_0\neq\emptyset$ and $\mathfrak{k}$ be a positive integer. Let, $\mathbb{F}:\mathbb{C}^{\mathfrak{k}}\to CB(\mathbb{D})$ be a mapping such that*

$$\begin{aligned}H(\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}}),\mathbb{F}(\mathbf{u}_2,\mathbf{u}_3,\cdots,\mathbf{u}_{\mathfrak{k}+1}))\leq&\sum_{i=1}^{\mathfrak{k}}(\gamma_i\varphi(\Im(\mathbf{u}_i,\mathbf{u}_{i+1})))\\&-\varpi\big(max\{\Im(\mathbf{u}_i,\mathbf{u}_{i+1}):i=1,2,\cdots,\mathfrak{k}\}\big)\end{aligned}$$

*for all $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}+1}\in\mathbb{C}$ with $max\{\Im(\mathbf{u}_i,\mathbf{u}_{i+1}):i=1,2,\cdots,\mathfrak{k}\}<\theta$ where $\gamma_i$s are non-negative constants such that $\sum_{i=1}^{\mathfrak{k}}\gamma_i<1$. Suppose that the following assertions hold:*
*i) There exist $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}+1}\in\mathbb{C}$ such that $max\{\Im(\mathbf{u}_i,\mathbf{u}_{i+1}):i=1,2,\cdots,\mathfrak{k}\}<\theta$ and $\mathbb{F}(\mathbf{u}_2,\mathbf{u}_3,\cdots,\mathbf{u}_{\mathfrak{k}+1})\subseteq\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}})$.*
*ii) $\mathbb{F}(\mathbb{C}_0^{\mathfrak{k}})\subseteq\mathbb{D}_0$ and the pair $(\mathbb{C},\mathbb{D})$ satisfies the property such that*

$$\begin{aligned}&\Im(\mathbf{u}_{\mathfrak{k}+1},\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}}))=dist(\mathbb{C},\mathbb{D})=\Im(\mathbf{u}_{\mathfrak{k}+2},\mathbb{F}(\mathbf{u}_2,\mathbf{u}_3,\cdots,\mathbf{u}_{\mathfrak{k}+1}))\\&\Rightarrow\Im(\mathbf{u}_{\mathfrak{k}+1},\mathbf{u}_{\mathfrak{k}+2})\leq H(\mathbb{F}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{\mathfrak{k}}),\mathbb{F}(\mathbf{u}_2,\mathbf{u}_3,\cdots,\mathbf{u}_{\mathfrak{k}+1}))\end{aligned}$$

*iii) There exist* $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}) \in \mathbb{C}_0^\mathfrak{k}$ *and* $\mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}_0$ *such that*

$$\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k})) = dist(\mathbb{C}, \mathbb{D})$$

*iv)* $\mathbb{F}$ *is continuous.*
*Then,* $\mathbb{F}$ *has a best proximity point in* $\mathbb{C}^\mathfrak{k}$*.*

*Proof.* We consider the graph with $\mathbb{V}(\mathbb{G}) = \mathbb{Y}$ and

$$\mathbb{E}(\mathbb{G}) = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{C} \times \mathbb{C} : \Im(\mathbf{u}, \mathbf{v}) < \theta\}$$

Afterthat, we can easily prove this from Theorem(3.1). ∎

**Corollary 6.2.** *Let,* $(\mathbb{C}, \mathbb{D})$ *be a pair of non-empty closed subsets of a complete* $\theta$*-chainable space* $(\mathbb{Y}, \Im)$ *such that* $\mathbb{C}_0$ *is non-empty and* $\mathfrak{k}$ *be a positive integer. Let,* $\mathbb{F} : \mathbb{C}^\mathfrak{k} \to \mathbb{D}$ *be a mapping such that*

$$\Im(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) \leq \sum_{i=1}^{\mathfrak{k}} (\gamma_i \varphi(\Im(\mathbf{u}_i, \mathbf{u}_{i+1}))) - \varpi\big(max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}\big)$$

*for all* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}$ *with* $max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\} < \theta$ *where* $\gamma_i$*s are non-negative constants such that* $\sum_{i=1}^{\mathfrak{k}} \gamma_i < 1$*. Suppose that the following assertions hold:*
*i) There exist* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}$ *such that* $max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\} < \theta$*.*
*ii)* $\mathbb{F}(\mathbb{C}_0^\mathfrak{k}) \subseteq \mathbb{D}_0$ *and the pair* $(\mathbb{C}, \mathbb{D})$ *satisfies the* $P$*-property.*
*iii) There exist* $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}) \in \mathbb{C}_0^\mathfrak{k}$ *and* $\mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}_0$ *such that*

$$\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k})) = \Im(\mathbb{C}, \mathbb{D})$$

*iv)* $\mathbb{F}$ *is continuous.*
*Then,* $\mathbb{F}$ *has a unique best proximity point in* $\mathbb{C}^\mathfrak{k}$*.*

**Corollary 6.3.** *Assume that* $(\mathbb{Y}, \Im)$ *be a complete* $\theta$*-chainable space such that* $\mathfrak{k}$ *be a positive integer. Suppose,* $\mathbb{F} : \mathbb{Y}^\mathfrak{k} \to \mathbb{Y}$ *be a continuous mapping such that*

$$\Im(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) \leq \sum_{i=1}^{\mathfrak{k}} (\gamma_i \varphi(\Im(\mathbf{u}_i, \mathbf{u}_{i+1}))) - \varpi\big(max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}\big)$$

*for all* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}+1} \in \mathbb{Y}$ *with* $max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\} < \theta$ *where* $\gamma_i$*s are non-negative constants such that* $\sum_{i=1}^{\mathfrak{k}} \gamma_i < 1$*. Suppose there exist* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}+1} \in \mathbb{Y}$ *such that* $max\{d(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, k\} < \theta$ *and* $\mathbf{u}_{\mathfrak{k}+1} = \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k})$*. Thus,* $\mathbb{F}$ *has a unique fixed point in* $\mathbb{Y}^\mathfrak{k}$*.*

Now, we define the following:

**Definition 6.1.** Suppose, $\mathbb{C}$ and $\mathbb{D}$ are two non-empty closed subsets of an ordered metric space $(\mathbb{Y}, \Im, \subseteq)$ which is complete such that $\mathbb{C}_0 \neq \emptyset$ and $\mathfrak{k} \geq 1$ such that $\mathfrak{k} \in \mathbf{N}$. Let, $\mathbb{F} : \mathbb{C}^\mathfrak{k} \to CB(\mathbb{D})$ be a mapping. Assume that for every non-decreasing sequence $\{\mathbf{u}_i\}_{i=1}^{\mathfrak{k}+1}$ with respect to $\subseteq$, the following conditions are satisfied:
i) There exist non-negative constants $\gamma_i$s such that $\sum_{i=1}^{\mathfrak{k}} \gamma_i < 1$ so that

$$H(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathfrak{k}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})) \leq \sum_{i=1}^{\mathfrak{k}} (\gamma_i \varphi(\Im(\mathbf{u}_i, \mathbf{u}_{i+1}))) - \varpi\big(max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}\big)$$

ii) If $\mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})$ and $\mathbb{F}(\mathbf{u}_3, \mathbf{u}_4, \cdots, \mathbf{u}_{\mathfrak{k}+2}) \subseteq \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1})$ are such that $\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) < max\{\Im(\mathbf{u}_i, \mathbf{u}_{i+1}) : i = 1, 2, \cdots, \mathfrak{k}\}$, then $(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) \in E(\mathbb{G})$.

**Theorem 6.4.** *Let us assume that $\mathbb{C}$ and $\mathbb{D}$ are two non-empty closed subsets of a complete ordered metric space $(\mathbb{Y}, \Im, \subseteq)$ such that $\mathbb{C}_0 \neq \emptyset$ and $\mathfrak{k} \geq 1$ such that $\mathfrak{k} \in \mathbf{N}$. Let, $\mathbb{F} : \mathbb{C}^{\mathfrak{k}} \to CB(\mathbb{D})$ be a mapping satisfying the above two conditions of the Definition(6.1). Suppose that the following assertions hold:*
*i) There exists a non-decreasing sequence $\{\mathbf{u}_i\}_{i=1}^{\mathfrak{k}+1}$ with respect to $\subseteq$ such that $\mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}) \subseteq \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})$.*
*ii) $\mathbb{F}(\mathbb{C}_0^{\mathfrak{k}}) \subseteq \mathbb{D}_0$ and the pair $(\mathbb{C}, \mathbb{D})$ satisfies the property such that*

$$\begin{aligned}&\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D}) = \Im(\mathbf{u}_{\mathfrak{k}+2}, \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}))\\ &\Rightarrow \Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbf{u}_{\mathfrak{k}+2}) \leq H(\mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}), \mathbb{F}(\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{\mathfrak{k}+1}))\end{aligned}$$

*iii) There exist $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}}) \in \mathbb{C}_0^{\mathfrak{k}}$ and $\mathbf{u}_{\mathfrak{k}+1} \in \mathbb{C}_0$ such that*

$$\Im(\mathbf{u}_{\mathfrak{k}+1}, \mathbb{F}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{\mathfrak{k}})) = dist(\mathbb{C}, \mathbb{D})$$

*iv) $\mathbb{F}$ is continuous.*
*Then, $\mathbb{F}$ has a best proximity point in $\mathbb{C}^{\mathfrak{k}}$.*

*Proof.* Let us consider the graph with $\mathbb{V}(\mathbb{G}) = \mathbb{Y}$ and

$$\mathbb{E}(\mathbb{G}) = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{C} \times \mathbb{C} : \mathbf{u} \subseteq \mathbf{v}\}$$

Now, we can easily prove this from Theorem(3.1). ∎

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## DECLARATIONS

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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