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INEQUALITIES INVOLVING A_∞ WEIGHTS BY EXTRAPOLATIONS

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ABSTRACT. We generalize the extrapolation theorem from A_p weights to A_∞ weights on the setting of weighted Morrey spaces by using the Rubio de Francia algorithm and ideas in a paper by D. Cruz-Uribe et al. First we have proved the classical Hardy-Littlewood maximal operator is bounded on the weighted Morrey spaces if the weight $w(x)$ is in A_∞ and then we have obtained inequalities involving the maximal operator, vector-valued maximal operator, the sharp maximal operator, and A_∞ weights.

Key words and phrases: A_p weights; A_∞ weights; Extrapolation Theorem; Maximal Operator; Morrey Space; Sharp Operator.

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1. INTRODUCTION

Since B. Muckenhoupt in [5] introduced A_p weights or functions, many authors have done a lot of work related to A_p functions on different settings and with different operators other than the classical Hardy operator.

In [1], D. Cruz-Uribe et al generalized the A_p extrapolation theorem of Rubio de Francia to A_∞ weights. With the ideas in the paper and the beautiful algorithm of Rubio de Francia, we have proved that in the setting of weighted Morrey spaces, some results are still true.

Let w be nonnegative, locally integrable with respect to the Lebesgue measure. the weighted Morrey space $L^{p,\lambda}(w)$ is the space of all $f(x)$ defined on $\mathbf{R^n}$ such that

$$\|f\|_{L^{p,\lambda}(w)} = \sup_Q \left(\frac{1}{|Q|^\lambda} \int_Q |f(x)|^p w(x) dx \right)^{1/p} < \infty$$

where the supremum is taken over all bounded cubes of $\mathbf{R^n}$.

The purpose of the paper is to extend the extrapolation theorem with A_p weights to extrapolation theorem with A_∞ weights.

2. PRELIMINARIES

For our convenience, we recall some necessary materials that we are going to need in the proofs of the later context. The following definitions and characterizations can be found in [2], [8], and references therein.

Let Q be a cube in $\mathbf{R^n}$. The classical Hardy-Littlewood maximal operator, M and the sharp maximal operator $M^\sharp$, on a locally integrable function f on $\mathbf{R^n}$ are defined as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(t)| dt.$$

and

$$M^\sharp f(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(t) - f_Q| dt.$$

where $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$, the average of $f(x)$ on Q.

For $\vec{f}(x) = (f_1(x), f_2(x), \cdots)$ or $\vec{f}(x) = (f_1(x), f_2(x), \cdots, f_N(x))$, we define the vector-valued maximal operator $\overline{M}_q$ by

$$\overline{M}_q \vec{f}(x) = \left(\sum_{i=1}^{\infty} (Mf_i(x))^q \right)^{1/q}$$

or

$$\overline{M}_q \vec{f}(x) = \left(\sum_{i=1}^{N} (Mf_i(x))^q \right)^{1/q}$$

For $1 < p < \infty$, the A_p function $w(x)$ is nonnegative, locally integrable with the following property, for any cube Q,

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w(x)} \right)^{\frac{1}{p-1}} dx \right)^{p-1} \le C$$

where the constant C is independent of cube Q.

The A_1 function is $w(x)$ such that $Mw \le Cw$, a.e. $x \in \mathbf{R^n}$.

The set of all A_∞ functions is

$$A_\infty = \cup_{1\le p<\infty} A_p.$$

Throughout this paper, C is used for a positive constant which is not depending on the main factors and C might be different at each occurrence.

3. Main results and their proofs

In this section we present all main results and their proofs.

Theorem 3.1. *The following statements are equivalent.*

(a) For some p_0, $0 < p_0 < \infty$ and every $w \in A_\infty$

$$\|Mf\|_{L^{p_0,\lambda}(w)} \le C\|f\|_{L^{p_0,\lambda}(w)}. \tag{3.1}$$

(b) For all q, $0 < q < p_0$, and $w \in A_1$,

$$\|Mf\|_{L^{q,\lambda}(w)} \le C\|f\|_{L^{q,\lambda}(w)}. \tag{3.2}$$

Theorem 3.2. *If for all q, $0 < q < p_0$, and $w \in A_1$,*

$$\|Mf\|_{L^{q,\lambda}(w)} \le C\|f\|_{L^{q,\lambda}(w)}.$$

then for all p, $0 < p < \infty$ and $w \in A_\infty$

$$\|Mf\|_{L^{p,\lambda}(w)} \le C\|f\|_{L^{p,\lambda}(w)}.$$

Theorem 3.3. *If for some p_0, $0 < p_0 < \infty$ and every $w \in A_\infty$*

$$\|Mf\|_{L^{p_0,\lambda}(w)} \le C\|f\|_{L^{p_0,\lambda}(w)}.$$

then for all $0 < p < \infty$ and $w \in A_\infty$

$$\|Mf\|_{L^{p,\lambda}(w)} \le C\|f\|_{L^{p,\lambda}(w)}. \tag{3.3}$$

Theorem 3.4. *For all $0 < p, q < \infty$ and $w \in A_\infty$, if $f_i(x) \in L^{p,\lambda}(w),\ i = 1, 2, \ldots$, then*

$$\|\overline{M}_q\vec{f}\|_{L^{p,\lambda}(w)} \le C\|\vec{f}\|_{L^{p,\lambda}(w)} \tag{3.4}$$

Theorem 3.5. *The following statements are equivalent.*

(a) For some p_0, $0 < p_0 < \infty$ and every $w \in A_\infty$,

$$\|Mf\|_{L^{p_0,\lambda}(w)} \le C\|M^\sharp f\|_{L^{p_0,\lambda}(w)} \tag{3.5}$$

(b) For all q, $0 < q < p_0$ and every $w \in A_1$,

$$\|Mf\|_{L^{p_0,\lambda}(w)} \le C\|M^\sharp f\|_{L^{p_0,\lambda}(w)} \tag{3.6}$$

(c) For all $0 < p < \infty$ and every $w \in A_\infty$,

$$\|Mf\|_{L^{p_0,\lambda}(w)} \le C\|M^\sharp f\|_{L^{p_0,\lambda}(w)} \tag{3.7}$$

Proof of Theorem 3.1, (a) $\Rightarrow$ (b) We assume that $\|Mf\|_{L^{q,\lambda}(w)} > 0$, otherwise we have nothing to prove. Let $s = p_0/q > 1$. For s', the conjugate of s, since $w \in A_1 \subset A_{s'}$, by [10], M is bounded on $L^{s',\lambda}(w)$. Denote the operator norm of M on $L^{s',\lambda}(w)$ by $\|M\|_{L^{s',\lambda}(w)}$. For $h \in L^{s',\lambda}(w)$, $h \ge 0$, we use the algorithm of Rubio de Francia to define

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{s',\lambda}(w)}}, \tag{3.8}$$

where M^k is the operator M iterated k times if $k \ge 1$, and for $k = 0$ is just the identity. From the definition of R, it is immediate that

(i) $h(x) \le R(h(x))$.
(ii) $\|R(h(x))\|_{L^{s',\lambda}(w)} \le 2\|h\|_{L^{s',\lambda}(w)}$.
(iii) $MR(h(x)) \le 2\|M\|_{L^{s',\lambda}(w)} Rh(x)$.

So $Rh \in A_1$ with constant independent of h. Since Mf, f belong to $L^{q,\lambda}(w)$ and have positive norms, by (ii), we define

$$H(x) = R\left[\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'} + \left(\frac{f}{\|f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right](x) \in L^{s',\lambda}(w). \tag{3.9}$$

By (i), we know that

$$\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'} \leq H(x), \quad \left(\frac{f}{\|f\|_{L^{q,\lambda}(w)}}\right)^{q/s'} \leq H(x). \tag{3.10}$$

So $H(x) > 0$ whenever $MH(x) > 0$. Further, H is finite a.e. on the set where $w(x) > 0$ because $H(x) \in L^{s',\lambda}(w)$.

Therefore, by Hölder inequality, for a cube Q of $\mathbf{R^n}$, we have

$$\begin{aligned}
& \tfrac{1}{|Q|^\lambda}\int_Q |Mf(x)|^q w(x)dx \\
= & \tfrac{1}{|Q|^\lambda}\int_Q |Mf(x)|^q H(x)^{-1}H(x)w(x)^{1/s}w(x)^{1/s'}dx \\
\leq & \tfrac{1}{|Q|^\lambda}\left(\int_Q \left[|Mf(x)|^q H(x)^{-1}w(x)^{1/s}\right]^s dx\right)^{1/s}\left(\int_Q \left[H(x)w(x)^{1/s'}\right]^{s'} dx\right)^{1/s'} \\
= & \left(\tfrac{1}{|Q|^\lambda}\int_Q |Mf(x)|^{p_0}H(x)^{-s}w(x)dx\right)^{1/s}\left(\tfrac{1}{|Q|^\lambda}\int_Q H(x)^{s'}w(x)dx\right)^{1/s'} \\
= & A\cdot B.
\end{aligned}$$

First let us estimate A. Since $w(x), H(x) \in A_1$ (by (c)), and $1+s > 1$, by the factorization property of A_p weights, $w(x)H(x)^{-s} = w(x)H(x)^{1-(1+s)} \in A_{1+s} \subset A_\infty$.

Applying 3.1 to estimate A, we have

$$\begin{aligned}
A &= \left(\frac{1}{|Q|^\lambda}\int_Q |Mf(x)|^{p_0}H(x)^{-s}w(x)dx\right)^{1/s} \\
&\leq \sup_Q \left(\frac{1}{|Q|^\lambda}\int_Q |Mf(x)|^{p_0}H(x)^{-s}w(x)dx\right)^{1/s} \\
&\leq C\sup_Q \left(\frac{1}{|Q|^\lambda}\int_Q |f(x)|^{p_0}H(x)^{-s}w(x)dx\right)^{1/s} \\
&\leq C\sup_Q \left(\frac{1}{|Q|^\lambda}\int_Q |f(x)|^{p_0}\left(\frac{|f(x)|}{\|f\|_{L^{q,\lambda}(w)}}\right)^{-sq/s'} w(x)dx\right)^{1/s} \\
&= C\|f\|^q_{L^{q,\lambda}(w)}
\end{aligned}$$

To estimate B, computations give us

$$\begin{aligned}
&\left(\frac{1}{|Q|^\lambda}\int_Q H(x)^{s'}w(x)dx\right)^{1/s'}\\
\leq\ & \|H\|_{L^{s',\lambda}(w)}\\
=\ & \left\|R\left[\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'}+\left(\frac{f}{\|f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right](x)\right\|_{L^{s',\lambda}(w)}\\
\leq\ & \left\|R\left[\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right](x)\right\|_{L^{s',\lambda}(w)}+\left\|R\left[\left(\frac{f}{\|f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right](x)\right\|_{L^{s',\lambda}(w)}\\
\leq\ & 2\left\|\left[\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right](x)\right\|_{L^{s',\lambda}(w)}+2\left\|\left[\left(\frac{f}{\|f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right](x)\right\|_{L^{s',\lambda}(w)}\\
=\ & 2\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q\left[\left(\frac{Mf(x)}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right]^{s'}w(x)dx\right)^{1/s'}\\
& +2\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q\left[\left(\frac{f(x)}{\|f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right]^{s'}w(x)dx\right)^{1/s'}\\
=\ & 2\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q\frac{Mf(x)^q}{\|Mf\|^q_{L^{q,\lambda}(w)}}w(x)dx\right)^{1/s'}+2\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q\frac{f(x)^q}{\|f\|^q_{L^{q,\lambda}(w)}}w(x)dx\right)^{1/s'}\\
=\ & 2\|Mf\|^{-q/s'}_{L^{q,\lambda}(w)}\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q|Mf(x)|^q w(x)dx\right)^{1/s'}\\
& +2\|f\|^{-q/s'}_{L^{q,\lambda}(w)}\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q|f(x)|^q w(x)dx\right)^{1/s'}\\
=\ & 2\|Mf\|^{-q/s'}_{L^{q,\lambda}(w)}\cdot\|Mf\|^{q/s'}_{L^{q,\lambda}(w)}+2\|f\|^{-q/s'}_{L^{q,\lambda}(w)}\cdot\|f\|^{q/s'}_{L^{q,\lambda}(w)}\\
=\ & 4.
\end{aligned}$$

The result follows by the combination of estimates of both A and B.

Proof of Theorem 3.2 For $0 < p < \infty$ and $w \in A_\infty$, since $A_{p_1} \subset A_{p_2}$ when $p_1 < p_2$, there exists q with $0 < q < p$ such that $w(x) \in A_s$, $s = p/q > 1$. Since $w \in A_s$, $w^{1-s'} \in A_{s'}$, s' is the conjugate of s.

For $g(x) \in L^{s'}(w^{1-s'})$, $g(x) \geq 0$ we use the Rubio de Francia algorithm to define

$$Rg(x) = \sum_{k=0}^{\infty}\frac{M^k g(x)}{2^k\|M\|^k_{L^{s'}(w^{1-s'})}}, \tag{3.11}$$

where $\|M\|^k_{L^{s'}(w^{1-s'})}$ is the operator norm of M on $L^{s'}(w^{1-s'})$; this is finite because $w^{1-s'} \in A_{s'}$ From the definition of R, it is immediate that

(i) $g(x) \leq R(g(x))$.

(ii) $\|R(g(x))\|_{L^{s'}(w^{1-s'})} \leq 2\|g\|_{L^{s'}(w^{1-s'})}$.

(iii) $MR(g(x)) \leq 2\|M\|_{L^{s'}(w^{1-s'})}Rg(x)$. So $R(g)(x) \in A_1$ with constant independent of $g(x)$

For a fixed cube Q_0, we consider

$$\begin{aligned} & \left[\frac{1}{|Q_0|^\lambda}\int_{Q_0}|Mf(x)|^p w(x)dx\right]^{q/p} \\ = & \frac{1}{|Q_0|^{\lambda(q/p)}}\left[\int_{Q_0}\left(|Mf(x)|^q\right)^{p/q}w(x)dx\right]^{q/p} \end{aligned}$$

Applying duality of the space $L^s_{Q_0}(w)$, we know that there exists a function $h_{Q_0}(x)\in L^{s'}_{Q_0}(w)$ on Q_0 with $h_{Q_0}(x)\geq 0$ and $\|h_{Q_0}\|_{L^{s'}(w)}=1$ such that

$$\begin{aligned} & \left[\int_{Q_0}\left(|Mf(x)|^q\right)^{p/q}w(x)dx\right]^{q/p} \\ = & \int_{Q_0}|Mf(x)|^q h_{Q_0}(x)w(x)dx \\ = & \int_{Q_0}|Mf(x)|^q h(x)w(x)dx \end{aligned}$$

where

$$h(x)=\begin{cases} h_{Q_0}(x), & \text{if } x\in Q_0\ , \\ 0, & \text{if } x\notin Q_0\ . \end{cases}$$

Then $h(x)w(x)\in L^{s'}(w^{1-s'})$ and $\|hw\|_{L^{s'}(w^{1-s'})}=1$.

By the property (i) of $R(g)(x)$ and $h(x)w(x)\in L^{s'}(w^{1-s'})$, we have

$$\begin{aligned} & \frac{1}{|Q_0|^{\lambda(q/p)}}\int_{Q_0}|Mf(x)|^q h(x)w(x)dx \\ \leq & \frac{1}{|Q_0|^{\lambda(q/p)}}\int_{Q_0}|Mf(x)|^q R(hw)(x)dx \\ = & \left\{\left[\frac{1}{|Q_0|^{\lambda(q/p)}}\int_{Q_0}|Mf(x)|^q R(hw)(x)dx\right]^{1/q}\right\}^q \\ \leq & \|Mf\|^q_{L^{q,\frac{\lambda q}{p}}(R(hw))} \end{aligned}$$

We have, by the hypothesis,

$$\begin{aligned} & \|Mf\|^q_{L^{q,\frac{\lambda q}{p}}(R(hw))} \\ \leq & C\|f\|^q_{L^{q,\frac{\lambda q}{p}}(R(hw))} \\ = & C\sup_Q \frac{1}{|Q|^{\frac{\lambda q}{p}}}\int_Q|f(x)|^q R(hw)(x)dx \\ \leq & C\sup_Q\left[\frac{1}{|Q|^\lambda}\int_Q|f(x)|^p w(x)dx\right]^{1/s}\left[\int_{R^n}R(hw)(x)^{s'}w(x)^{1-s'}dx\right]^{1/s'} \\ \leq & \|f\|^q_{L^p(w)}. \end{aligned}$$

We have completed the proof of Theorem 3.2.

The Theorem 3.1, **(b)** $\Rightarrow$ **(a)** is immediate from the proof of Theorem 3.2 and the proof of Theorem 3.3 follows from the combination of those of Theorems 3.1 and 3.2 since (a) and (b) in Theorem 3.1 are equivalent.

Proof of Theorem 3.4 For $0 < q < \infty$ and by the Monotone convergence theorem it suffices to show that the vector-valued inequality is true only for finite sums.

For $N \geq 1$, we define

$$\vec{f}_q(x) = \left(\sum_{i=1}^{N} f_i(x)^q \right)^{1/q}.$$

Then for every $w \in A_\infty$, we obtain

$$\begin{aligned} & \|\overline{M}_q \vec{f}_q(x)\|^q_{L^{q,\lambda}(w)} \\ = & \; \| \left(\sum_{i=1}^{N} M f_i(x)^q \right)^{1/q} \|^q_{L^{q,\lambda}(w)} \\ = & \; \sup_Q \left(\frac{1}{|Q|^\lambda} \int_Q \sum_{i=1}^{N} M f_i(x)^q w(x) dx \right) \\ \leq & \; C \|\vec{f}_q\|^q_{L^{q,\lambda}(w)}. \end{aligned}$$

This inequality shows that the hypotheses of Theorem 3.1 are satisfied with $p_0 = q$. Therefore by Theorem 3.1, we are done.

Proof of Theorem 3.5, (a) $\Rightarrow$ (b) We assume $\|Mf\|_{L^{q,\lambda}} > 0$ or we are done. Let $s = p_0/q > 1$. For s', the conjugate of s, since $w \in A_1 \subset A_{s'}$, M is bounded on $L^{s',\lambda}(w)$. For $h \in L^{s',\lambda}(w)$, using the algorithm of Rubio de Francia, we define

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{s',\lambda}(w)}}, \tag{3.12}$$

where $\|M\|^k_{L^{s',\lambda}(w)}$ is the operator norm of M on $L^{s',\lambda}(w)$. From the definition of R, it is immediate that

(i) $h(x) \leq R(h(x))$.

(ii) $\|R(h(x))\|_{L^{s',\lambda}(w)} \leq 2\|h\|_{L^{s',\lambda}(w)}$.

(iii) $MR(h(x)) \leq 2\|M\|_{L^{s',\lambda}(w)} Rh(x)$. So $R(h)(x) \in A_1$ with constant independent of $h(x)$ Since $Mf, M^\sharp f \in L^{q,\lambda}(w)$ and have positive norms, we define

$$H(x) = R\left[\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}} \right)^{q/s'} + \left(\frac{M^\sharp f}{\|M^\sharp f\|_{L^{q,\lambda}(w)}} \right)^{q/s'} \right](x) \in L^{s',\lambda}(w) \tag{3.13}$$

By (i) we know that

$$\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}} \right)^{q/s'} \leq H(x), \quad \text{and} \quad \left(\frac{M^\sharp f}{\|M^\sharp f\|_{L^{q,\lambda}(w)}} \right)^{q/s'} \leq H(x) \tag{3.14}$$

So $H(x) > 0$ whenever $Mf(x) > 0$. Moreover H is finite a.e. on the set where $w(x) > 0$ because $H(x) \in L^{s',\lambda}(w)$.

For a cube Q of R^n and the Hölder inequality we get

$$\begin{aligned} & \frac{1}{|Q|^\lambda} \int_Q Mf(x)^q w(x) dx \\ \leq & \; \left\{ \frac{1}{|Q|^\lambda} \int_Q Mf(x)^{p_0} H(x)^{-s} w(x) dx \right\}^{1/s} \cdot \left\{ \frac{1}{|Q|^\lambda} \int_Q H(x)^{s'} w(x) dx \right\}^{1/s'} \\ = & \; A \cdot B. \end{aligned}$$

Let us first estimate A. Since $w(x),\ H(x) \in A_1$ and by the factorization property of A_1 weights, $H(x)^{-s}w(x) = w(x)H(x)^{1-(1+s)} \in A_{1+s} \subset A_\infty$. Now we need to make sure $A < \infty$. By 3.14, we have

$$\begin{aligned} A &\le \frac{1}{|Q|^\lambda}\int_Q Mf(x)^{p_0}\left(\frac{Mf(x)}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{-sq/s'} w(x)dx \\ &= \|Mf\|^{sq/s'}_{L^{q,\lambda}(w)}\frac{1}{|Q|^\lambda}\int_Q Mf(x)^q w(x)dx \\ &= \|Mf\|^{p_0}_{L^{q,\lambda}(w)} < \infty. \end{aligned}$$

Applying 3.5 to A, we obtain

$$\begin{aligned} A &= \sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q Mf(x)^{p_0}H(x)^{-s}w(x)dx\right)^{1/s} \\ &\le C\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q M^\sharp f(x)^{p_0}H(x)^{-s}w(x)dx\right)^{1/s} \\ &\le C\sup_Q\left[\frac{1}{|Q|^\lambda}\int_Q M^\sharp f(x)^{p_0}\left(\frac{M^\sharp f(x)}{\|M^\sharp f\|_{\ L^{q,\lambda}(w)}}\right)^{-sq/s'} w(x)dx\right]^{1/s} \\ &= \|M^\sharp f\|^q_{L^{q,\lambda}(w)}. \end{aligned}$$

To estimate B, we have the following computations.

$$\begin{aligned} B &= \left\{\frac{1}{|Q|^\lambda}\int_Q H(x)^{s'}w(x)dx\right\}^{1/s'} \\ &\le \|H\|_{L^{s',\lambda}(w)} \\ &\le \left\|R\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right\|_{L^{s',\lambda}(w)} + \left\|R\left(\frac{M^\sharp f}{\|M^\sharp f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right\|_{L^{s',\lambda}(w)} \\ &\le 2\left\|\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right\|_{L^{s',\lambda}(w)} + 2\left\|\left(\frac{M^\sharp f}{\|M^\sharp f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right\|_{L^{s',\lambda}(w)} \\ &\le 2\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q\left[\left(\frac{Mf}{\|Mf\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right]^{s'}\right)^{1/s'} \\ &\quad +2\sup_Q\left(\frac{1}{|Q|^\lambda}\int_Q\left[\left(\frac{M^\sharp f}{\|M^\sharp f\|_{L^{q,\lambda}(w)}}\right)^{q/s'}\right]^{s'}\right)^{1/s'} \\ &= 4. \end{aligned}$$

Combining the estimates of A and B gives us the desired result.

Proof of Theorem 3.5, (b) $\Rightarrow$ (c) We consider a $p,\ 0 < p < \infty$ and choose $q,\ 0 < q < \min(p, p_0)$ such that $w(x) \in A_s$, where $s = p/q > 1$ since $A_{p_1} \subset A_{p_2}$ when $p_1 < p_2$. Since $w(x) \in A_s$, we have $w^{1-s'} \in A_{s'}$, where s' is the conjugate of s.

For $g(x) \in L^{s'}(w^{1-s'})$, $g(x) \geq 0$, we use the Rubio de Francia algorithm to define

$$R(g)(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{2^k \|M\|^k_{L^{s'}(w^{1-s'})}}$$

We have the properties

(i) $g(x) \leq R(g)(x)$.

(ii) $\|R(g)\|_{L^{s'}(w^{1-s'})} \leq 2\|g\|_{L^{s'}(w^{1-s'})}$.

(iii) $M(R(g))(x) \leq 2\|M\|_{L^{s'}(w^{1-s'})} R(g)(x)$, so $R(g)(x) \in A_1$, where A_1 constant independent of $g(x)$.

For a fixed cube Q_0, we consider

$$\begin{aligned} &\left[\frac{1}{|Q_0|^\lambda} \int_{Q_0} Mf(x)^p w(x)dx\right]^{1/s} \\ = \;& \frac{1}{|Q_0|^{\lambda/s}} \left[\int_{Q_0} (Mf(x)^q)^s \, w(x)dx\right]^{1/s} \end{aligned}$$

By duality of the space $L^s_{Q_0}(w)$, there exists $h_{Q_0}(x) \in L^{s'}_{Q_0}(w)$, $h_{Q_0}(x) \geq 0$ and $\|h_{Q_0}\|_{L^{s'}_{Q_0}(w)} = 1$ such that

$$\begin{aligned} &\left[\int_{Q_0} (Mf(x)^q)^s \, w(x)dx\right]^{1/s} \\ = \;& \int_{Q_0} Mf(x)^q h_{Q_0}(x) w(x)dx \\ = \;& \int_{Q_0} Mf(x)^q h(x) w(x)dx \end{aligned}$$

where $h(x) = h_{Q_0}(x)$, when $x \in Q_0$ and $h(x) = 0$, otherwise.

Since $h \in L^{s'}(w)$ and $w \in A_s$, $hw \in L^{s'}(w^{1-s'})$ and $\|hw\|_{L^{s'}(w^{1-s'}} = 1$.

With the help of property (i) of R, we get

$$\begin{aligned} &\left[\frac{1}{|Q_0|^\lambda} \int_{Q_0} Mf(x)^p w(x)dx\right]^{1/s} \\ = \;& \frac{1}{|Q_0|^{\lambda/s}} \left[\int_{Q_0} Mf(x)^q h(x) w(x)dx\right] \\ \leq \;& \frac{1}{|Q_0|^{\lambda/s}} \left[\int_{Q_0} Mf(x)^q R(hw)(x)dx\right] \\ \leq \;& \|Mf\|^q_{L^{q,\frac{\lambda}{s}}(R(hw))} \\ \leq \;& C\|M^\sharp f\|^q_{L^{q,\frac{\lambda}{s}}(R(hw))} \quad \text{by } (b) \end{aligned}$$

$$
\begin{aligned}
&= C\sup_Q \left[\frac{1}{|Q|^{\lambda/s}}\int_Q M^\sharp f(x)^q R(hw)(x)dx\right] \\
&\le C\sup_Q \left[\frac{1}{|Q|^{\lambda}}\int_Q M^\sharp f(x)^p w(x)dx\right]^{1/s}\left[\int_{R^n} R(hw)(x)^{s'} w(x)^{-s'/s}dx\right]^{1/s'} \\
&= \|M^\sharp f\|^q_{L^{p,\lambda}}\|R(hw)\|_{L^{s'}(w^{1-s'})} \\
&\le C\|M^\sharp f\|^q_{L^{p,\lambda}} \quad \text{by } (ii)
\end{aligned}
$$

.

Therefore we have

$$\|Mf\|_{L^{p,\lambda}} \le C\|M^\sharp f\|_{L^{p,\lambda}}$$

for all $0 < p < \infty$.

Proof of Theorem 3.5, (c) $\Rightarrow$ (a) It is obvious since (a) is a special case of (c).

This completes the entire proof of Theorem 3.5.

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