Aust. J. Math. Anal. Appl.
Vol. **21** (2024), No. 1, Art. 3, 13 pp.
AJMAA

BOUNDNESS OF THE POWER EXPONENTIAL FUNCTION $a^{2b} + b^{2a}$

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Received 19 February, 2023; accepted 26 January, 2024; published 16 February, 2024.

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ABSTRACT. In this paper, we consider the boundness of the power exponential function $a^{2b} + b^{2a}$ for nonnegative real numbers a and b. The author [3] proved that the function has the maximum value 1 for $a + b = \frac{1}{2}$, but it is no known that the minimum value for $a + b = \frac{1}{2}$. In this paper, we give the new proof of the function has the maximum value 1 and show that $a^{2b} + b^{2a} > 0.989905$ for $a + b = \frac{1}{2}$.

Key words and phrases: Boundness of power exponential functions; Monotonically increasing functions; Monotonically decreasing functions.

2010 *Mathematics Subject Classification.* Primary 26A09, 33B10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Cîrtoaje [1] [2] conjectured and showed the power exponential function $a^{2b}+b^{2a}$ has the maximum value 1, and the author [4] gave a lower boundness of the function for nonnegative real numbers a and b with $a+b=1$. Also, for nonnegative real numbers a and b with $a+b=\frac{1}{2}$, the author [3] proved that the function has the maximum value 1, but it is no known that a lower boundness of the function $a^{2b}+b^{2a}$. According to computer simulations, the minimum seems to be around 0.990131. In this paper, for nonnegative real numbers a and b with $a+b=\frac{1}{2}$, we give a new proof different from [3] and show that the power exponential function $a^{2b}+b^{2a}$ takes a maximum value 1, also we show that $a^{2b}+b^{2a}>0.989905$. Our main results are given as follows.

Theorem 1.1. *For nonnegative real numbers a and b with $a+b=\frac{1}{2}$, the power exponential function $a^{2b}+b^{2a}$ has the maximum value 1.*

Theorem 1.2. *For nonnegative real numbers a and b with $a+b=\frac{1}{2}$, the power exponential function $a^{2b}+b^{2a}$ is greater than*

$$\frac{237854403804852760744441611}{240280005622339350} \cong 0.989905\,.$$

Since we have $a^{2b}+b^{2a}=1$ for $(a,b)=(\frac{1}{4},\frac{1}{4})$, $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, we may assume that $0<a<\frac{1}{4}<b<\frac{1}{2}$. If $a=\frac{7}{100}$ and $b=\frac{43}{100}$ then, we can get

$$a^{2b}+b^{2a}=\frac{7^{\frac{43}{50}}}{10\cdot 10^{\frac{18}{25}}}+\frac{43^{\frac{7}{50}}}{10^{\frac{7}{25}}} \cong 0.990132\,,$$

therefore, it can be seen that the minimum value of $a^{2b}+b^{2a}$ is less than or equal to 0.990132. From Theorem 1.2, for nonnegative real numbers a and b with $a+b=\frac{1}{2}$, we can see that the minimum value of $a^{2b}+b^{2a}$ is between 0.990132 and 0.989905.

2. PROOF OF THEOREM 1.1

We will show two lemmas needed to prove Theorem 1.1.

Lemma 2.1. *We have*

$$\frac{405969-47752a-1184496a^2-320000\ln 2+1280000a\ln 2}{160000} > a^{-2a}$$

for $0<a<\frac{1}{4}$.

Proof. From

$$\begin{aligned}&405969-47752a-1184496a^2-320000\ln 2+1280000a\ln 2\\&>405969-47752\left(\frac{1}{4}\right)-1184496\left(\frac{1}{4}\right)^2-320000\ln 2+1280000\cdot 0\cdot \ln 2\\&>320000\,(1-\ln 2)\cong 98192.9\,,\end{aligned}$$

we can take the logarithm and we set

$$\begin{aligned}f(a)=&\ln\frac{405969-47752a-1184496a^2-320000\ln 2+1280000a\ln 2}{160000}-\ln a^{-2a}\\=&\ln\left(405969-47752a-1184496a^2-320000\ln 2+1280000a\ln 2\right)\\&-\ln 160000+2a\ln a\,.\end{aligned}$$

We have the derivatives of $f(a)$ are

$$f'(a)=\frac{2(382093-1232248a-1184496a^2+320000\ln 2+1280000a\ln 2)}{405969-47752a-1184496a^2-320000\ln 2+1280000a\ln 2}+2\ln a$$

and

$$f''(a)=\frac{2g(a)}{a(-405969+47752a+1184496a^2+320000\ln 2-1280000a\ln 2)^2},$$

where

$$\begin{aligned}g(a)=&164810828961-520780446752a-1016019112736a^2-1289906668032a^3\\&+1403030774016a^4-259820160000\ln 2+1510003200000a\ln 2\\&+2151987200000a^2\ln 2-3032309760000a^3\ln 2+102400000000(\ln 2)^2\\&-1638400000000a(\ln 2)^2+1638400000000a^2(\ln 2)^2\,.\end{aligned}$$

The derivative of $g(a)$ is

$$\begin{aligned}g'(a)=&-520780446752-2032038225472a-3869720004096a^2+5612123096064a^3\\&+1510003200000\ln 2+4303974400000a\ln 2-9096929280000a^2\ln 2\\&-1638400000000(\ln 2)^2+3276800000000a(\ln 2)^2\\<&-520780446752-2032038225472a-3869720004096a^2+5612123096064a^3\\&+1510003200000\left(\frac{6932}{10000}\right)+4303974400000a\left(\frac{6932}{10000}\right)\\&-9096929280000a^2\left(\frac{6931}{10000}\right)-1638400000000\left(\frac{6931}{10000}\right)^2\\&+3276800000000a\left(\frac{6932}{10000}\right)^2\\=&-261113288736+2526065211840a-10174801688064a^2+5612123096064a^3\\<&-261113288736+2526065211840a-10174801688064a^2\\&+5612123096064a^2\left(\frac{1}{4}\right)\\=&-261113288736+2526065211840a-8771770914048a^2=h(a)\,.\end{aligned}$$

The function $h(a)$ is convex upwards and takes the maximum value at $a=\frac{13156589645}{91372613688}$, so we have $g'(a)<h\left(\frac{13156589645}{91372613688}\right)=-\frac{30172508150555383832}{3807192237}$. Therefore, we have $g'(a)<0$ for $0<a<\frac{1}{4}$ and $g(a)$ is strictly decreasing for $0<a<\frac{1}{4}$. From $g\left(\frac{1}{4}\right)=-43559680000+204800000000\ln 2-204800000000(\ln 2)^2\cong 85328.2$, we have $g(a)>0$ for $0<a<\frac{1}{4}$ and $f'(a)$ is strictly increasing for $0<a<\frac{1}{4}$. By $f'(\frac{1}{4})=0$, $f'(a)<0$ for $0<a<\frac{1}{4}$ and $f(a)$ is strictly decreasing for $0<a<\frac{1}{4}$. From $f(\frac{1}{4})=0$, we obtain $f(a)>0$ for $0<a<\frac{1}{4}$. ∎

Lemma 2.2. *We have*

$$4(1-3b+4b^2-\ln 2+6b\ln 2-8b^2\ln 2)>b^{-2b}$$

for $\frac{1}{4}<b<\frac{1}{2}$.

Proof. From

$$\begin{aligned}
&1-3b+4b^2-\ln 2+6b\ln 2-8b^2\ln 2\\
&>1-3b+4b^2-\left(\frac{6932}{10000}\right)+6b\left(\frac{6931}{10000}\right)-8b^2\left(\frac{6932}{10000}\right)\\
&=\frac{1534+5793b-7728b^2}{5000}>\frac{1534+5793\left(\frac{1}{4}\right)-7728\left(\frac{1}{2}\right)^2}{5000}=\frac{4201}{20000},
\end{aligned}$$

we can take the logarithm and we set

$$\begin{aligned}
f(b)&=\ln 4(1-3b+4b^2-\ln 2+6b\ln 2-8b^2\ln 2)-\ln b^{-2b}\\
&=2\ln 2+\ln(1-3b+4b^2-\ln 2+6b\ln 2-8b^2\ln 2)+2b\ln b\,.
\end{aligned}$$

The derivatives of $f(b)$ are

$$f'(b)=\frac{1-2b-8b^2-4\ln 2+4b\ln 2+16b^2\ln 2}{-1+3b-4b^2+\ln 2-6b\ln 2+8b^2\ln 2}+2\ln b$$

and

$$f''(b)=\frac{g(b)}{b(-1+3b-4b^2+\ln 2-6b\ln 2+8b^2\ln 2)^2},$$

where $g(b)=2-13b+58b^2-80b^3+32b^4-4\ln 2+48b\ln 2-216b^2\ln 2+320b^3\ln 2-128b^4\ln 2+2(\ln 2)^2-44b(\ln 2)^2+200b^2(\ln 2)^2-320b^3(\ln 2)^2+128b^4(\ln 2)^2$. The derivative of $g(b)$ is

$$\begin{aligned}
g'(b)=&-13+116b-240b^2+128b^3+48\ln 2-432b\ln 2+960b^2\ln 2-512b^3\ln 2\\
&-44(\ln 2)^2+400b(\ln 2)^2-960b^2(\ln 2)^2+512b^3(\ln 2)^2\\
<&-13+116b-240b^2+128b^3+48\left(\frac{6932}{10000}\right)-432b\left(\frac{6931}{10000}\right)\\
&+960b^2\left(\frac{6932}{10000}\right)-512b^3\left(\frac{6931}{10000}\right)-44\left(\frac{6931}{10000}\right)^2+400b\left(\frac{6932}{10000}\right)^2\\
&-960b^2\left(\frac{6931}{10000}\right)^2+512b^3\left(\frac{6932}{10000}\right)^2\\
=&\frac{-21586371+219782400b-892502640b^2+479055872b^3}{25000000}\\
<&\frac{-21586371+219782400b-892502640b^2+479055872\left(\frac{1}{2}\right)^3}{25000000}=\frac{h(b)}{25000000},
\end{aligned}$$

where $h(b)=-21586371+219782400b-652974704b^2$. The function $h(b)$ is convex upwards and takes the maximum value at $b=\frac{6868200}{40810919}$, so we have $g'(b)<h\left(\frac{6868200}{40810919}\right)=-\frac{126204898544949}{40810919}$. Therefore, we have $g'(b)<0$ for $\frac{1}{4}<b<\frac{1}{2}$ and $g(b)$ is strictly decreasing for $\frac{1}{4}<b<\frac{1}{2}$. From $g(\frac{1}{4})=\frac{5}{4}-\ln 2-(\ln 2)^2\cong 0.0763998$ and $g(\frac{1}{2})=2-2\ln 2-2(\ln 2)^2\cong -0.3472$, there exists a unique real number b_0 such that $g(b)>0$ for $\frac{1}{4}<b<b_0$ and $g(b)<0$ for $b_0<b<\frac{1}{2}$. Hence, $f'(b)$ is strictly increasing for $\frac{1}{4}<b<b_0$ and strictly decreasing for $b_0<b<\frac{1}{2}$. By $f'(\frac{1}{4})=0$ and $f'(\frac{1}{2})=4-6\ln 2\cong -0.158883$, there exists a unique real number b_1 such that $f'(b)>0$ for $\frac{1}{4}<b<b_1$ and $f'(b)<0$ for $b_1<b<\frac{1}{2}$. Therefore, $f(b)$ is strictly increasing for $\frac{1}{4}<b<b_1$ and strictly decreasing for $b_1<b<\frac{1}{2}$. Since we have $f(\frac{1}{4})=0$ and $f(\frac{1}{2})=0$, we obtain $f(a)>0$ for $\frac{1}{4}<b<\frac{1}{2}$. ∎

Proof of Theorem 1.1. From Lemmas 2.1 and 2.2, for $0<a<\frac{1}{4}<b<\frac{1}{2}$, we have

$$\begin{aligned} a^{2b}+b^{2a} &= a^{1-2a}+b^{1-2b} \\ &< \frac{a}{160000}(405969-47752a-1184496a^2-320000\ln 2+1280000a\ln 2) \\ &\quad + 4b(1-3b+4b^2-\ln 2+6b\ln 2-8b^2\ln 2) = \frac{f(a)}{160000}\,, \end{aligned}$$

where $f(a)=160000-234031a+1872248a^2-3744496a^3+320000a\ln 2-2560000a^2\ln 2+5120000a^3\ln 2$. Since the derivative of $f(a)$ is $f'(a)=(1-4a)(1-12a)(-234031+320000\ln 2)$, we have $f'(a)<0$ for $0<a<\frac{1}{12}$ and $f'(a)>0$ for $\frac{1}{12}<a<\frac{1}{4}$. Hence, we have $f(\frac{1}{12})\le f(a)<f(0)=f(\frac{1}{4})=1$ and the proof of Theorem 1.1 is complete. ∎

3. PROOF OF THEOREM 1.2

We will show some lemmas needed to prove Theorem 1.2.

Lemma 3.1. *We have*

$$\frac{38618000000+298677429103a-576629716412a^2}{38624000000}<a^{-2a}$$

for $0<a<\frac{1}{20}$.

Proof. From

$$\begin{aligned} &38618000000+298677429103a-576629716412a^2 \\ &> 38618000000+298677429103a-576629716412\left(\frac{1}{20}\right)^2 \\ &= \frac{3717642570897}{100}+298677429103a\,, \end{aligned}$$

we can take the logarithm and we set

$$\begin{aligned} f(a) =&\ln\frac{38618000000+298677429103a-576629716412a^2}{38624000000}-\ln a^{-2a} \\ =&\ln\left(38618000000+298677429103a-576629716412a^2\right)-\ln 38624000000 \\ &+2a\ln a\,. \end{aligned}$$

The derivatives of $f(a)$ are

$$f'(a)=\frac{2(187956714552-277952287309a-576629716412a^2)}{38618000000+298677429103a-576629716412a^2}+2\ln a$$

and

$$f''(a)=\frac{2g(a)}{a(-38618000000-298677429103a+576629716412a^2)^2}\,,$$

where

$$\begin{aligned} g(a) =&1491349924000000000000-4380373980213744240685\,6a \\ &+21689791512168511463945\,7a^2-67695439233403979323061\,6a^3 \\ &+332501829849383542153744a^4\,. \end{aligned}$$

The derivative of $g(a)$ is

$$\begin{aligned} g'(a) =& -4380373980213744240 6856 + 4337958302433702292 78914a \\ & -2030863177002119379691848a^2 + 1330007319397534168614976a^3 \\ <& -4380373980213744240 6856 + 4337958302433702292 78914\left(\frac{1}{20}\right) \\ & -2030863177002119379691848a^2 + 1330007319397534168614976\left(\frac{1}{20}\right)^3 \\ =& -\frac{5486924343761059792958357}{250} - 2030863177002119379691848a^2\,. \end{aligned}$$

Hence, $g'(a) < 0$ for $0 < a < \frac{1}{20}$ and $g(a)$ is strictly decreasing for $0 < a < \frac{1}{20}$. Since we have $g(0) = 149134992400000000000 0$ and $g(\frac{1}{20}) = -\frac{5978336022696391518988 09}{2500}$, there exists a unique real number a_0 such that $g(a) > 0$ for $0 < a < a_0$ and $g(a) < 0$ for $a_0 < a < \frac{1}{20}$. Hence, $f'(a)$ is strictly increasing for $0 < a < a_0$ and strictly decreasing for $a_0 < a < \frac{1}{20}$. By $\lim_{a\to 0} f'(a) = -\infty$ and $f'(\frac{1}{20}) = \frac{8630876294776}{1302757429103} - 2\ln 20 \cong 0.633619$, there exists a unique real number a_1 such that $f'(a) < 0$ for $0 < a < a_1$ and $f'(a) > 0$ for $a_1 < a < \frac{1}{20}$. Thus, $f(a)$ is strictly decreasing for $0 < a < a_1$ and strictly increasing for $a_1 < a < \frac{1}{20}$. From $\lim_{a\to 0} f(a) = \ln\frac{19309}{19312}$ and

$$\begin{aligned} f\left(\frac{1}{20}\right) &= \ln\frac{5211029716417}{3862400000000} - \frac{\ln 20}{10} = \frac{1}{10}\ln\left(\frac{1}{20}\left(\frac{5211029716417}{3862400000000}\right)^{10}\right) \\ &\cong \frac{1}{10}\ln\frac{14765069}{14777551}\,, \end{aligned}$$

we obtain $f(a) < 0$ for $0 < a < \frac{1}{20}$. ∎

Lemma 3.2. *We have*

$$\frac{106503811 + 617083346a - 972394360a^2}{100000000} < a^{-2a}$$

for $\frac{1}{20} < a < \frac{1}{10}$.

Proof. From

$$\begin{aligned} & 106503811 + 617083346a - 972394360a^2 \\ & > 106503811 + 617083346a - 972394360\left(\frac{1}{10}\right)^2 \\ & = \frac{483899337}{5} + 617083346a\,, \end{aligned}$$

we can take the logarithm and we set

$$\begin{aligned} f(a) =& \ln\frac{106503811 + 617083346a - 972394360a^2}{100000000} - \ln a^{-2a} \\ =& \ln\left(106503811 + 617083346a - 972394360a^2\right) - \ln 100000000 + 2a\ln a\,. \end{aligned}$$

The derivatives of $f(a)$ are

$$f'(a) = \frac{4(207522742 - 177655507a - 486197180a^2)}{106503811 + 617083346a - 972394360a^2} + 2\ln a$$

and

$$f''(a)=\frac{2g(a)}{a(-106503811-617083346a+972394360a^2)^2}\,,$$

where

$$g(a)=11343061757523721-162516176982920606a+773712810941072356a^2 \\ -2145647521960466720a^3+945550791359809600a^4\,.$$

The derivative of $g(a)$ is

$$\begin{aligned} g'(a)&=-162516176982920606+1547425621882144712a \\ &\quad-6436942565881400160a^2+3782203165439238400a^3 \\ &<-162516176982920606+1547425621882144712\left(\frac{1}{10}\right) \\ &\quad-6436942565881400160a^2+3782203165439238400\left(\frac{1}{10}\right)^3 \\ &=-\frac{19957058146334482}{5}-6436942565881400160a^2\,. \end{aligned}$$

Hence, $g'(a)<0$ for $\frac{1}{20}<a<\frac{1}{10}$ and $g(a)$ is strictly decreasing for $\frac{1}{20}<a<\frac{1}{10}$. Since we have $g(\frac{1}{10})=\frac{3887398629089491}{5}$ and $f''(a)>0$ for $\frac{1}{20}<a<\frac{1}{10}$, $f'(a)$ is strictly increasing for $\frac{1}{20}<a<\frac{1}{10}$. From $f'(\frac{1}{20})=\frac{1974244737}{337317481}-2\ln 20\cong-0.138685$ and $f'(\frac{1}{10})=\frac{369790439}{79244101}-2\ln 10\cong 0.0613026$, there exists a unique real number a_0 such that $f'(a)<0$ for $\frac{1}{20}<a<a_0$ and $f'(a)>0$ for $a_0<a<\frac{1}{10}$. Thus, $f(a)$ is strictly decreasing for $\frac{1}{20}<a<a_0$ and strictly increasing for $a_0<a<\frac{1}{10}$. Since we have

$$\begin{aligned} f\left(\frac{1}{20}\right)&=\ln\frac{337317481}{250000000}-\frac{\ln 20}{10}=\frac{1}{10}\ln\left(\frac{1}{20}\left(\frac{337317481}{250000000}\right)^{10}\right) \\ &\cong\frac{1}{10}\ln\frac{19071659}{19073486} \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{1}{10}\right)&=\ln\frac{79244101}{50000000}-\frac{\ln 10}{5}=\frac{1}{5}\ln\left(\frac{1}{10}\left(\frac{79244101}{50000000}\right)^{5}\right) \\ &\cong\frac{1}{5}\ln\frac{3124889}{3125000}\,, \end{aligned}$$

we obtain $f(a)<0$ for $\frac{1}{20}<a<\frac{1}{10}$. ∎

Lemma 3.3. *We have*

$$\frac{6626567+33716264a-48890128a^2}{6000000}<a^{-2a}$$

for $\frac{1}{10}<a<\frac{1}{4}$.

Proof. From $6626567+33716264a-48890128a^2>6626567+33716264a-48890128(\frac{1}{4})^2=3570934+33716264a$, we can take the logarithm and we set

$$\begin{aligned} f(a)=&\ln\frac{6626567+33716264a-48890128a^2}{6000000}-\ln a^{-2a} \\ =&\ln\left(6626567+33716264a-48890128a^2\right)-\ln 6000000+2a\ln a\,. \end{aligned}$$

The derivatives of $f(a)$ are

$$f'(a) = \frac{2(23484699 - 15173864a - 48890128a^2)}{6626567 + 33716264a - 48890128a^2} + 2\ln a$$

and

$$f''(a) = \frac{2g(a)}{a(-6626567 - 33716264a + 48890128a^2)^2},$$

where

$$\begin{aligned} g(a) =& 43911390205489 - 445520773118048a + 2137231503098336a^2 \\ & - 5687029541139968a^3 + 2390244615856384a^4 . \end{aligned}$$

The derivative of $g(a)$ is

$$\begin{aligned} g'(a) =& - 445520773118048 + 4274463006196672a - 17061088623419904a^2 \\ & + 9560978463425536a^3 \\ < & - 445520773118048 + 4274463006196672a - 17061088623419904a^2 \\ & + 9560978463425536a^2 \left(\frac{1}{4}\right) \\ =& - 445520773118048 + 4274463006196672a - 14670844007563520a^2 = h(a) . \end{aligned}$$

The function $h(a)$ is convex upwards and takes the maximum value at $a = \frac{17289278921}{118680785720}$, so we have $g'(a) < h\left(\frac{17289278921}{118680785720}\right) = -\frac{515258990176565627676}{3840305}$. Therefore, we have $g'(a) < 0$ for $\frac{1}{10} < a < \frac{1}{4}$ and $g(a)$ is strictly decreasing for $\frac{1}{10} < a < \frac{1}{4}$. From $g(\frac{1}{10}) = \frac{9552264278195769}{625}$ and $g(\frac{1}{4}) = -13414777680000$, there exists a unique real number a_0 such that $g(a) > 0$ for $\frac{1}{10} < a < a_0$ and $g(a) < 0$ for $a_0 < a < \frac{1}{4}$. Thus, $f'(a)$ is strictly increasing for $\frac{1}{10} < a < a_0$ and strictly decreasing for $a_0 < a < \frac{1}{4}$. By $f'(\frac{1}{10}) = \frac{357973522}{79244101} - 2\ln 10 \cong -0.0878179$ and $f'(\frac{1}{4}) = \frac{13863}{5000} - 4\ln 2 \cong 0.0000112778$, there exists a unique real number a_1 such that $f'(a) < 0$ for $\frac{1}{10} < a < a_1$ and $f'(a) > 0$ for $a_1 < a < \frac{1}{4}$. Hence, $f(a)$ is strictly decreasing for $\frac{1}{10} < a < a_1$ and strictly increasing for $a_1 < a < \frac{1}{4}$. Since we have

$$\begin{aligned} f\left(\frac{1}{10}\right) &= \ln \frac{79244101}{50000000} - \frac{\ln 10}{5} = \frac{1}{5}\ln \left(\frac{1}{10}\left(\frac{79244101}{50000000}\right)^5\right) \\ &\cong \frac{1}{5}\ln \frac{3124889}{3125000} \end{aligned}$$

and $f(\frac{1}{4}) = 0$, we obtain $f(a) < 0$ for $\frac{1}{10} < a < \frac{1}{4}$. ∎

Lemma 3.4. *We have*

$$\frac{423 + 1302b - 1736b^2}{320} < b^{-2b}$$

for $\frac{1}{4} < b < \frac{2}{5}$.

Proof. From $423 + 1302b - 1736b^2 > \frac{3631}{25} + 1302b$, we can take the logarithm and we set

$$\begin{aligned} f(b) =& \ln \frac{423 + 1302b - 1736b^2}{320} - \ln b^{-2b} \\ =& \ln (423 + 1302b - 1736b^2) - \ln 320 + 2b\ln b . \end{aligned}$$

The derivatives of $f(b)$ are

$$f'(b)=\frac{4(537-217b-868b^2)}{423+1302b-1736b^2}+2\ln b$$

and

$$f''(b)=\frac{2g(b)}{b(-423-1302b+1736b^2)^2}\,,$$

where $g(b)=178929-480438b+2486820b^2-7534240b^3+3013696b^4$. The derivative of $g(b)$ is

$$\begin{aligned}
g'(b)=&-480438+4973640b-22602720b^2+12054784b^3\\
<&-480438+4973640b-22602720b^2+12054784b^2\left(\frac{2}{5}\right)\\
=&-480438+4973640b-\frac{88904032b^2}{5}\\
<&-480438+4973640b-\frac{88904032b}{5}\left(\frac{1}{4}\right)\\
=&-480438+\frac{2642192}{5}\left(\frac{2}{5}\right)=-\frac{6726566}{25}\,.
\end{aligned}$$

Hence, $g'(b)<0$ for $\frac{1}{4}<b<\frac{2}{5}$ and $g(b)$ is strictly decreasing for $\frac{1}{4}<b<\frac{2}{5}$. Since we have $g(\frac{1}{4})=\frac{216591}{2}$ and $g(\frac{2}{5})=-\frac{12747339}{625}$, there exists a unique real number b_0 such that $g(b)>0$ for $\frac{1}{4}<b<b_0$ and $g(b)<0$ for $b_0<a<\frac{2}{5}$. Thus, $f'(b)$ is strictly increasing for $\frac{1}{4}<b<b_0$ and strictly decreasing for $b_0<b<\frac{2}{5}$. By $f'(\frac{1}{4})=\frac{857}{320}-4\ln 2\cong-0.0944637$ and $f'(\frac{2}{5})=\frac{31132}{16651}-2\ln\frac{5}{2}\cong 0.037096$, there exists a unique real number b_1 such that $f'(b)<0$ for $\frac{1}{4}<b<b_1$ and $f'(b)>0$ for $b_1<a<\frac{2}{5}$. Hence, $f(b)$ is strictly decreasing for $\frac{1}{4}<b<b_1$ and strictly increasing for $b_1<b<\frac{2}{5}$. From $f(\frac{1}{4})=0$ and

$$\begin{aligned}
f\left(\frac{2}{5}\right)=&-\frac{4}{5}\ln\frac{5}{2}-\ln 320+\ln\frac{16651}{25}=\frac{1}{5}\ln\left(\left(\frac{2}{5}\right)^4\left(\frac{1}{320}\right)^5\left(\frac{16651}{25}\right)^5\right)\\
\cong&\frac{1}{5}\ln\frac{1279975}{1280000}\,,
\end{aligned}$$

we obtain $f(b)<0$ for $\frac{1}{4}<b<\frac{2}{5}$. ∎

Lemma 3.5. *We have*

$$\frac{2447+5016b-7020b^2}{1600}<b^{-2b}$$

for $\frac{2}{5}<b<\frac{9}{20}$.

Proof. From $2447+5016b-7020b^2>\frac{20509}{20}+5016b$, we can take the logarithm and we set

$$\begin{aligned}
f(b)=&\ln\frac{2447+5016b-7020b^2}{1600}-\ln b^{-2b}\\
=&\ln\left(2447+5016b-7020b^2\right)-\ln 1600+2b\ln b\,.
\end{aligned}$$

The derivatives of $f(b)$ are

$$f'(b)=\frac{2(4955-2004b-7020b^2)}{2447+5016b-7020b^2}+2\ln b$$

and

$$f''(b) = \frac{2g(b)}{b(-2447 - 5016b + 7020b^2)^2},$$

where $g(b) = 5987809 - 5209764b + 26016696b^2 - 119705040b^3 + 49280400b^4$. The derivative of $g(b)$ is

$$\begin{aligned} g'(b) =& - 5209764 + 52033392b - 359115120b^2 + 197121600b^3 \\ <& - 5209764 + 52033392b - 359115120b^2 + 197121600b^2 \left(\frac{9}{20}\right) \\ =& - 5209764 + 52033392b - 270410400b^2 \\ <& - 5209764 + 52033392b - 270410400b \left(\frac{2}{5}\right) \\ =& - 5209764 - 56130768b\,. \end{aligned}$$

Hence, $g'(b) < 0$ for $\frac{2}{5} < b < \frac{9}{20}$ and $g(b)$ is strictly decreasing for $\frac{2}{5} < b < \frac{9}{20}$. By $g(\frac{9}{20}) = \frac{9791509}{400}$, $g(b) > 0$ for $\frac{2}{5} < b < \frac{9}{20}$ and $f'(b)$ is strictly increasing for $\frac{2}{5} < b < \frac{9}{20}$. $f'(\frac{2}{5}) = \frac{30302}{16651} - 2\ln\frac{5}{2} \cong -0.0127508$ and $f'(\frac{9}{20}) = \frac{15038}{9379} - 2\ln\frac{20}{9} \cong 0.00635384$, there exists a unique real number b_0 such that $f'(b) < 0$ for $\frac{2}{5} < b < b_0$ and $f'(b) > 0$ for $b_0 < a < \frac{9}{20}$. Hence, $f(b)$ is strictly decreasing for $\frac{2}{5} < b < b_0$ and strictly increasing for $b_0 < b < \frac{9}{20}$. Since we have

$$\begin{aligned} f\left(\frac{2}{5}\right) =& -\frac{4}{5}\ln\left(\frac{5}{2}\right) - \ln 1600 + \ln\frac{16651}{5} = \frac{1}{5}\ln\left(\left(\frac{2}{5}\right)^4\left(\frac{1}{1600}\right)^5\left(\frac{16651}{5}\right)^5\right) \\ \cong& \ln\frac{1279975}{1280000} \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{9}{20}\right) =& -\frac{9}{10}\ln\frac{20}{9} - \ln 1600 + \ln\frac{65653}{20} = \frac{1}{10}\ln\left(\left(\frac{9}{20}\right)^9\left(\frac{1}{1600}\right)^{10}\left(\frac{65653}{20}\right)^{10}\right) \\ \cong& \ln\frac{5764058}{5764607}, \end{aligned}$$

we obtain $f(b) < 0$ for $\frac{2}{5} < b < \frac{9}{20}$. ∎

Lemma 3.6. *We have*

$$\frac{65693 + 106324b - 155420b^2}{40000} < b^{-2b}$$

for $\frac{9}{20} < b < \frac{1}{2}$.

Proof. From $65693 + 106324b - 155420b^2 > 26838 + 106324b$, we can take the logarithm and we set

$$\begin{aligned} f(b) =& \ln\frac{65693 + 106324b - 155420b^2}{40000} - \ln b^{-2b} \\ =& \ln(65693 + 106324b - 155420b^2) - \ln 40000 + 2b\ln b\,. \end{aligned}$$

The derivatives of $f(b)$ are

$$f'(b) = \frac{2(118855 - 49096b - 155420b^2)}{65693 + 106324b - 155420b^2} + 2\ln b$$

and

$$f''(b)=\frac{2g(b)}{b(-65693-106324b+155420b^2)^2}\,,$$

where $g(b)=4315570249-1892917484b+7409656936b^2-57205128560b^3+24155376400b^4$. The derivatives of $g(b)$ are

$$\begin{aligned}g'(b)=&-1892917484+14819313872b-171615385680b^2+96621505600b^3\\<&-1892917484+14819313872b-171615385680b^2+96621505600b^2\left(\frac{1}{2}\right)\\=&-1892917484+14819313872b-123304632880b^2\\<&-1892917484+14819313872b-123304632880b\left(\frac{9}{20}\right)\\=&-1892917484-40667770924b\,.\end{aligned}$$

Hence, $g'(b)<0$ for $\frac{9}{20}<b<\frac{1}{2}$ and $g(b)$ is strictly decreasing for $\frac{9}{20}<b<\frac{1}{2}$. By $g(\frac{9}{20})=\frac{59353357937}{80}$ and $g(\frac{1}{2})=-419404304$, there exists a unique real number b_0 such that $g(b)>0$ for $\frac{9}{20}<b<b_0$ and $g(b)<0$ for $b_0<a<\frac{1}{2}$. Therefore, $f'(b)$ is strictly increasing for $\frac{9}{20}<b<b_0$ and strictly decreasing for $b_0<b<\frac{1}{2}$. $f'(\frac{9}{20})=\frac{522314}{328265}-2\ln\frac{20}{9}\cong-0.00588018$ and $f'(\frac{1}{2})=\frac{13863}{10000}-2\ln 2\cong 5.63888\cdot 10^{-6}$, there exists a unique real number b_1 such that $f'(b)<0$ for $\frac{9}{20}<b<b_1$ and $f'(b)>0$ for $b_1<b<\frac{1}{2}$. Hence, $f(b)$ is strictly decreasing for $\frac{9}{20}<b<b_1$ and strictly increasing for $b_1<b<\frac{1}{2}$. Since we have

$$\begin{aligned}f\left(\frac{9}{20}\right)&=\ln\frac{65653}{32000}-\frac{9}{10}\ln\frac{20}{9}=\frac{1}{10}\ln\left(\left(\frac{9}{20}\right)^9\left(\frac{65653}{32000}\right)^{10}\right)\\&\cong\frac{1}{10}\ln\frac{576405}{576460}\end{aligned}$$

and $f\left(\frac{1}{2}\right)=0$, we obtain $f(b)<0$ for $\frac{9}{20}<b<\frac{1}{2}$. ∎

Proof of Theorem 1.2. From Lemmas 3.1 and 3.6, for $0<a<\frac{1}{20}$, we have

$$\begin{aligned}a^{2b}+b^{2a}=&\,a^{1-2a}+b^{1-2b}\\>&\,\frac{a(38618000000+298677429103a-576629716412a^2)}{38624000000}\\&+\frac{(\frac{1}{2}-a)\left(65693+106324(\frac{1}{2}-a)-155420(\frac{1}{2}-a)^2\right)}{40000}=\frac{f(a)}{38624000000}\,,\end{aligned}$$

where $f(a)=38624000000-14926451200a+176233555503a^2-426556164412a^3$. The derivative of $f(a)$ is $f'(a)=-14926451200+352467111006a-1279668493236a^2$ and the function $f'(a)$ is convex upwards and takes the maximum value at $a=\frac{58744518501}{426556164412}\cong 0.137718$, so we have $f'(a)<f'\left(\frac{1}{20}\right)=-\frac{50226688279}{100}$. Therefore, we have $f(a)$ is strictly decreasing for $0<a<\frac{1}{20}$. From $f(\frac{1}{20})=\frac{19132470904103}{500}$, we have

$$a^{2b}+b^{2a}>\frac{19132470904103}{500\cdot 38624000000}\cong 0.990704$$

for $0 < a < \frac{1}{20}$. From Lemmas 3.2 and 3.5, for $\frac{1}{20} < a < \frac{1}{10}$, we have

$$\begin{aligned} a^{2b} + b^{2a} &= a^{1-2a} + b^{1-2b} \\ &> \frac{a(106503811 + 617083346a - 972394360a^2)}{100000000} \\ &\quad + \frac{(\frac{1}{2} - a)\left(2447 + 5016(\frac{1}{2} - a) - 7020(\frac{1}{2} - a)^2\right)}{1600} = \frac{g(a)}{100000000}, \end{aligned}$$

where $g(a) = 100000000 - 30871189a + 272458346a^2 - 533644360a^3$. The derivative of $g(a)$ is $g'(a) = -30871189 + 544916692a - 1600933080a^2$. Since we have $g'(\frac{136229173 \pm \sqrt{6202710654005899}}{800466540}) = 0$ and $\frac{136229173 - \sqrt{6202710654005899}}{800466540} \cong 0.071798$, we have

$$\begin{aligned} g(a) &\geq g\left(\frac{136229173 - \sqrt{6202710654005899}}{800466540}\right) \\ &= \frac{48059389442734184398136932 - 6202710654005899\sqrt{6202710654005899}}{480560011244678700}. \end{aligned}$$

By $6202710654005899 < 6202710728144100 = 78757290^2$, we can get

$$\begin{aligned} a^{2b} + b^{2a} &> \frac{48059389442734184398136932 - 6202710654005899\sqrt{6202710654005899}}{480560011244678700 \cdot 100000000} \\ &> \frac{48059389442734184398136932 - 6202710654005899 \cdot 78757290}{480560011244678700 \cdot 100000000} \\ &= \frac{237854403804852760744441611}{240280005622339350} \cong 0.989905 \end{aligned}$$

for $\frac{1}{20} < a < \frac{1}{10}$. From Lemmas 3.3 and 3.4, for $\frac{1}{10} < a < \frac{1}{4}$, we have

$$\begin{aligned} a^{2b} + b^{2a} &= a^{1-2a} + b^{1-2b} \\ &> \frac{a(6626567 + 33716264a - 48890128a^2)}{6000000} \\ &\quad + \frac{(\frac{1}{2} - a)\left(423 + 1302(\frac{1}{2} - a) - 1736(\frac{1}{2} - a)^2\right)}{320} = \frac{h(a)}{6000000}, \end{aligned}$$

where $h(a) = 6000000 - 1304683a + 9303764a^2 - 16340128a^3$. The derivative of $h(a)$ is $h'(a) = -1304683 + 18607528a - 49020384a^2$ and the function $h'(a)$ is convex upwards and takes the minimum value at $a = \frac{1}{10}$ or $a = \frac{1}{4}$. By $h'(\frac{1}{10}) = \frac{1646649}{25}$ and $h'(\frac{1}{4}) = 283425$, we have $h'(a) > 0$ for $\frac{1}{10} < a < \frac{1}{4}$. Since $h(a)$ is strictly increasing for $\frac{1}{10} < a < \frac{1}{4}$ and $h(\frac{1}{10}) = \frac{1486557303}{250}$, we have

$$a^{2b} + b^{2a} > \frac{1486557303}{250 \cdot 6000000} = \frac{495519101}{500000000} \cong 0.991038$$

for $\frac{1}{10} < a < \frac{1}{4}$. Therefore, for $0 < a < \frac{1}{4}$, we obtain

$$a^{2b} + b^{2a} > \frac{237854403804852760744441611}{240280005622339350} \cong 0.989905.$$

Therefore, the proof of Theorem 1.2 is complete. ∎

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