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LEBESGUE ABSOLUTELY CONTINUOUS FUNCTIONS TAKING VALUES ON $C[a, b]$

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ABSTRACT. In this paper a version of absolutely continuous function using Lebesgue partition instead of Riemann partition over the field of $\mathcal{C}[a, b]$ will be introduced and we call this function as Lebesgue absolutely continuous function on $[f, g]$. Moreover, some properties of Lebesgue absolutely continuous function on $[f, g]$ will be presented together with its relationship with the other continuous functions.

Key words and phrases: $\mathcal{C}[a, b]$ space-valued function; Lebesgue Absolutely Continuous Function; Continuity; Lipschitz Continuous.

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1. INTRODUCTION

Absolute continuous function had been studied by Vitali in 1905 where he gave a definition of absolute continuity for a class of functions. In 2018, [5] Ali and Adeeb introduced the Lebesgue absolutely continuous function using Lebesgue partition and study some of its properties. In this paper we change the way to define the domain of the function by choosing a closed interval $[f, g]$ with $f, g \in \mathcal{C}[a, b]$.

2. PRELIMINARIES

Throughout, we consider the space $\mathcal{C}[a, b]$ of all continuous real-valued functions defined on $[a, b]$. For more details of the space $\mathcal{C}[a, b]$, see [2], [3] or [4].

Let $[f, g]$ be a closed interval of $\mathcal{C}[a, b]$. A **partition** of $[f, g]$ is any finite set $\{h_0, h_1, \dots, h_n\} \subset [f, g]$ such that

$$h_0 = f,\ h_n = g \text{ and } h_{i-1} < h_i$$

for all $i = 1, 2, \dots, n$.

Definition 2.1. [5] A real-valued function f which is defined on a closed, bounded interval $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ provided that for each $\epsilon > 0$, there exists $\delta > 0$ such that for every finite disjoint collection of open intervals $(x_i, y_i)_{i=1}^n$ in $[a, b]$, if $\sum_{i=1}^n |y_i - x_i| < \delta$, then

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon.$$

The set of all absolutely continuous functions on $[a, b]$ is denoted by $AC[a, b]$. Another definition of absolutely continuous functions is given using Lebesgue integrable functions.

Theorem 2.1. [5] (Cousin's Lemma) *A function $f(x)$ is said to be absolutely continuous in $[a, b]$ if $f(x) \in L^1$, where L^1 is the set of all Lebesgue integrable functions satisfying*

$$\int_a^b |f(x)|\, dx < \infty.$$

An absolutely continuous function has a relationship with the other kind of functions. For example, an absolutely continuous function is continuous and uniformly continuous.

Definition 2.2. [5] A real-valued function f which is defined on a closed, bounded interval $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that for every finite Lebesgue partition $\{A_i\}_{i=1}^n$ on $[a, b]$, if $\sum_{i=1}^n m(A_i) < \delta$, then

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon.$$

The set of all absolutely continuous functions on [a,b] denote by $LAC[a, b]$.

Remark 2.1. [5] Any Riemann partition of closed, bounded interval is Lebesgue partition. Then every absolutely continuous function is Lebesgue absolutely continuous function in that interval.

Proposition 2.2. [5] Lebesgue absolutely continuous function $f : [a, b] \to \mathbb{R}$ on $[a, b]$ is continuous on $[a, b]$.

Proposition 2.3. [5] Lebesgue absolutely continuous function $f : [a, b] \to \mathbb{R}$ on $[a, b]$ is uniformly continuous on $[a, b]$.

3. THE SPACE $\mathcal{C}[a,b]$

Denote the set of all real-valued continuous functions on $[a,b]$ by $\mathcal{C}[a,b]$; that is,

$$\mathcal{C}[a,b] = \{f \mid f : [a,b] \to \mathbb{R} \text{ is continuous on } [a,b]\}.$$

We denote the zero function in $\mathcal{C}[a,b]$ by θ. If $f, g \in \mathcal{C}[a,b]$ we define

$$f \le g \Leftrightarrow f(x) \le g(x),\ f < g \Leftrightarrow f(x) < g(x),\ \text{and } f = g \Leftrightarrow f(x) = g(x)$$

for every $x \in [a,b]$. Also $\mathcal{C}[a,b]$ is a Banach space, see [3], with norm defined by

$$|f| = \max_{x\in[a,b]} |f(x)|.$$

Ubaidillah et. al. [4] showed that the space $\mathcal{C}[a,b]$ is a commutative Riesz algebra with e as its unit element, that is, if $f \le g$ and the relation "$\le$" is a partial ordering in $\mathcal{C}[a,b]$, we have the following:

- $f \le g \Rightarrow f + h \le g + h$ for every $h \in \mathcal{C}[a,b]$
- $f \le g \Rightarrow \alpha f \le \alpha g$ for every $\alpha \in \mathbb{R}^+$
- $(fg)(x) = f(x)g(x)$ for every $x \in [a,b]$.

Definition 3.1. [4] We say that $f, g \in \mathcal{C}[a,b]$ are $\mathbf{comparable}$ if $f \le g$ or $g \le f$. If neither $f \le g$ nor $g \le f$, then f and g are $\mathbf{non-comparable}$.

Definition 3.2. [4] Let $f, g \in \mathcal{C}[a,b]$ with $f \le g$. We define the following:

- $(f,g) = \{h \in \mathcal{C}[a,b] : f < h < g\}$, is called an $\mathbf{open\,interval}$;
- $[f,g] = \{h \in \mathcal{C}[a,b] : f \le h \le g\}$, is called a $\mathbf{closed\,interval}$;
- $[f,g) = \{h \in \mathcal{C}[a,b] : f \le h < g\}$, is called $\mathbf{half-closed\,half-open\ interval}$; and
- $(f,g] = \{h \in \mathcal{C}[a,b] : f < h \le g\}$, is called $\mathbf{half-open\,half-closed\ interval}$.

For $h, k \in \mathcal{C}[a,b]$, we define $\dfrac{h}{k}$, $h \vee k$, $h \wedge k$ and $|h|$ as follows :

$$\begin{aligned}
\left(\frac{h}{k}\right)(x) &= \frac{h(x)}{k(x)}, \quad \text{for all } x \in [a,b],\ k(x) \neq 0, \\
(h \vee k)(x) &= \sup\{h(x), k(x)\}, \quad \text{for all } x \in [a,b]. \\
(h \wedge k)(x) &= \inf\{h(x), k(x)\}, \quad \text{for all } x \in [a,b]. \\
|h|(x) &= |h(x)|, \quad \text{for all } x \in [a,b].
\end{aligned}$$

Definition 3.3. A subset $S \subset \mathcal{C}[a,b]$ is said to be $\mathbf{bounded}$ if there exists $K > 0$ such that for all $h \in S$,

$$|h| \le K \cdot e.$$

Definition 3.4. A sequence $\{f_n\}$ of elements of $\mathcal{C}[a,b]$ is said to be $\mathbf{convergent}$ to $f \in \mathcal{C}[a,b]$ if for every $\epsilon > 0$ there is a positive integer K such that for every $n \ge K$, the terms f_n satisfy

$$|f_n - f| < \epsilon \cdot e.$$

A sequence $\{f_n\}$ which converges to f in $\mathcal{C}[a,b]$, will be written

$$f_n \to f \qquad\qquad n \to \infty.$$

Definition 3.5. A subset $S \subset \mathcal{C}[a,b]$ is said to be $\mathbf{closed}$ if for all $h \in S$, there exists a sequence $\{h_n\}$ in S such that $h_n \to h$.

4. LEBESGUE ABSOLUTELY CONTINUOUS FUNCTION ON $\mathcal{C}[a, b]$

In this section the definition of Lebesgue Absolutely Continuous Function on $[f, g]$ is presented.

Definition 4.1. A function $F : [f, g] \to \mathcal{C}[a, b]$ which is defined on a closed, bounded interval $[f, g]$ is said to be **Lebesgue Absolutely Continuous** on $[f, g]$ if for each $\epsilon > 0$, there exists $\delta(h) > \theta$ such that for any finite Lebesgue partition $\{A_i\}_{i=1}^n$ on $[f, g]$, if $\sum_{i=1}^n m(A_i) < \delta(h)$, then

$$\sum_{\substack{i=1 \\ h_i, h_{i-1} \in A_i}}^{n} |F(h_i) - F(h_{i-1})| < \epsilon \cdot e.$$

The set of all absolutely continuous functions on $[f, g]$ will be denoted by $LAC[f, g]$.

Remark 4.1. Any Lebesgue partition of closed, bounded interval on $[a, b]$ is Lebesgue partition on $[f, g]$. Then every absolutely continuous function on $[f, g]$ is Lebesgue absolutely continuous function in that interval.

5. LEBESGUE ABSOLUTELY CONTINUOUS FUNCTION ON $[f, g]$ AND OTHER CONTINUOUS FUNCTIONS ON $[f, g]$

In this section we study a relationship between Lebesgue absolutely continuous function on $[f, g]$, continuous function $[f, g]$, uniformly continuous function on $[f, g]$ and Lipschitz function on $[f, g]$.

Definition 5.1. A function $F : [f, g] \to \mathcal{C}[a, b]$ is **continuous** at $h_0 \in [f, g]$, if for any $\epsilon > 0$ there exists $\delta_0(h_0) : [a, b] \to \mathbb{R}^+$ such that whenever $h \in [f, g]$ with $|h - h_0| < \delta_0(h_0)$, we have

$$\big|F(h) - F(h_0)\big| < e \cdot \epsilon.$$

F is said to be **uniformly continuous** on $[f, g]$, if for any $\epsilon > 0$ there exists $\delta : [a, b] \to \mathbb{R}^+$ such that whenever $h, h' \in [f, g]$ with $|h' - h| < \delta$, we have

$$\big|F(h') - F(h)\big| < e \cdot \epsilon.$$

Proposition 5.1. Lebesgue Absolutely Continuous Function $F : [f, g] \to \mathcal{C}[a, b]$ on $[f, g]$ is continuous on $[f, g]$.

Proof Let $\epsilon > 0$ be given. Suppose that a function $F : [f, g] \to \mathcal{C}[a, b]$ is Lebesgue absolutely continuous on $[f, g]$. Then we choose $\delta(\epsilon, h) > \theta$ such that

$$\sum_{\substack{i=1 \\ h_i, h_{i-1} \in A_i}}^{n} |F(h_i) - F(h_{i-1})| < \epsilon \cdot e \qquad \text{whenever} \quad \sum_{i=1}^{n} m(A_i) < \delta(\epsilon, h),$$

for any finite Lebesgue partition $\{A_i\}_{i=1}^n$ on $[f, g]$. Take $n = 1$. Then for $h_1 \in [f, g]$, we have

$$|F(h_1) - F(h_0)| < \epsilon \cdot e$$

whenever $|h_1 - h_0| \leq m(A_1) < \delta(\epsilon, h)$. Hence, $F(h)$ is continuous $[f, g]$. ∎

Theorem 5.2. Let $[f, g]$ be a closed bounded interval and let $F : [f, g] \to \mathcal{C}[a, b]$ be continuous on $[f, g]$. Then F is uniformly continuous on $[f, g]$.

Proof Let $\epsilon > 0$ be given. Since F is continuous at $h_0 \in [f, g]$, there exists $\delta_0(h_0) > \theta$ such that whenever $h \in [f, g]$ with $|h - h_0| \leq 2\delta_0(h_0)$, then

$$|F(h) - F(h_0)| < \frac{\epsilon}{2} \cdot e.$$

Thus, δ_0 is a gauge on $[f,g]$. By Cousin's Lemma, there exists a δ_0-fine tagged division $\mathcal{D} = \left\{([h_{i-1},h_i],t_i)\right\}_{i=1}^{n}$ on $[f,g]$. Let $\delta = \min\{\delta_0(t_1)\ldots,\delta_0(t_n)\}$. Now, suppose that $h,h' \in [f,g]$ with $|h-h'| \leq \delta$. Then $|h-h'| \leq \delta_0(t_i)$, for all i. This means that there exists $i \in \{1,\ldots,n\}$ such that $h_{i-1} \leq h \leq h_i$ with $|h-t_i| < \delta_0(t_i)$. Since

$$|h'-t_i| \leq |h'-h| + |h-t_i| \leq \delta + \delta_0(t_i) \leq 2\delta_0(t_i),$$

it follows that

$$|F(h')-F(t_i)| < \frac{\epsilon}{2}\cdot e \quad \text{and} \quad |F(h)-F(t_i)| < \frac{\epsilon}{2}\cdot e.$$

Hence,

$$|F(h)-F(h')| \leq |F(h)-F(t_i)| + |F(t_i)-F(h')| \leq \frac{\epsilon}{2}\cdot e + \frac{\epsilon}{2}\cdot e = \epsilon e.$$

Therefore, F is uniformly continuous on $[f,g]$. ∎

Proposition 5.3. Lebesgue Absolutely Continuous Function $F : [f,g] \to \mathcal{C}[a,b]$ on $[f,g]$ is uniformly continuous on $[f,g]$.

Proof follows from Theorem 5.2 and Proposition 5.1. ∎

Theorem 5.4. A function $F : [f,g] \to \mathcal{C}[a,b]$ is $\mathbf{Lipschitz\,Continuous}$ on $[f,g]$ if there exists $K < \infty$ such that

$$|F(h_i) - F(h_{i-1})| \leq K|h_i - h_{i-1}|.$$

Proof Suppose that $F : [f,g] \to \mathcal{C}[a,b]$ is $\mathbf{Lipschitz\,Continuous}$ on $[f,g]$ with $K > 0$. Then given $\epsilon > 0$, choose $\delta(h,\epsilon) = \frac{\epsilon}{K}\cdot e$. Now, by Definition 4.1, for any finite Lebesgue partition $\{A_i\}_{i=1}^n$ on $[f,g]$ such that $\sum_{i=1}^{n} m(A_i) < \delta(h)$, we have

$$\begin{aligned}\sum_{\substack{i=1\\h_i,h_{i-1}\in A_i}}^{n} |F(h_i)-F(h_{i-1})| &< K\sum_{i=1}^{n} |h_i - h_{i-1}| \\ &< K\sum_{i=1}^{n} m(A_i) \\ &< K\cdot\delta(h) \\ &= \epsilon\cdot e.\end{aligned}$$

Moreover, $F(h)$ is Lebesgue absolutely continuous on $[f,g]$. ∎

Remark 5.1. A function $F : [f,g] \to \mathcal{C}[a,b]$ which is Lebesgue absolute continuous on $[f,g]$ may not be Lipschitz continuous on $[f,g]$.

Theorem 5.5. Let $\{F_k\}_{k=1}^n$ be a sequence of Lebesgue absolutely continuous functions on $[f,g]$. Then the functions $\max\{F_1,F_2,\ldots,F_k\}$ and $\min\{F_1,F_2,\ldots,\,F_k\}$ are also Lebesgue absolutely continuous function on $[f,g]$.

Proof Let $F_i, F_j \in \{F_k\}_{k=1}^n$. Then given $\epsilon_1,\epsilon_2 > 0$, choose $\delta_i(h),\delta_j(h) > \theta$ such that for any finite Lebesgue partition $\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n$ on $[f,g]$, we have

$$\sum_{\substack{i=1\\h_i,h_{i-1}\in A_i}}^{n} |F_i(h_i)-F_i(h_{i-1})| < \epsilon_1\cdot e \quad \text{and} \quad \sum_{\substack{i=1\\h_i,h_{i-1}\in B_i}}^{n} |F_j(h_i)-F_j(h_{i-1})| < \epsilon_2\cdot e$$

whenever

$$\sum_{i=1}^{n} m(A_i) < \delta_i(h) \qquad \text{and} \qquad \sum_{i=1}^{n} m(B_i) < \delta_j(h).$$

Define $\delta(h) = \min\{\delta_i(h), \delta_j(h)\}$ such that $\sum_{i=1}^{n} m(C_i) < \delta(h)$, where $\{C_i\}_{i=1}^{n}$ is a refinement Lebesgue partition on $[f,g]$ for $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$. Thus $m(C_i) < m(A_i)$ and $m(C_i) < m(B_i)$ showing that

$$\sum_{i=1}^{n} m(C_i) < \sum_{i=1}^{n} m(A_i) \leq \delta(h) < \delta_i(h)$$

and

$$\sum_{i=1}^{n} m(C_i) < \sum_{i=1}^{n} m(B_i) < \delta(h) \leq \delta_j(h).$$

Now, let $F_p = \min\{F_1, F_2\}$ and consider

$$\sum_{\substack{i=1\\ h_i, h_{i-1} \in C_i}} |F_p(h_i) - F_p(h_{i-1})| < \epsilon \cdot e \qquad \text{whenever} \qquad \sum_{i=1}^{n} m(C_i) < \delta(h).$$

Hence, the $\min\{F_1, F_2, \ldots, F_k\}$ is Lebesgue absolutely continuous on $[f,g]$.
Similarly, $\delta'(h) = \max\{\delta_1(h), \delta_2(h)\}$ such that $\sum_{i=1}^{n} m(D_i) < \delta'(h)$, where $\{D_i\}_{i=1}^{n}$ is a refinement Lebesgue partition on $[f,g]$ for $\{A_i\}_{i=1}^{n}$, $\{B_i\}_{i=1}^{n}$. Thus $m(A_i) < m(D_i)$ and $m(B_i) < m(D_i)$ showing that

$$\sum_{i=1}^{n} m(A_i) < \sum_{i=1}^{n} m(D_i) < \delta_i(h) \leq \delta'(h)$$

and

$$\sum_{i=1}^{n} m(B_i) < \sum_{i=1}^{n} m(D_i) < \delta_j(h) \leq \delta'(h).$$

Now, let $F_q = \max\{F_1, F_2\}$ and consider

$$\sum_{\substack{i=1\\ h_i, h_{i-1} \in D_i}} |F_q(h_i) - F_q(h_{i-1})| < \epsilon \cdot e \quad \text{whenever} \quad \sum_{i=1}^{n} m(D_i) < \delta'(h).$$

Hence, the $\max\{F_1, F_2, \ldots, F_k\}$ is Lebesgue absolutely continuous on $[f,g]$. ∎

6. PROPERTIES OF LEBESGUE ABSOLUTE CONTINUOUS FUNCTION

In this section, a version of some algebraic properties of Lebesgue absolutely continuous functions taking values on $\mathcal{C}[a,b]$ will be presented.

Theorem 6.1. Let $F, G : [f,g] \to \mathcal{C}[a,b]$ be Lebesgue absolutely continuous functions on $[f,g]$. Then:

(1) $F(h) + G(h)$ is Lebesgue absolutely continuous functions on $[f,g]$;
(2) $\beta F(h)$ is Lebesgue absolutely continuous functions on $[f,g]$, β is scalar in $\mathbb{R}$;
(3) $F(h) - G(h)$ is Lebesgue absolutely continuous functions on $[f,g]$;
(4) $|F(h)|$ is Lebesgue absolutely continuous functions on $[f,g]$;
(5) $FG(h)$ is Lebesgue absolutely continuous functions on $[f,g]$.

Proof (1) Let $F(h)$ and $G(h)$ be Lebesgue absolutely continuous functions on $[f,g]$. Then given any $\epsilon > 0$, there exist $\delta_F(h) > \theta$ and $\delta_G(h) > \theta$ such that for any finite Lebesgue partition $\{A_i\}_{i=1}^{n}, \{B_i\}_{i=1}^{n}$ on $[f,g]$, we have

$$\sum_{\substack{i=1\\ h_i, h_{i-1} \in D_i}} |F(h_i) - F(h_{i-1})| < \frac{\epsilon}{2} \cdot e \quad \text{and} \quad \sum_{\substack{i=1\\ h_i, h_{i-1} \in D_i}} |G(h_i) - G(h_{i-1})| < \frac{\epsilon}{2} \cdot e$$

whenever $\sum_{i=1}^{n} m(A_i) < \delta_F(h)$ and $\sum_{i=1}^{n} m(B_i) < \delta_G(h)$. Define $\delta(h) = \min\{\delta_F(h), \delta_G(h)\}$ such that $\sum_{i=1}^{n} m(C_i) < \delta(h)$, where $\{C_i\}_{i=1}^{n}$ is a refinement Lebesgue partition on $[f, g]$ for $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$. Thus

$$m(C_i) \leq m(A_i) \qquad \text{and} \qquad m(C_i) \leq m(B_i)$$

showing that

$$\sum_{i=1}^{n} m(C_i) \leq \sum_{i=1}^{n} m(A_i) < \delta(h) \leq \delta_F(h)$$

and

$$\sum_{i=1}^{n} m(C_i) \leq \sum_{i=1}^{n} m(B_i) < \delta(h) \leq \delta_G(h).$$

Let $H(h) = F(h) + G(h)$. Now

$$\begin{aligned} \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |H(h_i) - H(h_{i-1})| &= \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |(F(h_i) + G(h_i)) - (F(h_{i-1}) \\ &\qquad + G(h_{i-1}))| \\ &= \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |(F(h_i) - F(h_{i-1})) \\ &\qquad - (G(h_i) - G(h_{i-1}))| \\ &\leq \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |F(h_i) - F(h_{i-1})| \\ &\qquad + \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |G(h_i) - G(h_{i-1})| \\ &< \frac{\epsilon}{2} \cdot e + \frac{\epsilon}{2} \cdot e = \epsilon \cdot e. \end{aligned}$$

Hence, $F(h) + G(h)$ is Lebesgue absolutely continuous functions on $[f, g]$.

(2) If $\beta = 0$, then we are done. Suppose that $\beta \neq 0$. Then $|\beta| > 0$. Now, assume that $F : [f, g] \to \mathcal{C}[a, b]$ is a Lebesgue absolutely continuous on $[f, g]$. Given any $\epsilon > 0$, choose $\delta(h) > \theta$ such that for any finite Lebesgue partition $\{A_i\}_{i=1}^{n}$ on $[f, g]$ whenever $\sum_{i=1}^{n} m(A_i) < \delta(h)$, we have

$$\sum_{\substack{i=1 \\ h_i, h_{i-1} \in A_i}} |F(h_i) - F(h_{i-1})| < \frac{\epsilon}{|\beta|} \cdot e.$$

Now,

$$\begin{aligned} \sum_{\substack{i=1 \\ h_i, h_{i-1} \in A_i}} |\beta F(h_i) - F(h_{i-1})| &\leq |\beta| \sum_{\substack{i=1 \\ h_i, h_{i-1} \in A_i}} |F(h_i) - F(h_{i-1})| \\ &< |\beta| \cdot \frac{\epsilon}{|\beta|} \cdot e \\ &= \epsilon \cdot e. \end{aligned}$$

Hence, $\beta F(h)$ is Lebesgue absolutely continuous function on $[f, g]$.

(3) Since $G(h)$ is Lebesgue absolutely continuous function on $[f, g]$, by (2), $(-1)G(h) = -G(h)$ is also a Lebesgue absolutely continuous function on $[f, g]$. By (1), $F(h) + (-G(h)) = F(h) - G(h)$ is again a Lebesgue absolutely continuous function on $[f, g]$.

(4) Since $F(h)$ is Lebesgue absolutely continuous function on $[f, g]$, by Theorem 5.5 it follows that $F^+(h) = \max\{F(h), \theta\}$ and $F^-(h) = \max\{-F(h), \theta\}$ are also Lebesgue absolutely continuous functions on $[f, g]$. Hence by (1), $|F(h)| = F^+(h) + F^-(h)$ is again a Lebesgue absolutely continuous on $[f, g]$.

(5) Since $F(h)$ and $G(h)$ are Lebesgue absolutely continuous functions on $[f, g]$, given any $\epsilon > 0$, there exists $\delta_F(h), \delta_G(h) > \theta$ such that for any finite Lebesgue partition $\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n$, on $[f, g]$, we have

$$\sum_{\substack{i=1 \\ h_i, h_{i-1} \in A_i}} |F(h_i) - F(h_{i-1})| < \frac{\epsilon}{2} \cdot e \quad \text{and} \quad \sum_{\substack{i=1 \\ h_i, h_{i-1} \in B_i}} |G(h_i) - G(h_{i-1})| < \frac{\epsilon}{2} \cdot e$$

whenever

$$\sum_{i=1}^n m(A_i) < \delta_F(h) \quad \text{and} \quad \sum_{i=1}^n m(B_i) < \delta_G(h).$$

Define $\delta(h) = \min\{\delta_F(h), \delta_G(h)\}$ such that $\sum_{i=1}^n m(C_i) < \delta(h)$, where $\{C_i\}_{i=1}^n$ is a refinement Lebesgue partition on $[a, b]$ for $\{A_i\}_{n=1}^n$ and $\{B_i\}_{i=1}^n$. Thus $m(C_i) \leq m(A_i)$ and $m(C_i) \leq m(B_i)$ showing that

$$\sum_{i=1}^n m(C_i) \leq \sum_{i=1}^n m(A_i) < \delta(h) \leq \delta_F(h)$$

and

$$\sum_{i=1}^n m(C_i) \leq \sum_{i=1}^n m(B_i) < \delta(h) \leq \delta_G(h).$$

Now, let $H(h) = F(h)G(h)$. Now

$$\begin{aligned}
\sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |H(h_i) - H(h_{i-1})| &= \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |(FG)(h_i) - (FG)(h_{i-1})| \\
&= \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |F(h_i)G(h_i) - F(h_{i-1})G(h_i) + F(h_{i-1})G(h_i) \\
&\qquad - F(h_{i-1})G(h_{i-1})| \\
&\leq |G(h_i)| \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |F(h_i) - F(h_{i-1})| \\
&\qquad + |F(h_{i-1})| \sum_{\substack{i=1 \\ h_i, h_{i-1} \in C_i}} |G(h_i) - G(h_{i-1})| \\
&\leq M_1 \cdot \frac{\epsilon}{2} \cdot e + M_2 \cdot \frac{\epsilon}{2} \cdot e \\
&= \frac{\epsilon}{2} \cdot (M_1 + M_2) \cdot e.
\end{aligned}$$

Since $F(h)$ and $G(h)$ are continuous on $[f, g]$, there exists $M_1, M_2 > 0$ such that $|F(h)| \leq M_1 \cdot e$ and $|G(h)| \leq M_2 \cdot e$. Therefore, $FG(h)$ is Lebesgue absolutely continuous function on $[f, g]$. ∎

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