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TWO FURTHER METHODS FOR DERIVING FOUR RESULTS CONTIGUOUS TO KUMMER'S SECOND THEOREM

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ABSTRACT. In the theory of generalized hypergeometric function, transformation and summation formulas play a key role. In particular, in one of the Kummer's transformation formulas, Kim, et al. in 2012, have obtained ten contiguous results in the form of a single result with the help of generalization of Gauss's second summation theorem obtained earlier by Lavoie, et al.. In this paper, we aim at presenting four of such results by the technique of contiguous function relations and integral method developed by MacRobert.

Key words and phrases: Generalized hypergeometric function; Transformation formulas; Summation formulas; Contiguous results.

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1. INTRODUCTION

The term 'hypergeometric series' was first used by John Wallis (1655) in his book 'Arithmetica Infinitorum'. Hypergeometric series were studied by the mathematician Euler but a systematic treatment was given by the famous mathematician Gauss [4] in 1812. He defined his series as follows:

$$1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots$$

Here the parameters a, b and c may be real or complex numbers with an exception that c should not be zero or a negative integer. Also, z is called the variable of the series.

The above series is called Gauss series or simply the ordinary hypergeometric series and is usually represented by the symbol ${}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix}; z\right]$ and is known as the hypergeometric function.

Thus we have

$$(1.1)\qquad {}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix}; z\right]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z}{n!},$$

where $(a)_n$ denotes the well-known Pochhammer symbol (or the raised factorial or the shifted factorial, since $(1)_n = n!$) defined for any complex number $a(\neq 0)$ by

$$(a)_n=\begin{cases}a(a+1)\dots(a+n-1), & n\in\mathbb{N}\\ 1, & n=0.\end{cases}$$

From the definition, it is clear that:

(i) ${}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix}; z\right]$ is symmetric in the numerator parameters a and b.
(ii) If one or both of the numerator parameters equal to zero, then the value of ${}_2F_1$ is one.
(iii) For $a = 1$ and $b = c$ or for $b = 1$ and $a = c$, the series reduces to the well-known Geometric series and thus from this fact this series is called 'Hypergeomtric series'.
(iv) For the numerator parameters a or b or both is a negative integer, then the series becomes a polynomial (i.e. containing a finite number of terms) and question of convergence does not arise.

Here, we verify that the series (1.1)

(i) is convergent for all value of z provided $|z| < 1$ and divergent provided $|z| > 1$.
(ii) is convergent for $z = 1$ provided $\mathrm{Re}(c-a-b) > 0$ and divergent provided $\mathrm{Re}(c-a-b) \leq 0$.
(iii) is absolutely convergent for $z = -1$ provided $\mathrm{Re}(c-a-b) > 0$ and convergent but not absolutely provided $-1 < \mathrm{Re}(c-a-b) \leq 0$ and divergent provided $\mathrm{Re}(c-a-b) < -1$.

It is interesting to mention that limiting case of (1.1) is worth of mentioning here. For this, if we replace z by $\frac{z}{b}$ in (1.1) and take the limit as $b\to\infty$, then since $\frac{(b)_n}{b^n}z^n \to z^n$, we arrive at the following series which is in the literature known as the Kummer's series or the confluent hypergeometric series [9] viz.

$$(1.2)\qquad {}_1F_1\left[\begin{matrix}a\\ c\end{matrix}; z\right]=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}.$$

Moreover, it is evident that almost all elementary fuctions available in mathematics or mathematical physics are special cases or limiting cases of the hypergeometric function ${}_2F_1$ or the confluent hypergeometric function ${}_1F_1$. For detail see [16].

Next, we consider the definition of the generalized hypergeometric function [1, 17] with p numerator and q denumerator parameters by the following expression:

$$(1.3)\qquad {}_pF_q\left[\begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},$$

For more details about its convergence conditions (including absolutely convergence) and its various properties, we refer the standard texts [17].

It is well known that whenever a hypergeometric function or a generalized hypergeometric function reduces to the gamma function, the results are very important from the application point of view. Thus the classical summation theorems such as those of Gauss, Gauss's second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschuitz for the series ${}_3F_2$ and generalizations obtained earlier br Rakha and Rathie [18], Lavoie, et al. [10, 11, 12] and Kim, et al. [7] play an important role. In this regard, we refer an interesting paper by Bailey [2]. On the other hand, transformation formulas (including quadratic and cubic) play an important role in the theory of hypergeometric and generlized hypergeometric function.

In our present investigation, we are interested in the following transformation formula which is in the literature known as the Kummer's second theorem [3] viz.

$$(1.4)\qquad e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha \end{matrix}; x\right] = {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix}; \frac{x^2}{16}\right].$$

Kummer [9] has obtained the result (1.4) from the theory of differential equations. Bailey [2] established the results (1.4) with the help of the following Gauss's second summation theorem [3] viz.

$$(1.5)\qquad {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) & \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}.$$

MacRobert [13] established Kummer's second theorem (1.4) with the help of the following integral formula viz.

$$(1.6)\qquad \int_{-1}^{1} e^{zx}(1-x^2)^{\alpha-1}\,dx = \frac{\Gamma(\frac{1}{2})\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})}\,{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix}; \frac{z^2}{4}\right].$$

In 1998, Rathie and Choi [19] derived the Kummer's second theorem (1.4) with the help of the following Gauss's summation theorems [3, 4] viz.

$$(1.7)\qquad {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

provided Re$(c-a-b) > 0$.

In 2001, Malani and Choi [14] deduced the Kummer's second theorem (1.4) with the help of the following Preece's identity involving product of generalized hypergeometric function [2] viz.

$$(1.8)\qquad \left\{{}_1F_1\left[\begin{matrix}\alpha\\2\alpha\end{matrix};x\right]\right\}^2=e^x\,{}_1F_2\left[\begin{matrix}\alpha\\\alpha+\frac{1}{2},\,2\alpha\end{matrix};\frac{x^2}{4}\right].$$

In 1995, Rathie and Nagar [21] established the following two results contigous to that of Kummer's second theorem (1.4) as follows:

$$(1.9)\qquad e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix}\alpha\\2\alpha+1\end{matrix};x\right]={}_0F_1\left[\begin{matrix}-\\\alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]-\frac{x}{2(2\alpha+1)}\,{}_0F_1\left[\begin{matrix}-\\\alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]$$

and

$$(1.10)\qquad e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix}\alpha\\2\alpha-1\end{matrix};x\right]={}_0F_1\left[\begin{matrix}-\\\alpha-\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]+\frac{x}{2(2\alpha-1)}\,{}_0F_1\left[\begin{matrix}-\\\alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right].$$

They have obtained the results (1.9) and (1.10) with the help of following two results contigous to Gauss's second summation theorem obtained earier by Lavoie, Grondin and Rathie [12] viz.

$$(1.11)\qquad {}_2F_1\left[\begin{matrix}a,\quad b\\\frac{1}{2}(a+b+2)\end{matrix};\frac{1}{2}\right]$$
$$=\frac{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)}{(a-b)}\left\{\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}-\frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}\right\}$$

and

$$(1.12)\qquad {}_2F_1\left[\begin{matrix}a,\quad b\\\frac{1}{2}(a+b)\end{matrix};\frac{1}{2}\right]$$
$$=\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b)\left\{\frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}+\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}\right\}.$$

In 1998, Kim, et al.[6] established the results (1.9) and (1.10) by following three different methods. In the first method, they have used Gauss's summation theorem (1.7). In the second method they have used the following contiguous function relations viz.

$$(1.13)\qquad {}_1F_1\left[\begin{matrix}\alpha\\2\alpha+1\end{matrix};x\right]={}_1F_1\left[\begin{matrix}\alpha\\2\alpha\end{matrix};x\right]-\frac{x}{2(2\alpha+1)}\,{}_1F_1\left[\begin{matrix}\alpha+1\\2\alpha+2\end{matrix};x\right]$$

and

$$(1.14)\qquad {}_1F_1\left[\begin{matrix}\alpha\\2\alpha-1\end{matrix};x\right]={}_1F_1\left[\begin{matrix}\alpha-1\\2\alpha-2\end{matrix};x\right]+\frac{x}{2(2\alpha-1)}\,{}_1F_1\left[\begin{matrix}\alpha\\2\alpha\end{matrix};x\right].$$

In the third method, they have used the following two results closely related to the Preece's identity (1.8) obtained earlier by Rathie and Choi [20] viz.

$$(1.15)\qquad \left\{{}_1F_1\left[\begin{matrix}\alpha\\2\alpha+1\end{matrix};x\right]\right\}^2=e^x\left\{{}_1F_2\left[\begin{matrix}\alpha\\\alpha+\frac{1}{2},\,2\alpha\end{matrix};\frac{x^2}{4}\right]\right.$$
$$-\frac{x}{(2\alpha+1)}\,{}_1F_2\left[\begin{matrix}\alpha+1\\\alpha+\frac{3}{2},\,2\alpha+1\end{matrix};\frac{x^2}{4}\right]$$
$$\left.+\frac{x^2}{4(2\alpha+1)^2}\,{}_1F_2\left[\begin{matrix}\alpha+1\\\alpha+\frac{3}{2},\,2\alpha+2\end{matrix};\frac{x^2}{4}\right]\right\}$$

and

$$\left\{ {}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha-1\end{matrix};x\right]\right\}^2 = e^x\left\{ {}_1F_2\left[\begin{matrix}\alpha-1\\ \alpha-\frac{1}{2},\,2\alpha-2\end{matrix};\frac{x^2}{4}\right] + \frac{x}{(2\alpha-1)}\,{}_1F_2\left[\begin{matrix}\alpha\\ \alpha+\frac{1}{2},\,2\alpha-1\end{matrix};\frac{x^2}{4}\right] + \frac{x^2}{4(2\alpha-1)^2}\,{}_1F_2\left[\begin{matrix}\alpha\\ \alpha+\frac{1}{2},\,2\alpha\end{matrix};\frac{x^2}{4}\right]\right\}. \tag{1.16}$$

In 2002, Malani, et al. [15] established the results (1.9) and (1.10) with the help of the result (1.6) and its following contiguous result viz.

$$\int_{-1}^{1} x e^{zx}(1-x^2)^{\alpha-1}\,dx = z\frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{3}{2}\end{matrix};\frac{z^2}{4}\right]. \tag{1.17}$$

In 2015, Kodavanji, et al. [8] estabilshed the results (1.9) and (1.10) from the theory of differential equations.

In 2010, Kim, et al. [5] generalized the well-known and useful Kummer's second theorem (1.4) and obtained explicit expressions of

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+i\end{matrix};x\right]$$

for $i = 0, \pm1, \pm2, \cdots, \pm5$.

As special cases, they have obtained ten results closely related to Kummer's second theorem (1.4).

In this paper, we aim at presenting four results (for $i = \pm2$ and ±3) by following two different methods. The same will be discussed in the subsequent sections.

2. Derivations of Four results Using Contiguous Functions Relations

In this section, we shall establish the following four results closely related to Kummer's second theorem (1.4) by employing contiguous functions relations. These are asserted in the following theorems.

Theorem 2.1. *For $2\alpha+2$ neither zero nor a negative integer, the following results holds true.*

$$e^{-\frac{1}{2}x}\,{}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+2\end{matrix};x\right] = {}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right] - \frac{x}{2(\alpha+1)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right] + \frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\alpha+3)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]. \tag{2.1}$$

Theorem 2.2. *For $2\alpha-2$ neither zero nor a negative integer, the following results holds true.*

$$e^{-\frac{1}{2}x}\,{}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha-2\end{matrix};x\right] = {}_0F_1\left[\begin{matrix}-\\ \alpha-\frac{3}{2}\end{matrix};\frac{x^2}{16}\right] + \frac{x}{2(\alpha-1)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha-\frac{1}{2}\end{matrix};\frac{x^2}{16}\right] + \frac{(\alpha-2)x^2}{4(\alpha-1)(2\alpha-1)(2\alpha-3)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]. \tag{2.2}$$

Theorem 2.3. *For $2\alpha+3$ neither zero nor a negative integer, the following results holds true.*

$$
\begin{aligned}
(2.3)\qquad & e^{-\frac{1}{2}x}\,{}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+3\end{matrix};x\right]\\
&= {}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]-\frac{3x}{2(2\alpha+3)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\\
&\quad+\frac{3\alpha x^2}{4(\alpha+2)(2\alpha+1)(2\alpha+3)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]\\
&\quad-\frac{\alpha x^3}{8(\alpha+2)(2\alpha+3)^2(2\alpha+5)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{7}{2}\end{matrix};\frac{x^2}{16}\right].
\end{aligned}
$$

Theorem 2.4. *For $2\alpha-3$ neither zero nor a negative integer, the following results holds true.*

$$
\begin{aligned}
(2.4)\qquad & e^{-\frac{1}{2}x}\,{}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha-3\end{matrix};x\right]\\
&= {}_0F_1\left[\begin{matrix}-\\ \alpha-\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]+\frac{3x}{2(2\alpha-3)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha-\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\\
&\quad+\frac{3(\alpha-3)x^2}{4(\alpha-1)(2\alpha-3)(2\alpha-5)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha-\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]\\
&\quad+\frac{(\alpha-3)x^3}{8(\alpha-1)(2\alpha-1)(2\alpha-3)^2}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right].
\end{aligned}
$$

Proof. **(a) Derivation of the result** (2.1)**.**

In order to establish the result (2.1) asserted in Theorem 2.1, we shall first establish the following contiguous relation involving three ${}_1F_1$ viz.

$$
(2.5)\qquad {}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+2\end{matrix};x\right]={}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+1\end{matrix};x\right]-\frac{\alpha x}{2(\alpha+1)(2\alpha+1)}\,{}_1F_1\left[\begin{matrix}\alpha+1\\ 2\alpha+3\end{matrix};x\right].
$$

For this, denoting the left-hand side of (2.5) by S, we have

$$
S={}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+2\end{matrix};x\right].
$$

Expressing ${}_1F_1$ with the help of the definition (1.3), we have

$$
\begin{aligned}
S&=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(2\alpha+2)_n}\frac{x^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{(\alpha)_n\,x^n}{n!}\frac{\Gamma(2\alpha+2)}{\Gamma(2\alpha+2+n)}\\
&=\sum_{n=0}^{\infty}\frac{(\alpha)_n\,x^n}{n!}\frac{(2\alpha+1)\Gamma(2\alpha+1)}{(2\alpha+1+n)\Gamma(2\alpha+1+n)}.
\end{aligned}
$$

This means

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(2\alpha+1)_n} \frac{x^n}{n!} \left\{ \frac{2\alpha+1}{2\alpha+1+n} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(2\alpha+1)_n} \frac{x^n}{n!} \left\{ \frac{(2\alpha+1+n)-n}{2\alpha+1+n} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(2\alpha+1)_n} \frac{x^n}{n!} \left\{ 1 - \frac{n}{2\alpha+1+n} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(2\alpha+1)_n} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(\alpha)_n \, x^n}{(2\alpha+1)_n (2\alpha+1+n)(n-1)!}. \end{aligned}$$

Thus,

$$\begin{aligned} S &= {}_1F_1 \left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x \right] - \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1} \, x^{m+1}}{(2\alpha+1)_{m+1}(2\alpha+m+2)m!} \\ &= {}_1F_1 \left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x \right] - \frac{\alpha x}{(2\alpha+1)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m \, x^m}{(2\alpha+2)_m (2\alpha+2+m)m!} \\ &= {}_1F_1 \left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x \right] - \frac{\alpha x}{(2\alpha+1)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m \, x^m \Gamma(2\alpha+2)}{(2\alpha+2+m)\Gamma(2\alpha+2+m)m!} \\ &= {}_1F_1 \left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x \right] - \frac{\alpha x}{(2\alpha+1)(2\alpha+2)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m \, x^m \Gamma(2\alpha+3)}{\Gamma(2\alpha+3+m)m!} \\ &= {}_1F_1 \left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x \right] - \frac{\alpha x}{2(\alpha+1)(2\alpha+2)} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m \, x^m}{(2\alpha+3)_m m!} \\ &= {}_1F_1 \left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x \right] - \frac{\alpha x}{2(\alpha+1)(2\alpha+2)} \, {}_1F_1 \left[\begin{matrix} \alpha+1 \\ 2\alpha+3 \end{matrix} ; x \right], \end{aligned}$$

which is the right-hand side of (2.5). This completes the proof of the contiguous relation involving three ${}_1F_1$.

We are now ready to establish the result (2.1). For this, multiply both sides of the relation (2.5) by $e^{-\frac{x}{2}}$, we have

$$\begin{aligned} e^{-\frac{x}{2}} {}_1F_1 & \left[\begin{matrix} \alpha \\ 2\alpha+2 \end{matrix} ; x \right] \\ &= e^{-\frac{x}{2}} {}_1F_1 \left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x \right] - \frac{\alpha x}{2(\alpha+1)(2\alpha+2)} e^{-\frac{x}{2}} {}_1F_1 \left[\begin{matrix} \alpha+1 \\ 2\alpha+3 \end{matrix} ; x \right]. \end{aligned} \tag{2.6}$$

We now observe that in the right-hand side of (2.6), if we apply the result (1.9) and after little simplification, we easily arive at the right-hand side of (2.1) asserted in Theorem 2.1.

This completes the proof of the results (2.1) asserted in Theorem 2.1.

(b) Derivation of the result (2.2)**.**

In order to establish the result (2.2) asserted in Theorem 2.2, we shall use the following contiguous relation involving three ${}_1F_1$, which can be easily derived a similar lines that of (2.5):

$$(2.7)\qquad {}_1F_1\left[\begin{matrix}\alpha\\2\alpha-2\end{matrix};x\right]={}_1F_1\left[\begin{matrix}\alpha-1\\2\alpha-3\end{matrix};x\right]+\frac{(\alpha-2)x}{2(\alpha-1)(2\alpha-3)}{}_1F_1\left[\begin{matrix}\alpha\\2\alpha-1\end{matrix};x\right].$$

Now multiply both sides of the equation (2.7) by $e^{-\frac{x}{2}}$, we have

$$\begin{aligned}&e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha\\2\alpha-2\end{matrix};x\right]\\&=e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha-1\\2\alpha-3\end{matrix};x\right]+\frac{(\alpha-2)x}{2(\alpha-1)(2\alpha-3)}e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha\\2\alpha-1\end{matrix};x\right].\end{aligned}$$

Now in the right-hand, applying the result (1.10) and after little simplification, we easily arive at the right-hand side of (2.2).

This completes the proof of the results (2.2) asserted in Theorem 2.2.

(c) Derivation of the result (2.3)**.**

For establishing the result (2.3) asserted in Theorem 2.3, we shall use the following contiguous relation involving three ${}_1F_1$ viz.

$$(2.8)\qquad {}_1F_1\left[\begin{matrix}\alpha\\2\alpha+3\end{matrix};x\right]={}_1F_1\left[\begin{matrix}\alpha\\2\alpha+2\end{matrix};x\right]-\frac{\alpha x}{2(\alpha+1)(2\alpha+3)}{}_1F_1\left[\begin{matrix}\alpha+1\\2\alpha+4\end{matrix};x\right].$$

The result (2.3) follows by multiplying both sides of (2.5) by $e^{-\frac{x}{2}}$ and making use of the result (2.2). We omit the details.

(d) Derivation of the result (2.4)**.**

For establishing the result (2.4) asserted in Theorem 2.4, we shall use the following contiguous relation involving three ${}_1F_1$ viz.

$$(2.9)\qquad {}_1F_1\left[\begin{matrix}\alpha\\2\alpha-3\end{matrix};x\right]={}_1F_1\left[\begin{matrix}\alpha-1\\2\alpha-4\end{matrix};x\right]+\frac{(\alpha-3)x}{2(\alpha-2)(2\alpha-3)}{}_1F_1\left[\begin{matrix}\alpha\\2\alpha-2\end{matrix};x\right].$$

The result (2.4) follows by multiplying both sides of (2.9) by $e^{-\frac{x}{2}}$ and making use of the result (2.3). We left the details for the interested readers. ∎

Remark 2.1. The results (2.1) and (2.2) have also been obtained by Ainkooran, et al. [1] by using the classical Gauss's summation theorem (1.7)

3. Derivations of the results (2.1) to (2.4) by Integral Method Developed by MacRobert

In this section, we shall establish the results (2.1) to (2.4) asserted in the four theorems given in the previous section. For this, we shall establish the following two results closely related to the results (1.6) and (1.17) that are required (and presumably new) in our present investigation.

$$(3.1)\qquad \begin{aligned}&\int_{-1}^{1}x^2e^{zx}(1-x^2)^{\alpha-1}\,dx\\&=\frac{z^2}{2}\frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{5}{2})}\,{}_0F_1\left[\begin{matrix}-\\\alpha+\frac{5}{2}\end{matrix};\frac{z^2}{4}\right]+\frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})}\,{}_0F_1\left[\begin{matrix}-\\\alpha+\frac{3}{2}\end{matrix};\frac{z^2}{4}\right]\end{aligned}$$

and

$$\tag{3.2} \int_{-1}^{1} x^3 e^{zx}(1-x^2)^{\alpha-1}\,dx = \frac{z^3}{4}\,\frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{7}{2})}\,{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{7}{2}\end{matrix};\,\frac{z^2}{4}\right] + \frac{3z}{2}\,\frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{5}{2})}\,{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{5}{2}\end{matrix};\,\frac{z^2}{4}\right].$$

Proof. In order to establish the result (3.1), we proceed as follows. Differentiating the result (1.17) with respect to z, we have

$$\tag{3.3} \int_{-1}^{1} x^2 e^{zx}(1-x^2)^{\alpha-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})}\,{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2}\end{matrix};\,\frac{z^2}{4}\right] + z\frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})}\,\frac{d}{dz}{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2}\end{matrix};\,\frac{z^2}{4}\right].$$

But it is not much difficult to see that

$$\frac{d}{dz}{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2}\end{matrix};\,\frac{z^2}{4}\right] = \frac{z^2}{2}\,\frac{\Gamma(\alpha)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{5}{2})}\,{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{5}{2}\end{matrix};\,\frac{z^2}{4}\right].$$

Thus from (3.3), we have the result (3.1).

In exactly the same manner, the result (3.2) can be derived from the result (3.1). We, however, prefer to omit the details.

Now we are ready to establish the results (2.1) to (2.4) by employing the integral method developed by MacRobert.

(a) Derivation of the result (2.1)

In order to derive the result (2.1) asserted in Theorem 2.1, we proceed as follows. It is easy to see that the result (2.1) can be re-written in the following form:

$$\tag{3.4} {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+2\end{matrix};\,2x\right] = e^{x}\Big\{{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2}\end{matrix};\,\frac{x^2}{4}\right] - \frac{x}{(\alpha+1)}\,{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2}\end{matrix};\,\frac{x^2}{4}\right] + \frac{\alpha x^2}{(\alpha+1)(2\alpha+1)(2\alpha+3)}\,{}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{5}{2}\end{matrix};\,\frac{x^2}{4}\right]\Big\}.$$

Now, if we start with the right-hand side of (3.4) and making use of the following integral representation of ${}_1F_1$ viz

$${}_1F_1\left[\begin{matrix} \alpha \\ \rho\end{matrix};\,x\right] = \frac{\Gamma(\rho)}{\Gamma(\alpha)\Gamma(\rho-\alpha)}\int_0^1 e^{xt}t^{\alpha-1}(1-t)^{\rho-\alpha-1}dt$$

provided $\mathrm{Re}(\rho) > \mathrm{Re}(\alpha) > 0$, we have

$${}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+2\end{matrix};\,2x\right] = \frac{\Gamma(2\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+2)}\int_0^1 e^{2xt}t^{\alpha-1}(1-t)^{\alpha+1}dt$$

which, upon putting $T = 2t-1$ and using the Legendre's duplication formula for the gamma function, leads to

$$ {}_1F_1\left[\begin{matrix}\alpha\\2\alpha+2\end{matrix};2x\right]=e^x\frac{\Gamma(\alpha+1)\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}\Gamma(\alpha)\Gamma(\alpha+2)}\int_{-1}^{1}e^{xT}(1-T^2)^{\alpha-1}(1-2T+T^2)dT. $$

Now, separating the integral into three integrals and applying the known results (1.6), (1.17) and (3.1) immediately leads to the right-hand side of (2.1). This completes the derivation of the result (2.1).

(b) Derivation of the result (2.2)

Proceeding exactly as explained in the cases (a) above, it is not difficult to see that

$$ \begin{aligned} &{}_1F_1\left[\begin{matrix}\alpha\\2\alpha-2\end{matrix};2x\right]\\ &=e^x\frac{\Gamma(\alpha-1)\Gamma(\alpha-\frac{3}{2})}{\sqrt{\pi}\Gamma(\alpha)\Gamma(\alpha-2)}\int_{-1}^{1}e^{xT}(1-T^2)^{\alpha-3}(1+2T+T^2)dT. \end{aligned} $$

Now, separating the integral into three integrals and applying the known results (1.6), (1.17) and (3.1), we arrive easily at the right-hand side of (2.2). This completes the derivation of the result (2.2).

(c) Derivation of the result (2.3)

Proceeding exactly as explained in the case (a) and (b) above, it is not difficult to arrive at the following expression:

$$ \begin{aligned} &{}_1F_1\left[\begin{matrix}\alpha\\2\alpha+3\end{matrix};2x\right]\\ &=e^x\frac{\Gamma(\alpha+2)\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}\Gamma(\alpha)\Gamma(\alpha+3)}\int_{-1}^{1}e^{xT}(1-T^2)^{\alpha-1}(1-3T+3T^2-T^3)dT. \end{aligned} $$

Now, separating the integral into four integrals and making use of the known results (1.6), (1.17) and (3.2), we easily reach at the right-hand side of (2.3). This completes the proof of the result (2.3).

(d) Derivation of the result (2.4)

Proceeding exactly as explained above in the cases (a), (b) and (c), it is quite simple to reach at the following expression:

$$ \begin{aligned} &{}_1F_1\left[\begin{matrix}\alpha\\2\alpha-3\end{matrix};2x\right]\\ &=e^x\frac{\Gamma(\alpha-1)\Gamma(\alpha-\frac{3}{2})}{\sqrt{\pi}\Gamma(\alpha)\Gamma(\alpha-3)}\int_{-1}^{1}e^{xT}(1-T^2)^{\alpha-1}(1+3T+3T^2+T^3)dT. \end{aligned} $$

Now, separating the above integral into four integrals and making use of the known results (1.6), (1.17) and (3.2), we easily arrive at the right-hand side of (2.4). This completes the proof of the result (2.4). ∎

4. CONCLUDING REMARK

In this paper, we have established four well-known results closely related to the Kummer's second theorem by two different methods. In the first method, we have used the technique of contiguous functions relations involving three ${}_1F_1$ while in the second method, we have employed the technique developed by MacRobert.

We conclude this paper by remarking that by the same technique developed by MacRobert, the derivation of the general result of the form ${}_1F_1\left[\begin{matrix}\alpha\\2\alpha\pm i\end{matrix};2x\right]$ for $i=0,1,2,\cdots$ are under investigation and will form a part of the subsequent paper in this direction.

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