Aust. J. Math. Anal. Appl.
Vol. **20** (2023), No. 2, Art. 7, 9 pp.
AJMAA

DIFFERENTIAL EQUATIONS FOR INDICATRICES, SPACELIKE AND TIMELIKE CURVES

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Received 30 March, 2023; accepted 14 September, 2023; published 25 October, 2023.

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ABSTRACT. Motivated by the recent work of Deshmukh et al. [20], in this paper we show that Tangent, Binormal, and Principal Normal indicatrices do not form non-trivial differential equations. Finally, we obtain the 4th-order differential equations for spacelike and timelike curves.

Key words and phrases: Helix; Slant helix; Tangent, Binormal, Principal Normal indicatrices; Spacelike curves; and Timelike curves.

2010 *Mathematics Subject Classification.* Primary 53A04, Secondary 53B20.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In differential geometry, the curve is among one of the fascinating topics. Helices, spherical curves, and rectifying curves are a few important types of curves that appear in many important applications. For example, helical structures arise in seashells, vines, carbon nanotubes, DNA double, and nano-springs, etc. Though many authors [2, 3, 7, 12, 13, 16, 19, 23, 15] studied curves from the last several decades nevertheless curves are still a relevant and significant area of the research. In the study of curves, the notion of associated curves is pretty exciting. If there exist a mathematical relation between two or more curves, then the curves are known as associated curves.

Izumiya et al. [21] introduced some special curves which are known as a slant helix and conical geodesic curves in Euclidean 3-space. Besides, Izumiya and Takeuchi gave some classifications of the special developable surfaces and obtained an example of a slant helix. In [11], Kula et al. studied the spherical images of the tangent indicatrix and binormal indicatrix of a slant helix. Moreover, they obtained that the spherical images of the slant helices are spherical helices and a curve of constant precession is a slant helix.

In [1], Ali obtained the position vector of a general helix $(\tau/\kappa = m)$ associated with Frenet frame and represented the general helix in terms of curvature (κ) and torsion (τ) through a standard frame of Euclidean 3-space, where m is a constant given by $m = cos[\phi]$, here ϕ denotes the angle between the axis of a general helix and the tangent of the curve. In [2], Ali et al. extended the concept of a slant helix to Euclidean space of dimension n, and gave the necessary and sufficient conditions for a curve in Euclidean n-space to be a slant helix. Moreover, Ali also gave an example of a slant helix in Euclidean 5-space.

Recently, Sahiner [3] defined the associated curves as integral curves of a vector field produced by Frenet vectors of the tangent indicatrix of a curve in Euclidean 3-space and obtained some relations between curvatures and Frenet vectors. Besides, he gave a few techniques to obtain helices and slant helices from special spherical curves and constructed some examples of it. In [4], B. Y. Chen investigated the characterization and classification of the rectifying curves. On the other hand in [5], B. Y. Chen studied via rectifying curves that all geodesics on an arbitrary cone in Euclidean space of dimension 3, are not necessarily a circular cone.

In [6], Yilmaz et al. used the system of linear ordinary differential equations to construct the slant helices. Also, using integration in Minkowski 3-space, they obtained the position vectors for slant helices. In [7], Camci et al. studied and obtained a spherical slant helix and gave some examples of the spherical slant helices in Euclidean 3-space. In [8], Arroyo et al. investigated the unit speed curves contained in a real space form of arbitrary dimension m. Moreover, they gave a classification of semi-Riemannian Hopf cylinders of $H_1^3(-1)$ and Hopf cylinders of S^3 with proper mean curvature function.

In [9], Choi et al. introduced the concept of the principal-direction curve and principal-donor curve of a Frenet curve in Euclidean 3-space. Moreover, Choi et al. constructed a canonical method for associated curves and characterized some associated curves in Euclidean 3-space. Kula et al. [10] obtained a relationship between a slant helix and a general helix. Furthermore, Kula et al. deduced some differential equations by characterizing of a slant helix and gave a few examples of slant helices in Euclidean 3-space.

In [17], Lucas et al. studied a weaker version of the classic slant helices in Minkowski 3-space and Euclidean 3-space which are known as general slant helices. Furthermore, Lucas showed that the classic slant helix is a general helix but the converse is not true. Also, he obtained equations that involve the torsion and curvature.

In [19], Deshmukh et al. investigated the rectifying curves via the dilation of the unit speed curve on S^2 (unit sphere) in Euclidean 3-space and obtained a necessary and sufficient condition for centrode of a unit speed curve in Euclidean 3-space. Moreover, Deshmukh and Chen proved that if a unit speed curve is neither a helix nor a planar curve, then its dilated centrode is always a rectifying curve. Deshmukh et al. [20] shown that for every Frenet curve in Euclidean 3-space, the distance function satisfies a 4th-order differential equation and using this they derived a new characterization of helices. In [22], Ozdemir et al. introduced the notion of type-3 slant helix according to the parallel transport frame in Euclidean 4-space.

Motivated by Deshmukh et al. [20] in this paper, we investigate the distance function. We show that Tangent, Binormal, and Principal Normal indicatrices do not form non-trivial differential equations, and obtain the 4th-order differential equations for spacelike and timelike curves.

2. Preliminaries

In this section, we recall some basic concepts of the curves and indicatrices in the Euclidean 3-space. Let $\beta : I \to \mathbb{R}^3$ represents the unit speed curve in the Euclidean 3-space and $T',\ N',\ B'$ be the three orthonormal vectors of the Frenet frame $\{T, N, B\}$, given by

$$T = \frac{d\beta}{ds}, \qquad N = \frac{T'}{\kappa}, \qquad B = T \times N$$

where T, N, B represent the unit tangent vector field, unit principal normal vector field and unit binormal vector field, respectively.

The Serret-Frenet formulae are given by

$$\begin{cases} T'(s) = \kappa(s) N(s) \\ N'(s) = -\kappa(s) T(s) + \tau(s) B(s) \\ B'(s) = -\tau(s) N(s) \end{cases} \tag{2.1}$$

where $\kappa(s) = \|T'(s)\|$ denote the curvature and $\tau(s) = -\langle B'(s), N(s)\rangle$ denote the torsion of the curve β. Here the curve β is parameterized in terms of the arc-length parameter s [18].

If the position vector of the curve β lies in the rectifying plane then the curve is known as a rectifying curve. The distance function $d(s) = \|\beta(s)\|$ of a rectifying curve β satisfies the following equation

$$d(s) = \sqrt{s^2 + c_1 s + c_2}$$

here c_1 and c_2 denote the arbitrary constants.

Furthermore, it can be shown that the unit speed curve β is also a rectifying curve if and only if the ratio of torsion τ and curvature κ verifies

$$\frac{\tau}{\kappa} = as + b$$

where $a \neq 0$ and b are constants [4].

Choi and Kim investigated the relationship between curvature and torsion of the principal-direction curve and principal-donor curve in [9].

Theorem 2.1. [9] *Let β be a Frenet curve in Euclidean 3-space with the curvature κ and the torsion τ and $\overline{\beta}$ be the principal-direction curve of the curve β. Then the curvature $\overline{\kappa}$ and torsion $\overline{\tau}$ of the principal-direction curve $\overline{\beta}$ are given by*

$$\overline{\kappa} = \sqrt{\kappa^2 + \tau^2} \qquad and \qquad \overline{\tau} = \frac{\kappa^2}{\kappa^2 + \tau^2}\left(\frac{\tau}{\kappa}\right)'$$

Theorem 2.2. [9] *Let β be a principal-donor curve of the curve $\overline{\beta}$ in Euclidean 3-space with the curvature $\overline{\kappa}$ and torsion $\overline{\tau}$. Then the curvature κ and torsion τ of the principal-donor curve β are given by*

$$\kappa = \overline{\kappa}\ |\cos\left(\int \overline{\tau} ds\right)| \quad and \quad \tau = \overline{\kappa}\sin\left(\int \overline{\tau} ds\right)$$

A curve β is said to be general helix if unit tangent $T(s)$ makes a constant angle with a fixed straight line. Likewise, if unit principal normal $N(s)$ makes a constant angle with a fixed straight line then a curve β is said to be slant helix.

Let β be a unit speed curve in Euclidean space with Frenet vectors T, N and B. The unit tangent vectors along the curve β generate a curve β_t on the unit sphere centered at the origin, called the tangent indicatrix of curve β. Similarly, we have the binormal indicatrix β_b and principal normal indicatrix β_n [10].

Deshmukh and B. Y. Chen shown that for every Frenet curve in Euclidean 3-space, the distance function satisfies a general differential equation. We recall the following proposition from [20].

Proposition 2.3. *If β be a unit speed curve then every unit speed Frenet curve satisfies the following equation:*

$$\rho\sigma h''' + (\rho\sigma' + 2\rho'\sigma)h'' + \left\{(\sigma\rho')' + \frac{\rho}{\sigma} + \frac{\sigma}{\rho}\right\} h' + \left(\frac{\sigma}{\rho}\right)' h = (\sigma\rho')' + \frac{\rho}{\sigma}, \tag{2.2}$$

where $\rho = \kappa^{-1}$, $\sigma = \tau^{-1}$, $h(s) = d(s)d'(s)$.

The Minkowski 3-space $\mathbb{E}^3_1$ is the Euclidean 3-space provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$.

Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}^3_1$ can have one of the three causal characters; it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and lightlike (null) if $g(v,v) = 0$ and $v \neq 0$. Analogously, an arbitrary curve $\beta = \beta(s)$ in $\mathbb{E}^3_1$ can locally be spacelike, timelike or lightlike (null), if all of its velocity vectors $\beta'(s)$ are respectively spacelike, timelike or lightlike. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$ and the spacelike or (timelike) curve $\beta(s)$ is said to be of unit speed if $g(\beta'(s), \beta'(s)) = \pm 1$ [14].

3. DERIVATION OF THE DIFFERENTIAL EQUATIONS

In this section, first we give some propositions for indicatrices, Serret-Frenet formulae, and a few useful results for spacelike and timelike curves. Finally, we obtain the 4th-order differential equations for spacelike and timelike curves.

Proposition 3.1. *If β be a unit speed curve then tangent indicatrix β_t of the curve β does not form a non-trivial differential equation.*

Proof. Since the tangent indicatrix β_t has constant norm equal to one. By differentiating the distance function $d(s) = \|\beta_t(s)\|$, we get $d'(s) = 0$.

Proposition 3.2. *If β be a unit speed curve then binormal indicatrix β_b of curve β does not form a non-trivial differential equation.*

Proof. Since the binormal indicatrix β_b has constant norm equal to one. By differentiating the distance function $d(s) = \|\beta_b(s)\|$, we get $d'(s) = 0$.

Proposition 3.3. *If β be a unit speed curve then principal normal indicatrix β_n of curve β does not form a non-trivial differential equation.*

Proof. Since the principal normal indicatrix β_n has constant norm equal to one. By differentiating the distance function $d(s) = \|\beta_n(s)\|$, we get $d'(s) = 0$.

Remark 3.1. Suppose β denote a spacelike curve with a spacelike principal normal N and β' be the tangent vector field, then the Serret-Frenet formulae are given by

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = \tau N \end{cases} \tag{3.1}$$

where $\langle T, T\rangle = 1$, $\langle N, N\rangle = 1$, $\langle B, B\rangle = -1$, $\langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0$.

From the above formula, we have the following

$$\begin{cases} \langle \beta, T\rangle' = 1 + \kappa \langle \beta, N\rangle \\ \langle \beta, N\rangle' = -\kappa \langle \beta, T\rangle + \tau \langle \beta, B\rangle \\ \langle \beta, B\rangle' = \tau \langle \beta, N\rangle \end{cases} \tag{3.2}$$

Theorem 3.4. *Suppose β denote a spacelike curve with a spacelike principal normal N, then the function $f(s) = d(s)d'(s)$ satisfies the following differential equation*

$$\begin{aligned} \frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} + \frac{\kappa}{\tau} - \frac{\tau}{\kappa}\right] f' + \left[\frac{\kappa}{\tau'} + \frac{\kappa'}{\tau}\right] f \\ = \left[\frac{1}{\tau\kappa'}\right]' - \frac{\tau}{\kappa} \end{aligned} \tag{3.3}$$

where $d(s) = \|\beta(s)\|$ is the distance function of β.

Proof. Differentiating $d^2(s) = \langle \beta(s), \beta(s)\rangle$ and making use of equation (3.1), we get

$$f = \langle \beta, T\rangle \tag{3.4}$$

Now, differentiating above equation and using (3.2), we get

$$f' - 1 = \kappa \langle \beta, N\rangle \tag{3.5}$$

Further, differentiating equation (3.5), yields

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' + \frac{\kappa}{\tau} f - \frac{1}{\tau\kappa'} = \langle \beta, B\rangle \tag{3.6}$$

Now, differentiating equation (3.6) and using (3.2), (3.5), we get the desired result.

Remark 3.2. Suppose β denote a spacelike curve with a timelike principal normal N and β' be the tangent vector field, then the Serret-Frenet formulae are given by

$$\begin{cases} T' = \kappa N \\ N' = \kappa T + \tau B \\ B' = \tau N \end{cases} \tag{3.7}$$

where $\langle T, T\rangle = 1$, $\langle N, N\rangle = -1$, $\langle B, B\rangle = 1$, $\langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0$.

From the above equation, we get the following

$$\begin{cases} \langle \beta, T \rangle' = 1 + \kappa \langle \beta, N \rangle \\ \langle \beta, N \rangle' = \kappa \langle \beta, T \rangle + \tau \langle \beta, B \rangle \\ \langle \beta, B \rangle' = \tau \langle \beta, N \rangle \end{cases} \tag{3.8}$$

Theorem 3.5. *Suppose β denote a spacelike curve with a timelike principal normal N, then the function $f(s) = d(s)d'(s)$ satisfies the following differential equation*

$$\frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} - \frac{\kappa}{\tau} - \frac{\tau}{\kappa}\right] f' - \left[\frac{\kappa}{\tau'} + \frac{\kappa'}{\tau}\right] f = \left[\frac{1}{\tau\kappa'}\right]' - \frac{\tau}{\kappa} \tag{3.9}$$

where $d(s) = \|\beta(s)\|$ is the distance function of β.

Proof. Differentiating $d^2(s) = \langle \beta(s), \beta(s) \rangle$ and making use of equation (3.7), we get

$$f = \langle \beta, T \rangle \tag{3.10}$$

Using (3.10) and (3.8), a simple computation gives

$$f' - 1 = \kappa \langle \beta, N \rangle \tag{3.11}$$

Now, differentiating (3.11), we get

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' - \frac{\kappa}{\tau} f - \frac{1}{\tau\kappa'} = \langle \beta, B \rangle \tag{3.12}$$

Finally, differentiating equation (3.12) and using (3.8), (3.11), we get the desired result.

Remark 3.3. Suppose β denote a spacelike curve with a lightlike principal normal N and β' be the tangent vector field, then the Serret-Frenet formulae are given by

$$\begin{cases} T' = \kappa N \\ N' = \tau N \\ B' = -\kappa T - \tau B \end{cases} \tag{3.13}$$

where $\langle T, T \rangle = 1, \langle N, B \rangle = 1, \langle N, N \rangle = \langle B, B \rangle = \langle T, N \rangle = \langle T, B \rangle = 0.$

From the above formula, we have the following

$$\begin{cases} \langle \beta, T \rangle' = 1 + \kappa \langle \beta, N \rangle \\ \langle \beta, N \rangle' = \tau \langle \beta, N \rangle \\ \langle \beta, B \rangle' = -\kappa \langle \beta, T \rangle - \tau \langle \beta, B \rangle \end{cases} \tag{3.14}$$

Theorem 3.6. *Suppose β denote a spacelike curve with a lightlike principal normal N, then the function $f(s)$ satisfies the following differential equation*

$$\frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'} - \frac{1}{\kappa}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} - \frac{1}{\kappa'}\right] f' = \left[\frac{1}{\tau\kappa'}\right]' - \frac{1}{\kappa'} \tag{3.15}$$

where $f(s) = d(s)d'(s)$, and $d(s) = \|\beta(s)\|$ is the distance function of the curve β.

Proof. Differentiating $d(s) = \|\beta(s)\|$ and using equation (3.13), we get

$$f = \langle \beta, T \rangle \tag{3.16}$$

From equations (3.16) and (3.14), we have

$$f' - 1 = \kappa \langle \beta, N \rangle \tag{3.17}$$

Now, differentiating (3.17), yields

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' - \frac{1}{\tau\kappa'} = \langle \beta, N \rangle \tag{3.18}$$

Differentiating equation (3.18) and using (3.16), we get the result.

Remark 3.4. Suppose β denote a timelike curve and β' be the tangent vector field, then the Serret-Frenet formulae are given by

$$\begin{cases} T' = \kappa N \\ N' = \kappa T + \tau B \\ B' = -\tau N \end{cases} \tag{3.19}$$

where $\langle T, T \rangle = -1$, $\langle N, N \rangle = 1$, $\langle B, B \rangle = 1$, $\langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0$.

From the above formula, we have the following

$$\begin{cases} \langle \beta, T \rangle' = -1 + \kappa \langle \beta, N \rangle \\ \langle \beta, N \rangle' = \kappa \langle \beta, T \rangle + \tau \langle \beta, B \rangle \\ \langle \beta, B \rangle' = -\tau \langle \beta, N \rangle \end{cases} \tag{3.20}$$

Theorem 3.7. *Suppose β denote a timelike curve, then the function $f(s) = d(s)d'(s)$ satisfies the following differential equation*

$$\begin{aligned} \frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} - \frac{\kappa}{\tau} + \frac{\tau}{\kappa}\right] f' - \left[\frac{\kappa}{\tau'} + \frac{\kappa'}{\tau}\right] f \\ = -\left[\frac{1}{\tau\kappa'}\right]' - \frac{\tau}{\kappa} \end{aligned} \tag{3.21}$$

where $d(s) = \|\beta(s)\|$ is the distance function of β.

Proof. Differentiating $d^2(s) = \langle \beta(s), \beta(s) \rangle$ and making use of equation (3.19), we get

$$f = \langle \beta, T \rangle \tag{3.22}$$

Now, differentiating above equation and using (3.20), we get

$$f' + 1 = \kappa \langle \beta, N \rangle \tag{3.23}$$

Further, differentiating equation (3.23), yields

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' - \frac{\kappa}{\tau} f + \frac{1}{\tau\kappa'} = \langle \beta, B \rangle \tag{3.24}$$

Now, differentiating equation (3.24) and using (3.20), (3.23), the result follows.

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