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ON THE EQUIFORM GEOMETRY OF THE INVOLUTE-EVOLUTE CURVE COUPLE IN HYPERBOLIC AND DE SITTER SPACES

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ABSTRACT. In this paper, we aim to investigate the equiform differential geometric properties of the involute-evolute curve couple with constant equiform curvatures in three-dimensional hyperbolic and de Sitter spaces. Also, we obtain some relations between the curvature functions of these curves and investigate some special curves with respect to their equiform curvatures. Finally, we defray two computational examples to support our main findings.

Key words and phrases: Equiform geometry; hyperbolic and de Sitter spaces; evolute-involute curves; Frenet apparatus.

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1. INTRODUCTION

The idea of a string involute is due to Christian Huygens (1658), who is also known for his work in optics. He found involutes while attempting to develop a more exact clock [1]. The involute of a given curve is a well-known idea in Euclidean 3-space. It is notable that, if a curve is differentiable at each point of an open interval, a set of mutually orthogonal unit vectors can be constructed and called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve defined curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curve. An evolute and its involute, are defined in mutual pairs. The evolute and involute of the curve pair is well known by mathematicians particularly the differential geometry researchers.
Let us define involute-evolute curve couple. The curve ϕ is an evolute of the curve ψ if ϕ lies on the tangent line to ψ at $\psi(s_0)$ and the tangents to ψ and ϕ at $\psi(s_0)$ and ϕ are perpendicular for each s_0. Also, ϕ is an evolute of ψ if ψ is an involute of ϕ, for more details, see for instance [2, 3, 4, 5, 6].
The geometry of space is associated with mathematical group. The idea of invariance of geometry under transformation group may imply that on some spacetimes of maximum symmetry there should be a principle of relativity, which requires the invariance of physical laws without gravity under transformations among inertial systems. Besides, the theory of curves and the curves of constant curvature in the equiform differential geometry of the isotropic spaces I_3^1 and I_3^2 and the Galilean space G_3 are described in [7, 8, 9], respectively. Although the equiform geometry has minor importance related to the usual one, the curves that appear here in the equiform geometry can be seen as generalizations of well-known curves from the above mentioned geometries and therefore could have been of research interest.
The equiform geometry of Cayley-Klein space is defined by mentioning that the similitude gathering of the space jam points among planes and lines, individually. Cayley-Klein's geometries have been read up for a long time. However, they have as of late become fascinating since their significance for different fields, like soliton theory, have been rediscovered.
We have found motivation for this work in [2, 7, 10, 11, 12], where the authors considered characterizations of general helices in the Minkowski space-time, double isotropic and Galilean spaces. Therefore, in this paper, we introduced a visualization for the equiform geometry of Frenet apparatus in three dimensional hyperbolic and de Sitter spaces. Also, we define the equiform geometry of involute-evolute curve couple in $\mathbb{H}^3_+(-1)$ and $\mathbb{S}^3_1$.

2. GEOMETRY OF HYPERBOLIC AND DE SITTER SPACES

In this section, we use the basic notions and results in Lorentzian geometry for Frenet frame in hyperbolic and de Sitter spaces. Also, we introduce some definitions and basic facts which are needed in the subsequent sections, for more details, see [1, 6, 13]).

2.1. **Hyperbolic 3-space.**

Let $\mathbb{R}^4$ be a four-dimensional vector space. For any $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, the pseudo-scalar product of x and y is defined by $\langle x, y\rangle = -x_1y_1+x_2y_2+x_3y_3 + x_4y_4$. We call $(R^4, \langle,\rangle)$ Minkowski 4-space and denoted by $\mathbb{E}^4_1$. We say that a vector $x \in \mathbb{E}^4_1$ is spacelike, lightlike or timelike if $\langle x_1, x_2\rangle > 0, \langle x_1, x_2\rangle = 0$ or $\langle x_1, x_2\rangle < 0$, respectively. The norm of the vector $x \in \mathbb{E}^4_1$ is defiend by $\|x\| = \sqrt{|\langle x, x\rangle|}$. The hyperbolic space is defined by

$$\mathbb{H}^3_+(-1) = \{x \in \mathbb{E}^4_1 \mid \langle x, x\rangle = -1, x_1 > 0\},$$

For any $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$ and $z = (z_1, z_2, z_3, z_4) \in \mathbb{E}^4_1$, the pseudo vector product of x, y and z is defined as follows:

$$\begin{aligned} x \wedge y \wedge z &= \begin{vmatrix} -i & j & k & l \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \\ &= \left(-\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix}, -\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \right). \end{aligned}$$

Let $\gamma : I \longrightarrow \mathbb{H}^3_+ \subset \mathbb{E}^4_1; \quad \gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a smooth regular curve in $\mathbb{H}^3_+$ $(i.e., \gamma'(t) \neq 0)$ for any $t \in I$ where I is an open interval. So that, $\langle \gamma'(t), \gamma'(t) \rangle > 0$ for any $t \in I$. The arc-length of γ, measured from $\gamma(t_\circ), t_\circ \in I$ is $s(t) = \int_{t_\circ}^{t} \|\gamma'(t)\| dt$. Then the parameter s is determined such that $\|\dot{\gamma}(s)\| = 1$, where $\dot{\gamma}(s) = \frac{d\gamma(s)}{ds}$. The spacelike curve γ is said to be parameterized by arc-length if it satisfies that $\|\dot{\gamma}(s)\| = 1$. In what follows, we denote the parameter s of γ as the arc-length parameter. Let us denote $\mathbf{T}(s) = \dot{\gamma}(s)$, and we call $\mathbf{T}(s)$ a unit tangent vector of γ at s.
Here, we construct the explicit differential geometry on curves in $\mathbb{H}^3_+(-1)$. Let $\gamma : I \longrightarrow \mathbb{H}^3_+(-1)$ be a regular curve. Since $\mathbb{H}^3_+(-1)$ is a Riemannian manifold, we can re-parameterize γ by the arc-length. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So we have the tangent vector $\mathbf{T}(s) = \dot{\gamma}(s)$ with $\|\mathbf{T}\| = 1$. When $\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq -1$, then we have a unit vector $\mathbf{N}(s) = \dfrac{\dot{\mathbf{T}}(s) - \gamma(s)}{\|\dot{\mathbf{T}}(s) - \gamma(s)\|}$. Moreover, we define $\mathbf{E}(s) = \gamma(s) \wedge \mathbf{T}(s) \wedge \mathbf{N}(s)$, then we have a pseudo orthonormal frame $\{\gamma(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{E}(s)\}$ of $\mathbb{H}^3_+$ along γ. By standard arguments and under the assumption that $\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq -1$, we have the following Frenet formulas:

$$\begin{cases} \dot{\gamma}(s) = \mathbf{T}(s), \\ \dot{\mathbf{T}}(s) = \gamma(s) + \kappa \mathbf{N}(s), \\ \dot{\mathbf{N}}(s) = -\kappa \mathbf{T}(s) + \tau \mathbf{E}(s), \\ \dot{\mathbf{E}}(s) = -\tau \mathbf{N}(s). \end{cases} \tag{2.1}$$

Or in the matrix form as follows:

$$\begin{bmatrix} \dot{\gamma}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{N}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{E}(s) \end{bmatrix},$$

where

$$\begin{aligned} \kappa &= \|\dot{\mathbf{T}}(s) - \gamma(s)\|, \\ \tau &= -\frac{det\,(\gamma(s), \dot{\gamma}(s), \ddot{\gamma}(s), \dddot{\gamma}(s))}{(\kappa(s))^2}, \end{aligned} \tag{2.2}$$

are the curvature and torsion of the curve γ, respectively. Since $\left\langle \dot{\mathbf{T}}(s)-\gamma(s), \dot{\mathbf{T}}(s)-\gamma(s) \right\rangle = \left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle + 1$, the condition:

$$\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq -1,$$

is equivalent to the condition $\kappa(s) \neq 0$. Moreover, we can show that the curve $\gamma(s)$ satisfies the condition $\kappa(s) \equiv 0$ if and only if there exists a lightlike vector c such that $\gamma(s) - c$ is a geodesic. Such a curve is called an equidistant curve.

2.2. **De Sitter 3-space .** Here, we define the de Sitter 3-space as follows:

$$\mathbb{S}^3_1 = \{x \in \mathbb{E}^4_1 \mid \langle x, x \rangle = 1\}.$$

Let $\gamma : I \longrightarrow \mathbb{S}^3_1$ be a smooth and regular spacelike curve in $\mathbb{S}^3_1$. We can parameterize it by arc length s, since $\mathbb{S}^3_1$ is a Riemannian manifold, we can re-parameterize γ by the arc-length. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So, we have the tangent vector $\mathbf{T}(s) = \dot{\gamma}(s)$ with $\|\mathbf{T}\| = 1$. In this case, we call γ a unit speed spacelike curve. If $\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq 1$, then $\|\dot{\mathbf{T}}(s) + \gamma(s)\| \neq 0$, and we define the unit vector $\mathbf{N}(s) = \dfrac{\dot{\mathbf{T}}(s) + \gamma(s)}{\|\dot{\mathbf{T}}(s) + \gamma(s)\|}$. Furthermore, $\mathbf{E}(s) = \gamma(s) \wedge \mathbf{T}(s) \wedge \mathbf{N}(s)$, then we have a pseudo orthonormal frame $\{\gamma(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{E}(s)\}$ of $\mathbb{E}^4_1$ along γ. Also, $\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq 1$, we have Frenet equations:

$$\begin{cases} \dot{\gamma}(s) = \mathbf{T}(s), \\ \dot{\mathbf{T}}(s) = -\gamma(s) + \kappa \mathbf{N}(s), \\ \dot{\mathbf{N}}(s) = -\delta(\gamma)\kappa \mathbf{T}(s) + \tau \mathbf{E}(s), \\ \dot{\mathbf{E}}(s) = \tau \mathbf{N}(s). \end{cases} \tag{2.3}$$

Or in the matrix form:

$$\begin{bmatrix} \dot{\gamma}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{N}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa & 0 \\ 0 & -\delta(\gamma)\kappa & 0 & \tau \\ 0 & 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{E}(s) \end{bmatrix},$$

where $\delta(\gamma) = sign(\mathbf{N}(s))$ (which we shall write as simply δ) and

$$\begin{cases} \kappa = \|\dot{\mathbf{T}}(s) + \gamma(s)\|, \\ \\ \tau = \dfrac{\delta \det(\gamma(s), \dot{\gamma}(s), \ddot{\gamma}(s), \dddot{\gamma}(s))}{\kappa(s)^2}, \end{cases} \tag{2.4}$$

are the curvatures of the curve γ in $\mathbb{S}^3_1$.
Since $\left\langle \dot{\mathbf{T}}(s)+\gamma(s), \dot{\mathbf{T}}(s)+\gamma(s) \right\rangle = \left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle - 1$, the condition $\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq 1$ is equivalent to the condition $\kappa(s) \neq 0$ (see [1, 2]).

3. EQUIFORM GEOMETRY OF CURVES IN HYPERBOLIC 3-SPACE

Let $\gamma(s) : I \to \mathbb{H}^3_+(-1)$ be a curve in hyperbolic 3-space. We define the equiform parameter of γ by

$$\sigma = \int \frac{ds}{\rho} = \int \kappa ds, \tag{3.1}$$

where $\rho = \frac{1}{\kappa}$, is the radius of curvature of γ. Eq. (3.1), leads to

$$\frac{ds}{d\sigma} = \rho. \tag{3.2}$$

Let h is a homothety with the center at the origin and the coefficient λ. So, if we put $\gamma^* = h\left(\gamma\right)$, then

$$s^* = \lambda s, \text{ and } \rho^* = \lambda \rho, \tag{3.3}$$

where s^* is the arclength parameter of γ^* and ρ^* is the radius of curvature of γ^*. Hence σ is an equiform invariant parameter of γ (see [7, 10, 14])

Notation . Let κ and τ be not invariants of the homothety group, then

$$\begin{aligned} \kappa^* &= \frac{1}{\lambda}\kappa, \\ \tau^* &= \frac{1}{\lambda}\tau. \end{aligned}$$

The vector

$$\mathbf{U}_1 = \frac{d\gamma\left(s\right)}{d\sigma}, \tag{3.4}$$

is called the tangent vector of γ in the equiform geometry of $\mathbb{H}^3_+$. Therefore, from Eqs. (3.2) and (3.4), we get

$$\mathbf{U}_1 = \frac{d\gamma\left(s\right)}{d\sigma} = \rho\,\frac{d\gamma\left(s\right)}{ds} = \rho\,\mathbf{T}. \tag{3.5}$$

Furthermore, we define the vectors $\mathbf{U}_2$ and $\mathbf{U}_3$ as follows:

$$\mathbf{U}_2 = \rho\,\mathbf{N}, \quad \mathbf{U}_3 = \rho\,\mathbf{E}. \tag{3.6}$$

It is easy to check that the tetrahedron $\{\gamma, \mathbf{U}_1,\ \mathbf{U}_2,\ \mathbf{U}_3\}$ is an equiform invariant tetrahedron of the curve γ.
Now, we find the derivatives of these vectors with respect to σ using Eqs. (2.1), (3.5) and (3.6) as follows:

$$\begin{aligned} \mathbf{U}_1' &= \frac{d}{d\sigma}\left(\mathbf{U}_1\right) = \rho\frac{d}{ds}\left(\rho\mathbf{T}\right) = \rho(\dot{\rho}\mathbf{T} + \rho\dot{\mathbf{T}}) = \rho(\dot{\rho}\mathbf{T} + \rho(\gamma + \kappa\mathbf{N})) \\ &= \rho^2\gamma + \dot{\rho}(\rho\mathbf{T}) + (\rho\mathbf{N}) = \rho^2\gamma + \dot{\rho}\mathbf{U}_1 + \mathbf{U}_2, \end{aligned} \tag{3.7}$$

where the derivative with respect to the arc-length s is denoted by a *(dot)* and respect to σ by a *(prime)*. Similarly, we obtain

$$\begin{aligned} \mathbf{U}_2' &= \frac{d}{d\sigma}\left(\mathbf{U}_2\right) = \rho\frac{d}{ds}\left(\rho\mathbf{N}\right) = \rho(\dot{\rho}\mathbf{N} + \rho\dot{\mathbf{N}}) = \rho(\dot{\rho}\mathbf{N} + \rho(-\kappa\mathbf{T} + \tau\mathbf{E})) \\ &= \rho\mathbf{T} + \dot{\rho}(\rho\mathbf{N}) + \frac{\tau}{\kappa}(\rho\mathbf{E}) = \mathbf{U}_1 + \dot{\rho}\mathbf{U}_2 + \frac{\tau}{\kappa}\mathbf{U}_3, \end{aligned} \tag{3.8}$$

therefore,

$$\begin{aligned} \mathbf{U}_3' &= \frac{d}{d\sigma}\left(\mathbf{U}_3\right) = \rho\frac{d}{ds}\left(\rho\mathbf{E}\right) = \rho(\dot{\rho}\mathbf{E} + \rho\dot{\mathbf{E}}) = \rho(\dot{\rho}\mathbf{E} + \rho(-\tau\mathbf{N})) \\ &= -\frac{\tau}{\kappa}(\rho\mathbf{N}) + \dot{\rho}(\rho\mathbf{E}) = -\frac{\tau}{\kappa}\mathbf{U}_2 + \dot{\rho}\mathbf{U}_3. \end{aligned} \tag{3.9}$$

Definition 3.1. The functions: $\mathcal{K}_i : I \to \mathcal{R}\ (i = 1, 2)$ defined by

$$\mathcal{K}_1 = \dot{\rho},\ \ \mathcal{K}_2 = \frac{\tau}{\kappa}, \tag{3.10}$$

are called the equiform curvatures of the curve γ. These functions are differential invariant of the group of equiform transformations, too.

Therefore, the formulas analogous to the famous Frenet formulas in the equiform geometry of the hyperbolic 3-space have the following form:

$$\begin{cases} \gamma' = \mathbf{U}_1, \\ \mathbf{U}_1' = \rho^2\gamma + \mathcal{K}_1\mathbf{U}_1 + \mathbf{U}_2, \\ \mathbf{U}_2' = \mathbf{U}_1 + \mathcal{K}_1\mathbf{U}_2 + \mathcal{K}_2\mathbf{U}_3, \\ \mathbf{U}_3' = -\mathcal{K}_2\mathbf{U}_2 + \mathcal{K}_1\mathbf{U}_3. \end{cases} \tag{3.11}$$

Or in the matrix form as follows:

$$\begin{bmatrix} \gamma'(\sigma) \\ \mathbf{U}_1'(\sigma) \\ \mathbf{U}_2'(\sigma) \\ \mathbf{U}_3'(\sigma) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \rho^2 & \mathcal{K}_1 & 1 & 0 \\ 0 & 1 & \mathcal{K}_1 & \mathcal{K}_2 \\ 0 & 0 & -\mathcal{K}_2 & \mathcal{K}_1 \end{bmatrix} \begin{bmatrix} \gamma(\sigma) \\ \mathbf{U}_1(\sigma) \\ \mathbf{U}_2(\sigma) \\ \mathbf{U}_3(\sigma) \end{bmatrix}.$$

Notation . The equiform parameter $\sigma = \int \kappa(s)ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of the Euclidean space. Also, the function $\frac{\tau}{\kappa}$ has interesting geometric interpretation.

According to the equiform Frenet formulas (3.11), we can write the following equalities regarding equiform curvatures:

$$\begin{cases} \mathcal{K}_1 = \frac{1}{\rho^2}\left\langle \mathbf{U}_j', \mathbf{U}_j \right\rangle;\ \ (j = 1, 2, 3), \\ \mathcal{K}_2 = \frac{1}{\rho^2}\left\langle \mathbf{U}_2', \mathbf{U}_3 \right\rangle = \frac{-1}{\rho^2}\left\langle \mathbf{U}_3', \mathbf{U}_2 \right\rangle. \end{cases} \tag{3.12}$$

Here, we characterize the equiform space and the curves using their equiform curvatures $\mathcal{K}_1$ and $\mathcal{K}_2$ in $\mathbb{H}^3_+$, which have some important geometric interpretations as follows:

(1) If $\mathcal{K}_2 = const.$, then the curve is an equiform general helix and vice versa. Here, we do not set conditions on $\mathcal{K}_1$ (for more details, see [15, 16, 17]).
(2) If the above condition holds and $\mathcal{K}_1$ is identically zero, then the curve is a W-curve (for more details, see [17, 18]).

According to [17], we have the following theorem.

Theorem 3.1. *Let γ be a curve in $\mathbb{H}^3_+$ with the equiform invariant tetrahedron $\{\gamma, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$ and equiform curvature $\mathcal{K}_1 \neq 0$. Then γ has $\mathcal{K}_2 \equiv 0$ if and only if γ lies fully in a 2-dimensional subspace of $\mathbb{H}^3_+(-1)$.*

Proposition 3.2. *Let γ be an equiform curve with an equiform invariant vector $\mathbf{U}_3$ in the equiform geometry of $\mathbb{H}^3_+$. Then, the curve γ is an equiform general helix if and only if*

$$\mathbf{U}_3'' - (\mathcal{K}_1^2 - \mathcal{K}_2^2 - \mathcal{K}_1')\mathbf{U}_3 = -\mathcal{K}_2\mathbf{U}_1 - 2\mathcal{K}_1\mathcal{K}_2\mathbf{U}_2. \tag{3.13}$$

Proof. Suppose that the curve γ is an equiform general helix. Thus, we have

$$\mathcal{K}_2 = const. \tag{3.14}$$

From Eqs. (3.11) and (3.14) , it is easy to prove that the equation (3.13) is satisfied.
Conversely, we assume that the equation (3.13) holds. Then from (3.11), we find

$$\mathbf{U}_3' = -\mathcal{K}_2\mathbf{U}_2 + \mathcal{K}_1\mathbf{U}_3. \tag{3.15}$$

Differentiating Eq.(3.15) with respect to σ, we get

$$\begin{aligned}\mathbf{U}_3'' &= -\mathcal{K}_2'\mathbf{U}_2 - \mathcal{K}_2\mathbf{U}_2' + \mathcal{K}_1'\mathbf{U}_3 + \mathcal{K}_1\mathbf{U}_3' \\ &= -\mathcal{K}_2'\mathbf{U}_2 - \mathcal{K}_2(\mathbf{U}_1 + \mathcal{K}_1\mathbf{U}_2 + \mathcal{K}_2\mathbf{U}_3) \\ &\quad + \mathcal{K}_1'\mathbf{U}_3 + \mathcal{K}_1(-\mathcal{K}_2\mathbf{U}_2 + \mathcal{K}_1\mathbf{U}_3),\end{aligned}$$

then,

$$\mathbf{U}_3'' = -\mathcal{K}_2\mathbf{U}_1 - (2\mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_2')\mathbf{U}_2 + (\mathcal{K}_1^2 - \mathcal{K}_2^2 + \mathcal{K}_1')\mathbf{U}_3.$$

So, we obtain

$$\mathcal{K}_2' = 0,$$

which completes the proof. □

4. EQUIFORM GEOMETRY OF CURVES IN DE SITTER 3-SPACE

In this section, we consider $\gamma(s) : I \to \mathbb{S}_1^3$ as a curve parameterized by arc-length s. Then we can write

$$\begin{aligned}\mathbf{V}_1 &= \rho\,\mathbf{T}, \\ \mathbf{V}_2 &= \rho\,\mathbf{N}, \\ \mathbf{V}_3 &= \rho\,\mathbf{E}.\end{aligned} \tag{4.1}$$

Thus, $\{\gamma, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ is an equiform invariant tetrahedron of γ. The derivatives of these vectors with respect to σ are

$$\begin{aligned}\mathbf{V}_1' &= \frac{d}{d\sigma}(\mathbf{V}_1) = \rho\frac{d}{ds}(\rho\mathbf{T}) = \rho(\dot{\rho}\mathbf{T} + \rho\dot{\mathbf{T}}) = \rho(\dot{\rho}\mathbf{T} + \rho(-\gamma + \kappa\mathbf{N})) \\ &= -\rho^2\gamma + \dot{\rho}(\rho\mathbf{T}) + (\rho\mathbf{N}) = -\rho^2\gamma + \dot{\rho}\mathbf{V}_1 + \mathbf{V}_2.\end{aligned} \tag{4.2}$$

Also, we obtain

$$\begin{aligned}\mathbf{V}_2' &= \frac{d}{d\sigma}(\mathbf{V}_2) = \rho\frac{d}{ds}(\rho\mathbf{N}) = \rho(\dot{\rho}\mathbf{N} + \rho\dot{\mathbf{N}}) = \rho(\dot{\rho}\mathbf{N} + \rho(-\delta\kappa\mathbf{T} + \tau\mathbf{E})) \\ &= -\delta\rho\mathbf{T} + \dot{\rho}(\rho\mathbf{N}) + \frac{\tau}{\kappa}(\rho\mathbf{E}) = -\delta\mathbf{V}_1 + \dot{\rho}\mathbf{V}_2 + \frac{\tau}{\kappa}\mathbf{V}_3,\end{aligned} \tag{4.3}$$

and so,

$$\begin{aligned}\mathbf{V}_3' &= \frac{d}{d\sigma}(\mathbf{V}_3) = \rho\frac{d}{ds}(\rho\mathbf{E}) = \rho(\dot{\rho}\mathbf{E} + \rho\dot{\mathbf{E}}) = \rho(\dot{\rho}\mathbf{E} + \rho(\tau\mathbf{N})) \\ &= \frac{\tau}{\kappa}(\rho\mathbf{N}) + \dot{\rho}(\rho\mathbf{E}) = \frac{\tau}{\kappa}\mathbf{V}_2 + \dot{\rho}\mathbf{V}_3.\end{aligned} \tag{4.4}$$

Hence, the Frenet equations in the equiform geometry of the de Sitter 3-space $\mathbb{S}_1^3$ can be written as

$$\begin{cases}\gamma' = \mathbf{V}_1, \\ \mathbf{V}_1' = -\rho^2\gamma + \mathcal{K}_1\mathbf{V}_1 + \mathbf{V}_2, \\ \mathbf{V}_2' = -\delta\mathbf{V}_1 + \mathcal{K}_1\mathbf{V}_2 + \mathcal{K}_2\mathbf{V}_3, \\ \mathbf{V}_3' = \mathcal{K}_2\mathbf{V}_2 + \mathcal{K}_1\mathbf{V}_3.\end{cases} \tag{4.5}$$

Or in the matrix form as follows:

$$\begin{bmatrix}\gamma'(\sigma) \\ \mathbf{V}_1'(\sigma) \\ \mathbf{V}_2'(\sigma) \\ \mathbf{V}_3'(\sigma)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0 \\ -\rho^2 & \mathcal{K}_1 & 1 & 0 \\ 0 & -\delta & \mathcal{K}_1 & \mathcal{K}_2 \\ 0 & 0 & \mathcal{K}_2 & \mathcal{K}_1\end{bmatrix}\begin{bmatrix}\gamma(\sigma) \\ \mathbf{V}_1(\sigma) \\ \mathbf{V}_2(\sigma) \\ \mathbf{V}_3(\sigma)\end{bmatrix}.$$

Therefore, according to the equiform Frenet formulas (4.5), the following equalities regarding equiform curvatures are given as follows:

$$\begin{cases} \mathcal{K}_1 = \frac{1}{\rho^2}\left\langle \mathbf{V}_j', \mathbf{V}_j \right\rangle; \quad (j=1,2,3), \\ \mathcal{K}_2 = \frac{1}{\rho^2}\left\langle \mathbf{V}_2', \mathbf{V}_3 \right\rangle = \frac{1}{\rho^2}\left\langle \mathbf{V}_3', \mathbf{V}_2 \right\rangle. \end{cases} \tag{4.6}$$

Corollary 4.1. *Let γ be an equifrom curve in $\mathbb{S}_1^3$ with the equiform invariant tetrahedron $\{\gamma, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ and equiform curvature $\mathcal{K}_1 \neq 0$. Then γ has $\mathcal{K}_2 \equiv 0$ if and only if γ lies fully in a 2-dimensional subspace of $\mathbb{S}_1^3$.*

Proposition 4.2. *Let γ be an equiform curve with an equiform invariant vector $\mathbf{V}_3$ in the equiform geometry of $\mathbb{S}_1^3$. Then, the curve γ is an equiform general helix if and only if*

$$\mathbf{V}_3'' - (\mathcal{K}_1^2 + \mathcal{K}_2^2 + \mathcal{K}_1')\mathbf{V}_3 = -\delta\mathcal{K}_2\mathbf{V}_1 + 2\mathcal{K}_1\mathcal{K}_2\mathbf{V}_2. \tag{4.7}$$

Proof. Suppose that the curve γ is an equiform general helix. Thus, we have

$$\mathcal{K}_2 = const. \tag{4.8}$$

From Eqs. (4.5) and (4.8) , it is easy to prove that the equation (4.7) is satisfied.
Conversely, we assume that the equation (4.7) holds. Then from (4.5), we obtain

$$\mathbf{V}_3' = \mathcal{K}_2\mathbf{V}_2 + \mathcal{K}_1\mathbf{V}_3. \tag{4.9}$$

Differentiating Eq.(4.9) with respect to σ, we get

$$\begin{aligned} \mathbf{V}_3'' =& \mathcal{K}_2'\mathbf{V}_2 + \mathcal{K}_2\mathbf{V}_2' + \mathcal{K}_1'\mathbf{V}_3 + \mathcal{K}_1\mathbf{V}_3' \\ =& \mathcal{K}_2'\mathbf{V}_2 + \mathcal{K}_2(-\delta\mathbf{V}_1 + \mathcal{K}_1\mathbf{V}_2 + \mathcal{K}_2\mathbf{V}_3) \\ &+ \mathcal{K}_1'\mathbf{V}_3 + \mathcal{K}_1(\mathcal{K}_2\mathbf{V}_2 + \mathcal{K}_1\mathbf{V}_3), \end{aligned}$$

then,

$$\mathbf{V}_3'' = -\delta\mathcal{K}_2\mathbf{V}_1 + (2\mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_2')\mathbf{V}_2 + (\mathcal{K}_1^2 + \mathcal{K}_2^2 + \mathcal{K}_1')\mathbf{V}_3.$$

So, we obtain

$$\mathcal{K}_2' = 0,$$

which completes the proof. □

5. EQUIFORM GEOMETRY OF INVOLUTE-EVOLUTE CURVE COUPLE IN $\mathbb{H}_+^3$

In this section, we introduce the equiform geometry for Frenet apparatus of an evolute curve according to the Frenet apparatus of an involute curve in $\mathbb{H}_+^3(-1)$.

Definition 5.1. Let $\psi : I \longrightarrow \mathbb{H}_+^3(-1)$ be a regular spacelike curve in $\mathbb{H}_+^3(-1)$ with arc-length parameter s so that κ and τ are not zero. Let $\gamma : I \longrightarrow \mathbb{H}_+^3(-1)$ be the evolute curve of ψ with arc-length parameter $\bar{s} = f(s)$. Denote $\{\gamma, \mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{E}_\gamma\}$ to be the Frenet frame along γ and κ_γ, τ_γ to be the curvatures of γ. Then

$$span\{\gamma, \mathbf{E}_\gamma\} = span\{\mathbf{T}, \mathbf{N}\}, \quad span\{\mathbf{T}_\gamma, \mathbf{N}_\gamma\} = span\{\gamma, \mathbf{E}\},$$

γ can be expressed as

$$\gamma(s) = \psi(s) + \lambda_1(s)\mathbf{N}_\psi(s) + \lambda_2(s)\mathbf{E}_\psi(s),$$

where λ_1 and λ_2 are C^∞ functions on I (see [19]).

Theorem 5.1. *Let γ and ψ be unit speed space-like curves and γ be an evolute of ψ. The equiform Frenet apparatus of γ: $\{\mathbf{U}_{1\gamma}; \mathbf{U}_{2\gamma}; \mathbf{U}_{3\gamma}, \mathcal{K}_{1\gamma}; \mathcal{K}_{2\gamma}\}$ can be formed according to Frenet apparatus of ψ: $\{\mathbf{T}_\psi; \mathbf{N}_\psi; \mathbf{E}_\psi; \kappa_\psi; \tau_\psi\}$.*

Proof. From the definition of involute-evolute curve couple in hyperbolic 3-space, we can write

$$\gamma(s) = \psi(s) + \lambda_1(s)\mathbf{N}_\psi(s) + \lambda_2(s)\mathbf{E}_\psi(s). \tag{5.1}$$

Differentiating both sides of Eq.(5.1) with respect to s, we obtain

$$\dot{f}\mathbf{T}_\gamma = \mathbf{T}_\psi + \dot{\lambda}_1\mathbf{N}_\psi + \lambda_1(s)\dot{\mathbf{N}}_\psi(s) + \dot{\lambda}_2\mathbf{E}_\psi + \lambda_2(s)\dot{\mathbf{E}}_\psi(s),$$

Using Eqs.(2.1), we find

$$\begin{aligned}\dot{f}\mathbf{T}_\gamma =& \mathbf{T}_\psi + \dot{\lambda}_1\mathbf{N}_\psi + \dot{\lambda}_2\mathbf{E}_\psi + \lambda_1(-\kappa_\psi\mathbf{T}_\psi + \tau_\psi\mathbf{E}_\psi) + \lambda_2(-\tau_\psi\mathbf{N}_\psi)\\ =&(1-\kappa_\psi\lambda_1)\mathbf{T}_\psi + (\dot{\lambda}_1 - \tau_\psi\lambda_2)\mathbf{N}_\psi + (\dot{\lambda}_2 + \tau_\psi\lambda_1)\mathbf{E}_\psi.\end{aligned}$$

Recalling the definition of involute and evolute curve couple, we can say that

$$\mathbf{T}_\gamma \perp \mathbf{T}_\psi,$$

then, we get

$$\dot{\lambda}_1 - \tau_\psi\lambda_2 = 0, \qquad 1 - \kappa_\psi\lambda_1 = 0. \tag{5.2}$$

By solving Eqs.(5.2), we obtain

$$\lambda_1 = \frac{1}{\kappa_\psi}, \qquad \lambda_2 = -\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}. \tag{5.3}$$

From Eq.5.1, we have

$$\gamma(s) = \psi(s) + \frac{1}{\kappa_\psi}\mathbf{N}_\psi(s) - \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\mathbf{E}_\psi(s). \tag{5.4}$$

Differentiating both sides of Eq.(5.4) with respect to s and using Eqs.(2.1), we find

$$\mathbf{T}_\gamma = \frac{1}{\dot{f}}\left(\frac{\tau_\psi}{\kappa_\psi} - \left(\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^{\cdot}\right)\mathbf{E}_\psi(s), \tag{5.5}$$

it leads to

$$\begin{aligned}\dot{\mathbf{T}}_\gamma =& -\frac{\tau}{\dot{f}^2}\left(\frac{\tau_\psi}{\kappa_\psi} - \left(\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^{\cdot}\right)\mathbf{N}_\psi(s) + \frac{1}{\dot{f}^2}\left(\frac{\tau_\psi}{\kappa_\psi} - \left(\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^{\cdot}\right)^{\cdot}\mathbf{E}_\psi(s)\\ &-\frac{\ddot{f}}{\dot{f}^2}\left(\frac{\tau_\psi}{\kappa_\psi} - \left(\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^{\cdot}\right)\mathbf{E}_\psi(s).\end{aligned} \tag{5.6}$$

From Eqs.(2.2), (5.4) and (5.6), where

$$\mathbf{N}_\gamma = \frac{\dot{\mathbf{T}}_\gamma - \gamma}{\|\dot{\mathbf{T}}_\gamma - \gamma\|},$$

we obtain

$$\begin{aligned}\mathbf{N}_\gamma =&[(\frac{\tau}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})^2 + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})^2 - 1]^{-\frac{1}{2}}(-\psi(s)\\ &-(\frac{\tau}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})\mathbf{N}_\psi(s) + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})\mathbf{E}_\psi(s)),\end{aligned} \tag{5.7}$$

where

$$\Omega = \left(\frac{\tau_\psi}{\kappa_\psi} - \left(\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^{\cdot}\right).$$

Also, from Eqs.(5.7) and (2.2), we get

$$\kappa_\gamma(s) = \sqrt{\left(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi}\right)^2 + \left(\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^2 - 1}, \tag{5.8}$$

which implies that

$$\rho_\gamma(s) = \left[\left(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi}\right)^2 + \left(\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^2 - 1\right]^{-\frac{1}{2}}, \tag{5.9}$$

where $\rho_\gamma = \frac{1}{\kappa_\gamma}$. By differentiating Eq.(5.9), we find

$$\begin{aligned}\dot{\rho}_\gamma =&[(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})^2 + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})^2 - 1]^{-\frac{3}{2}}((\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})^{\cdot} \\ &+(\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})(\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})^{\cdot}).\end{aligned} \tag{5.10}$$

Therefore, from Eqs.(5.4), (5.5), and (5.7), we have

$$\mathbf{E}_\gamma(s) = \rho_\gamma \begin{vmatrix} -\psi(s) & \mathbf{T}_\psi & \mathbf{N}_\psi & \mathbf{E}_\psi \\ 1 & 0 & \frac{1}{\kappa_\psi} & -\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi} \\ 0 & 0 & 0 & \frac{1}{\dot{f}}\Omega \\ -1 & 0 & -(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi}) & (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}) \end{vmatrix},$$

which can be written as

$$\mathbf{E}_\gamma(s) = -\frac{\tau_\psi}{\dot{f}^2}\Omega \left[\left(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi}\right)^2 + \left(\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^2 - 1\right]^{-\frac{1}{2}} \mathbf{T}_\psi(s). \tag{5.11}$$

Now, we need to find $\dddot{\gamma}(s)$, therefore, from Eq.(5.6), we get

$$\begin{aligned}\dddot{\gamma}(s) =&\frac{1}{\dot{f}^4}((\dot{f}\kappa_\psi\tau_\psi)\mathbf{T}_\psi(s) + ((2+\dot{f})\ddot{f}\tau_\psi\Omega - 2\dot{f}\tau_\psi\dot{\Omega} - \dot{f}\dot{\tau}_\psi\Omega)\mathbf{N}_\psi \\ &+ ((\ddot{f}^2 - \dot{f}\tau_\psi^2 - \dot{f}\dddot{f})\Omega - (\ddot{f}(2+\dot{f}))\dot{\Omega} + \dot{f}\ddot{\Omega})\mathbf{E}_\psi),\end{aligned} \tag{5.12}$$

or in the form

$$\dddot{\gamma}(s) = \mu_1\mathbf{T}_\psi + \mu_2\mathbf{N}_\psi + \mu_3\mathbf{E}_\psi, \tag{5.13}$$

where

$$\begin{cases} \mu_1 = \frac{1}{\dot{f}^3}\kappa_\psi\tau_\psi \\ \mu_2 = \frac{1}{\dot{f}^4}((2+\dot{f})\ddot{f}\tau_\psi\Omega - 2\dot{f}\tau_\psi\dot{\Omega} - \dot{f}\dot{\tau}_\psi\Omega) \\ \mu_3 = \frac{1}{\dot{f}^4}((\ddot{f}^2 - \dot{f}\tau_\psi^2 - \dot{f}\dddot{f})\Omega - (\ddot{f}(2+\dot{f}))\dot{\Omega} + \dot{f}\ddot{\Omega}). \end{cases} \tag{5.14}$$

Also, from Eqs.(2.2), (5.4), (5.5), (5.6), and (5.13), we have

$$\tau_\gamma(s) = -\rho_\gamma^2 \begin{vmatrix} 1 & 0 & \frac{1}{\kappa_\psi} & -\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi} \\ 0 & 0 & 0 & \frac{1}{\dot{f}}\Omega \\ 0 & 0 & -\frac{\tau_\psi}{\dot{f}^2}\Omega & \frac{1}{\dot{f}^2}(\dot{\Omega}-\ddot{f}\Omega) \\ 0 & \mu_1 & \mu_2 & \mu_3 \end{vmatrix},$$

then,

$$\tau_\gamma(s) = -\frac{\mu\tau_\psi}{\dot{f}^3}\rho_\gamma^2\Omega^2. \tag{5.15}$$

From Eqs.(5.5), (5.7), (5.11), (5.8), (5.10) and (5.15), we obtain the equiform geometry of Frenet apparatus of the evolute curve according to Frenet apparatus of the involute curve as follows:

$$\begin{aligned}
\mathbf{U}_{1\gamma} &= \rho_\gamma \mathbf{T}_\gamma = \frac{\rho_\gamma}{\dot{f}}\left(\frac{\tau_\psi}{\kappa_\psi} - \left(\frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^{\cdot}\right)\mathbf{E}_\psi(s),\\
\mathbf{U}_{2\gamma} &= \rho_\gamma \mathbf{N}_\gamma\\
&= \rho_\gamma[(\frac{\tau}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})^2 + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})^2 - 1]^{-\frac{1}{2}}(-\psi(s)\\
&\quad - (\frac{\tau}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})\mathbf{N}_\psi(s) + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})\mathbf{E}_\psi(s)),\\
\mathbf{U}_{3\gamma} &= \rho_\gamma \mathbf{E}_\gamma\\
&= -\frac{\tau_\psi}{\dot{f}^3}\rho_\gamma\Omega[(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})^2 + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})^2 - 1]^{-\frac{1}{2}}\mathbf{T}_\psi(s),
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_{1\gamma} =& \dot{\rho}_\gamma\\
=& [(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})^2 + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})^2 - 1]^{-\frac{3}{2}}((\frac{\tau_\psi}{\dot{f}^2}\Omega\\
&+ \frac{1}{\kappa_\psi})(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi})^{\cdot} + (\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})(\frac{1}{\dot{f}^2}\dot{\Omega}\\
&- \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi})^{\cdot}),\\
\mathcal{K}_{2\gamma} =& \frac{\tau_\gamma}{\kappa_\gamma}\\
=& -\frac{\mu\tau_\psi}{\dot{f}^3}\Omega^2\left[\left(\frac{\tau_\psi}{\dot{f}^2}\Omega + \frac{1}{\kappa_\psi}\right)^2 + \left(\frac{1}{\dot{f}^2}\dot{\Omega} - \frac{\ddot{f}}{\dot{f}^2}\Omega + \frac{\dot{\kappa}_\psi}{\kappa_\psi^2\tau_\psi}\right)^2 - 1\right]^{-\frac{3}{2}}.
\end{aligned}$$

Which completes the proof. $\square$

6. EQUIFORM GEOMETRY OF INVOLUTE-EVOLUTE CURVE COUPLE IN $\mathbb{S}_1^3$

Definition 6.1. Let $\phi : I \longrightarrow \mathbb{S}_1^3$ be a regular spacelike curve in $\mathbb{S}_1^3$ with arc-length parameter s, so that κ and τ are not to be zero. Let $\beta : I \longrightarrow \mathbb{S}_1^3$ be the evolute curve of ϕ with arc-length parameter $\tilde{s} = g(s)$. Denote $\{\beta, \mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{E}_\beta\}$ to be the Frenet frame along β and κ_β, τ_β to be the curvatures of β. Then

$$span\{\beta, \mathbf{E}_\beta\} = span\{\mathbf{T}_\phi, \mathbf{N}_\phi\}, \quad span\{\mathbf{T}_\beta, \mathbf{N}_\beta\} = span\{\phi, \mathbf{E}_\phi\},$$

β can be expressed as

$$\beta(s) = \phi(s) + \nu_1(s)\mathbf{N}_\phi(s) + \nu_2(s)\mathbf{E}_\phi(s),$$

where ν_1 and ν_2 are C^∞ functions on I, (for more details, we refer to [19]).

Theorem 6.1. *Let β and ϕ be unit speed space-like curves and β be an evolute of ϕ. The equiform Frenet apparatus of β: $\{\mathbf{V}_{1\beta}; \mathbf{V}_{2\beta}; \mathbf{V}_{3\beta}; \mathcal{K}_{1\beta}; \mathcal{K}_{2\beta}\}$ can be formed according to Frenet apparatus of ϕ: $\{\mathbf{T}_\phi; \mathbf{N}_\phi; \mathbf{E}_\phi; \kappa_\phi; \tau_\phi\}$.*

Proof. From the definition of involute-evolute curve couple in de Sitter 3-space, we can write

$$\beta(s) = \phi(s) + \nu_1(s)\mathbf{N}_\phi(s) + \nu_2(s)\mathbf{E}_\phi(s). \tag{6.1}$$

Differentiating both sides of Eq.(6.1) with respect to s, we obtain

$$\dot{g}\mathbf{T}_\beta = \mathbf{T}_\phi + \dot{\nu}_1\mathbf{N}_\phi + \nu_1(s)\dot{\mathbf{N}}_\phi(s) + \dot{\nu}_2\mathbf{E}_\phi + \nu_2(s)\dot{\mathbf{E}}_\phi(s),$$

from Eqs.(2.3), we have

$$\begin{aligned} \dot{g}\mathbf{T}_\beta =& \mathbf{T}_\phi + \dot{\nu}_1\mathbf{N}_\phi + \dot{\nu}_2\mathbf{E}_\phi + \nu_1(-\delta\kappa_\phi\mathbf{T}_\phi + \tau_\phi\mathbf{E}_\phi) + \nu_2(\tau_\phi\mathbf{N}_\phi) \\ =& (1 - \delta\kappa_\phi\nu_1)\mathbf{T}_\phi + (\dot{\nu}_1 + \tau_\phi\nu_2)\mathbf{N}_\phi + (\dot{\nu}_2 + \tau_\phi\nu_1)\mathbf{E}_\phi. \end{aligned}$$

Recalling the definition of involute and evolute curve couple, we can say that

$$\mathbf{T}_\beta \perp \mathbf{T}_\phi,$$

then, we get

$$\dot{\nu}_1 + \tau_\phi\nu_2 = 0, \qquad 1 - \delta\kappa_\phi\nu_1 = 0. \tag{6.2}$$

By solving Eqs.(6.2), we get

$$\nu_1 = \frac{1}{\delta\kappa_\phi}, \qquad \nu_2 = \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}. \tag{6.3}$$

Rewriting Eq.6.1, we obtain

$$\beta(s) = \phi(s) + \frac{1}{\delta\kappa_\phi}\mathbf{N}_\phi(s) + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\mathbf{E}_\phi(s). \tag{6.4}$$

Differentiating both sides of Eq.(6.4) with respect to s and then from Eqs.(2.3), we find

$$\mathbf{T}_\beta = \frac{1}{\dot{g}}\left(\frac{\tau_\phi}{\delta\kappa_\phi} + \left(\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^{\cdot}\right)\mathbf{E}_\phi(s). \tag{6.5}$$

Therefore, we have

$$\begin{aligned} \dot{\mathbf{T}}_\beta =& \frac{\tau_\phi}{\dot{g}^2}\left(\frac{\tau_\phi}{\delta\kappa_\phi} + \left(\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^{\cdot}\right)\mathbf{N}_\phi(s) + \frac{1}{\dot{g}^2}\left(\frac{\tau_\phi}{\delta\kappa_\phi} + \left(\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^{\cdot}\right)^{\cdot}\mathbf{E}_\phi(s) \\ & - \frac{\ddot{g}}{\dot{g}^2}\left(\frac{\tau_\phi}{\delta\kappa_\phi} + \left(\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^{\cdot}\right)\mathbf{E}_\phi(s). \end{aligned} \tag{6.6}$$

From Eqs.(2.2), (6.4) and (6.6), where

$$\mathbf{N}_\beta = \frac{\dot{\mathbf{T}}_\beta + \beta}{\|\dot{\mathbf{T}}_\beta + \beta\|},$$

one can get

$$\begin{aligned}\mathbf{N}_\beta =&[(\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi})^2+(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})^2+1]^{-\frac{1}{2}}(\phi(s)\\
&+(\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi})\mathbf{N}_\phi(s)+(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})\mathbf{E}_\phi(s)),\end{aligned}\tag{6.7}$$

where

$$\Upsilon=\left(\frac{\tau_\phi}{\delta\kappa_\phi}+\left(\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^{\cdot}\right).$$

Also, from Eqs.(6.7) and (2.4), we obtain

$$\kappa_\beta(s)=\sqrt{\left(\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi}\right)^2+\left(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^2+1},\tag{6.8}$$

it implies

$$\rho_\beta(s)=\left[\left(\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi}\right)^2+\left(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^2+1\right]^{-\frac{1}{2}},\tag{6.9}$$

where $\rho_\beta=\frac{1}{\kappa_\beta}$. By differentiating Eq.(6.9), we find

$$\begin{aligned}\dot{\rho}_\beta =&[(\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi})^2+(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})^2+1]^{-\frac{3}{2}}((\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi})(\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi})^{\cdot}\\
&+(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})^{\cdot}).\end{aligned}\tag{6.10}$$

Therefore, from Eqs.(6.4), (6.5), and (6.7), we have

$$\mathbf{E}_\beta(s)=\rho_\beta\begin{vmatrix}-\phi(s) & \mathbf{T}_\phi & \mathbf{N}_\phi & \mathbf{E}_\phi\\
1 & 0 & \frac{1}{\delta\kappa_\phi} & \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\\
0 & 0 & 0 & \frac{1}{\dot{g}}\Upsilon\\
1 & 0 & (\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi}) & (\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})\end{vmatrix},$$

which can be written as

$$\mathbf{E}_\beta(s)=\frac{\tau_\phi}{\dot{g}^3}\Upsilon^2\left[\left(\frac{\tau_\phi}{\dot{g}^2}\Upsilon+\frac{1}{\delta\kappa_\phi}\right)^2+\left(\frac{1}{\dot{g}^2}\dot{\Upsilon}-\frac{\ddot{g}}{\dot{g}^2}\Upsilon+\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^2+1\right]^{-\frac{1}{2}}\mathbf{T}_\phi(s).\tag{6.11}$$

Now, we need to find $\dddot{\beta}(s)$, so from Eq.(6.6), we find

$$\begin{aligned}\dddot{\beta}(s)=&\frac{1}{\dot{g}^4}((-\delta\dot{g}\kappa_\phi\tau_\phi)\mathbf{T}_\phi(s)-((2+\dot{g})\ddot{g}\tau_\phi\Upsilon-2\dot{g}\tau_\phi\dot{\Upsilon}-\dot{g}\dot{\tau}_\phi\Upsilon)\mathbf{N}_\phi\\
&+((-2\ddot{g}^2+\dot{g}\tau_\phi^2-\dot{g}\,\dddot{g})\Upsilon-(\ddot{g}(2+\dot{g}))\dot{\Upsilon}+\dot{g}\ddot{\Upsilon})\mathbf{E}_\phi),\end{aligned}\tag{6.12}$$

or in the form:

$$\dddot{\beta}(s)=\eta_1\mathbf{T}_\phi+\eta_2\mathbf{N}_\phi+\eta_3\mathbf{E}_\phi,\tag{6.13}$$

where

$$\begin{cases} \eta_1 = -\frac{1}{\dot{g}^3}\delta\kappa_\phi\tau_\phi \\ \eta_2 = -\frac{1}{\dot{g}^4}((2+\dot{g})\ddot{g}\tau_\phi\Upsilon - 2\dot{g}\tau_\phi\dot{\Upsilon} - \dot{g}\dot{\tau}_\phi\Upsilon) \\ \eta_3 = \frac{1}{\dot{g}^4}((-2\ddot{g}^2 + \dot{g}\tau_\phi^2 - \dot{g}\,\dddot{g})\Upsilon - (\ddot{g}(2+\dot{g}))\dot{\Upsilon} + \dot{g}\ddot{\Upsilon}). \end{cases} \tag{6.14}$$

Also, from Eqs.(2.4), (6.4), (6.5), (6.6), and (6.13), we obtain

$$\tau_\beta(s) = -\rho_\beta^2 \begin{vmatrix} 1 & 0 & \frac{1}{\delta\kappa_\phi} & \dfrac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi} \\ 0 & 0 & 0 & \frac{1}{\dot{g}}\Upsilon \\ 0 & 0 & \dfrac{\tau_\phi}{\dot{g}^2}\Upsilon & \frac{1}{\dot{g}^2}(\dot{\Upsilon} - \ddot{g}\Upsilon) \\ 0 & \eta_1 & \eta_2 & \eta_3 \end{vmatrix},$$

then, we get

$$\tau_\beta(s) = \frac{\eta\tau_\phi}{\dot{g}^3}\rho_\beta^2\Upsilon^2. \tag{6.15}$$

From Eqs.(6.5), (6.7), (6.11), (6.8), (6.9) and (6.15), we obtain

$$\begin{aligned} \mathbf{V}_{1\beta} &= \rho_\beta\mathbf{T}_\beta = \frac{\rho_\beta}{\dot{g}}\left(\frac{\tau_\phi}{\delta\kappa_\phi} + \left(\frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^{\cdot}\right)\mathbf{E}_\phi(s), \\ \mathbf{V}_{2\beta} &= \rho_\beta\mathbf{N}_\beta \\ &= \rho_\beta[(\frac{\tau_\phi}{\dot{g}^2}\Upsilon + \frac{1}{\delta\kappa_\phi})^2 + (\frac{1}{\dot{g}^2}\dot{\Upsilon} - \frac{\ddot{g}}{\dot{g}^2}\Upsilon + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})^2 + 1]^{-\frac{1}{2}}(\phi(s) \\ &\quad + (\frac{\tau_\phi}{\dot{g}^2}\Upsilon + \frac{1}{\delta\kappa_\phi})\mathbf{N}_\phi(s) + (\frac{1}{\dot{g}^2}\dot{\Upsilon} - \frac{\ddot{g}}{\dot{g}^2}\Upsilon + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})\mathbf{E}_\phi(s)), \\ \mathbf{V}_{3\beta} &= \rho_\beta\mathbf{E}_\beta \\ &= \frac{\tau_\phi}{\dot{g}^3}\Upsilon^2\left[\left(\frac{\tau_\phi}{\dot{g}^2}\Upsilon + \frac{1}{\delta\kappa_\phi}\right)^2 + \left(\frac{1}{\dot{g}^2}\dot{\Upsilon} - \frac{\ddot{g}}{\dot{g}^2}\Upsilon + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^2 + 1\right]^{-\frac{1}{2}}\mathbf{T}_\phi(s), \end{aligned}$$

$$\begin{aligned} \mathcal{K}_{1\beta} =& \dot{\rho}_\beta \\ =& [(\frac{\tau_\phi}{\dot{g}^2}\Upsilon + \frac{1}{\delta\kappa_\phi})^2 + (\frac{1}{\dot{g}^2}\dot{\Upsilon} - \frac{\ddot{g}}{\dot{g}^2}\Upsilon + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})^2 + 1]^{-\frac{3}{2}}((\frac{\tau_\phi}{\dot{g}^2}\Upsilon \\ & + \frac{1}{\delta\kappa_\phi})(\frac{\tau_\phi}{\dot{g}^2}\Upsilon + \frac{1}{\delta\kappa_\phi})^{\cdot} + (\frac{1}{\dot{g}^2}\dot{\Upsilon} - \frac{\ddot{g}}{\dot{g}^2}\Upsilon + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})(\frac{1}{\dot{g}^2}\dot{\Upsilon} \\ & - \frac{\ddot{g}}{\dot{g}^2}\Upsilon + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi})^{\cdot}), \\ \mathcal{K}_{2\beta} =& \frac{\tau_\beta}{\kappa_\beta} \\ =& \frac{\eta\tau_\phi}{\dot{g}^3}\Upsilon^2\left[\left(\frac{\tau_\phi}{\dot{g}^2}\Upsilon + \frac{1}{\delta\kappa_\phi}\right)^2 + \left(\frac{1}{\dot{g}^2}\dot{\Upsilon} - \frac{\ddot{g}}{\dot{g}^2}\Upsilon + \frac{\dot{\kappa}_\phi}{\delta\kappa_\phi^2\tau_\phi}\right)^2 + 1\right]^{-\frac{3}{2}}. \end{aligned}$$

Which completes the proof. □

7. EXAMPLES

Finally, in this section we present two computational examples to calculate Frenet apparatus of the two equiform curves in three-dimensional hyperbolic and de Sitter spaces.

Example 7.1. *We assume that the general involute helix ψ in $\mathbb{H}^3_+(-1)$ is given by*

$$\psi(s) = \left(\sqrt{5}\cosh(s), \sqrt{5}\sinh(s), 2\sin(s), 2\cos(s)\right). \tag{7.1}$$

From Eq.(7.1), the Frenet apparatus of the curve ψ is

$$\begin{cases} \mathbf{T}_\psi(s) = \left(\sqrt{5}\sinh(s), \sqrt{5}\cosh(s), 2\cos(s), -2\sin(s)\right), \\ \mathbf{N}_\psi(s) = (0, 0, -\cos(s), -\sin(s)), \\ \mathbf{E}_\psi(s) = \left(2\sqrt{5}\sinh(s), 2\sqrt{5}\cosh(s), -5\cos(s), 5\sin(s)\right), \\ \kappa_\psi = 4, \qquad \tau_\psi = 5. \end{cases} \tag{7.2}$$

Then, from Eqs.(5.4), (7.1) and (7.2), the evolute curve of ψ is

$$\gamma(s) = \left(\sqrt{5}\cosh(s), \sqrt{5}\sinh(s), \frac{7}{4}\sin(s), \frac{7}{4}\cos(s)\right). \tag{7.3}$$

From Eq.(7.3), the tangent vector of the curve γ is given as

$$\mathbf{T}_\gamma(s) = \left(\sqrt{5}\sinh(s), \sqrt{5}\cosh(s), \frac{7}{4}\cos(s), -\frac{7}{4}\sin(s)\right), \tag{7.4}$$

and, we get

$$\dot{\mathbf{T}}_\gamma(s) \left(\sqrt{5}\cosh(s), \sqrt{5}\sinh(s), -\frac{7}{4}\sin(s), -\frac{7}{4}\cos(s)\right). \tag{7.5}$$

From Eqs.(7.3) and (7.5), we obtain

$$\begin{aligned} \mathbf{N}_\gamma(s) &= \frac{\dot{\mathbf{T}}_\gamma(s) - \gamma(s)}{\|\dot{\mathbf{T}}_\gamma(s) - \gamma(s)\|} \\ &= (0, 0, -\sin(s), -\cos(s)), \end{aligned} \tag{7.6}$$

The curvature of γ is given by

$$\kappa_\gamma(s) = \|\dot{\mathbf{T}}_\gamma(s) - \gamma(s)\| = \frac{7}{2}.$$

Also, we get

$$\begin{aligned} \mathbf{E}_\gamma(s) &= \gamma(s) \wedge \mathbf{T}_\gamma(s) \wedge \mathbf{N}_\gamma(s) \\ &= \begin{vmatrix} -i & j & k & l \\ \sqrt{5}\cosh(s) & \sqrt{5}\sinh(s) & \frac{7}{4}\sin(s) & \frac{7}{4}\cos(s) \\ \sqrt{5}\sinh(s) & \sqrt{5}\cosh(s) & \frac{7}{4}\cos(s) & -\frac{7}{4}\sin(s) \\ 0 & 0 & -\sin(s) & -\cos(s) \end{vmatrix}, \end{aligned}$$

or in the form

$$\mathbf{E}_\gamma(s) = -\begin{vmatrix} \sqrt{5}\sinh(s) & \frac{7}{4}\sin(s) & \frac{7}{4}\cos(s) \\ \sqrt{5}\cosh(s) & \frac{7}{4}\cos(s) & -\frac{7}{4}\sin(s) \\ 0 & -\sin(s) & -\cos(s) \end{vmatrix} i - \begin{vmatrix} \sqrt{5}\cosh(s) & \frac{7}{4}\sin(s) & \frac{7}{4}\cos(s) \\ \sqrt{5}\sinh(s) & \frac{7}{4}\cos(s) & -\frac{7}{4}\sin(s) \\ 0 & -\sin(s) & -\cos(s) \end{vmatrix} j$$

$$+\begin{vmatrix} \sqrt{5}\cosh(s) & \sqrt{5}\sinh(s) & \frac{7}{4}\cos(s) \\ \sqrt{5}\sinh(s) & \sqrt{5}\cosh(s) & -\frac{7}{4}\sin(s) \\ 0 & 0 & -\cos(s) \end{vmatrix} k - \begin{vmatrix} \sqrt{5}\cosh(s) & \sqrt{5}\sinh(s) & \frac{7}{4}\sin(s) \\ \sqrt{5}\sinh(s) & \sqrt{5}\cosh(s) & \frac{7}{4}\cos(s) \\ 0 & 0 & -\sin(s) \end{vmatrix} l,$$

then, we have

$$\mathbf{E}_\gamma(s) = \left(\frac{7\sqrt{5}}{4}\sinh(s), \frac{7\sqrt{5}}{4}\cosh(s), -5\cos(s), 5\sin(s)\right).$$

Therefore, we obtain

$$\det(\gamma, \dot{\gamma}, \ddot{\gamma}, \dddot{\gamma}) = \begin{vmatrix} \sqrt{5}\cosh(s) & \sqrt{5}\sinh(s) & \frac{7}{4}\sin(s) & \frac{7}{4}\cos(s) \\ \sqrt{5}\sinh(s) & \sqrt{5}\cosh(s) & \frac{7}{4}\cos(s) & -\frac{7}{4}\sin(s) \\ \sqrt{5}\cosh(s) & \sqrt{5}\sinh(s) & -\frac{7}{4}\sin(s) & -\frac{7}{4}\cos(s) \\ \sqrt{5}\sinh(s) & \sqrt{5}\cosh(s) & -\frac{7}{4}\cos(s) & \frac{7}{4}\sin(s) \end{vmatrix},$$

then,

$$\det(\gamma, \dot{\gamma}, \ddot{\gamma}, \dddot{\gamma}) = -\frac{245}{4}.$$

and we have

$$\tau_\gamma(s) = -\frac{\det(\gamma, \dot{\gamma}, \ddot{\gamma}, \dddot{\gamma})}{\kappa_\gamma^2} = 5.$$

From Eqs. (3.5) *and* (3.6)*, the equiform invariant trihedron* $\{\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$ *of the evolute curve* γ *is obtained as follows:*

$$\mathbf{U}_1 = \left(\frac{2\sqrt{5}}{7}\sinh(s), \frac{2\sqrt{5}}{7}\cosh(s), \frac{1}{2}\cos(s), -\frac{1}{2}\sin(s)\right),$$

$$\mathbf{U}_2 = \left(0, 0, -\frac{2}{7}\sin(s), -\frac{2}{7}\cos(s)\right),$$

$$\mathbf{U}_3 = \left(\frac{\sqrt{5}}{2}\sinh(s), \frac{\sqrt{5}}{2}\cosh(s), -\frac{10}{7}\cos(s), \frac{10}{7}\sin(s)\right).$$

Hence, using Eqs. (3.10)*, we get*

$$\mathcal{K}_1 = 0, \quad \mathcal{K}_2 = \frac{10}{7}.$$

Example 7.2. *Consider the general involute helix* $\phi(s)$ *in* $\mathbb{S}_1^3$ *parameterized by*

$$\phi(s) = \left(\cosh(s), \sinh(s), \sqrt{2}\sin(s), \sqrt{2}\cos(s)\right). \tag{7.7}$$

From Eq. (7.7)*, the Frenet apparatus of the curve* ϕ *is calculated as follows:*

$$\begin{cases} \mathbf{T}_\phi(s) = \left(\sinh(s), \cosh(s), \sqrt{2}\cos(s), -\sqrt{2}\sin(s)\right), \\ \mathbf{N}_\phi(s) = \left(\cosh(s), \sinh(s), 0, 0\right), \\ \mathbf{E}_\phi(s) = \left(2\sinh(s), 2\cosh(s), -\sqrt{2}\cos(s), \sqrt{2}\sin(s)\right), \\ \kappa_\phi = 2, \qquad \tau_\phi = 2. \end{cases} \tag{7.8}$$

Therefore, from Eqs.(6.4), (7.7) and (7.8), the evolute curve of ϕ is

$$\beta(s) = \left(\frac{3}{2}\cosh(s), \frac{3}{2}\sinh(s), \sqrt{2}\sin(s), \sqrt{2}\cos(s)\right). \tag{7.9}$$

From Eq.(7.9), the tangent vector of β is given by

$$\mathbf{T}_\beta(s) = \left(\frac{3}{2}\sinh(s), \frac{3}{2}\cosh(s), \sqrt{2}\cos(s), -\sqrt{2}\sin(s)\right). \tag{7.10}$$

Also, we get

$$\dot{\mathbf{T}}_\beta(s) = \left(\frac{3}{2}\cosh(s), \frac{3}{2}\sinh(s), -\sqrt{2}\sin(s), -\sqrt{2}\cos(s)\right). \tag{7.11}$$

From Eqs. (7.9) and (7.11), we have

$$\mathbf{N}_\beta(s) = (\cosh(s), \sinh(s), 0, 0) \tag{7.12}$$

where

$$\mathbf{N}_\beta(s) = \frac{\dot{\mathbf{T}}_\beta(s) + \beta(s)}{\|\dot{\mathbf{T}}_\beta(s) + \beta(s)\|}.$$

Therefore, we can compute the curvature of β as follows:

$$\kappa_\beta(s) = \|\dot{\mathbf{T}}_\beta(s) + \beta(s)\| = 3.$$

Also, we get

$$\begin{aligned}\mathbf{E}_\beta(s) =& \beta(s) \wedge \mathbf{T}_\beta(s) \wedge \mathbf{N}_\beta(s)\\ =& \begin{vmatrix} -i & j & k & l \\ \frac{3}{2}\cosh(s) & \frac{3}{2}\sinh(s) & \sqrt{2}\sin(s) & \sqrt{2}\cos(s) \\ \frac{3}{2}\sinh(s) & \frac{3}{2}\cosh(s) & \sqrt{2}\cos(s) & -\sqrt{2}\sin(s) \\ \cosh(s) & \sinh(s) & 0 & 0 \end{vmatrix},\end{aligned}$$

or in the form

$$\mathbf{E}_\beta(s) = -\begin{vmatrix} \frac{3}{2}\sinh(s) & \sqrt{2}\sin(s) & \sqrt{2}\cos(s) \\ \frac{3}{2}\cosh(s) & \sqrt{2}\cos(s) & -\sqrt{2}\sin(s) \\ \sinh(s) & 0 & 0 \end{vmatrix} i - \begin{vmatrix} \frac{3}{2}\cosh(s) & \sqrt{2}\sin(s) & \sqrt{2}\cos(s) \\ \frac{3}{2}\sinh(s) & \sqrt{2}\cos(s) & -\sqrt{2}\sin(s) \\ \cosh(s) & 0 & 0 \end{vmatrix} j$$

$$+ \begin{vmatrix} \frac{3}{2}\cosh(s) & \frac{3}{2}\sinh(s) & \sqrt{2}\cos(s) \\ \frac{3}{2}\sinh(s) & \frac{3}{2}\cosh(s) & -\sqrt{2}\sin(s) \\ \cosh(s) & \sinh(s) & 0 \end{vmatrix} k - \begin{vmatrix} \frac{3}{2}\cosh(s) & \frac{3}{2}\sinh(s) & \sqrt{2}\sin(s) \\ \frac{3}{2}\sinh(s) & \frac{3}{2}\cosh(s) & \sqrt{2}\cos(s) \\ \cosh(s) & \sinh(s) & 0 \end{vmatrix} l,$$

then, we get

$$\mathbf{E}_\beta(s) = \left(2\sinh(s), 2\cosh(s), -\frac{3\sqrt{2}}{2}\cos(s), \frac{3\sqrt{2}}{2}\sin(s)\right). \tag{7.13}$$

By differentiating Eq. (7.11), *we have*

$$\dddot{\beta}(s) = \left(\frac{3}{2}\sinh(s), \frac{3}{2}\cosh(s), -\sqrt{2}\cos(s), \sqrt{2}\sin(s)\right). \tag{7.14}$$

Therefore, from Eqs.(7.9), (7.10), (7.11), and (7.14), we find

$$\det(\beta, \dot{\beta}, \ddot{\beta}, \dddot{\beta}) = -18,$$

then, we get

$$\tau_\beta(s) = -\frac{\det(\beta, \dot{\beta}, \ddot{\beta}, \dddot{\beta})}{\kappa_\beta^2} = 2.$$

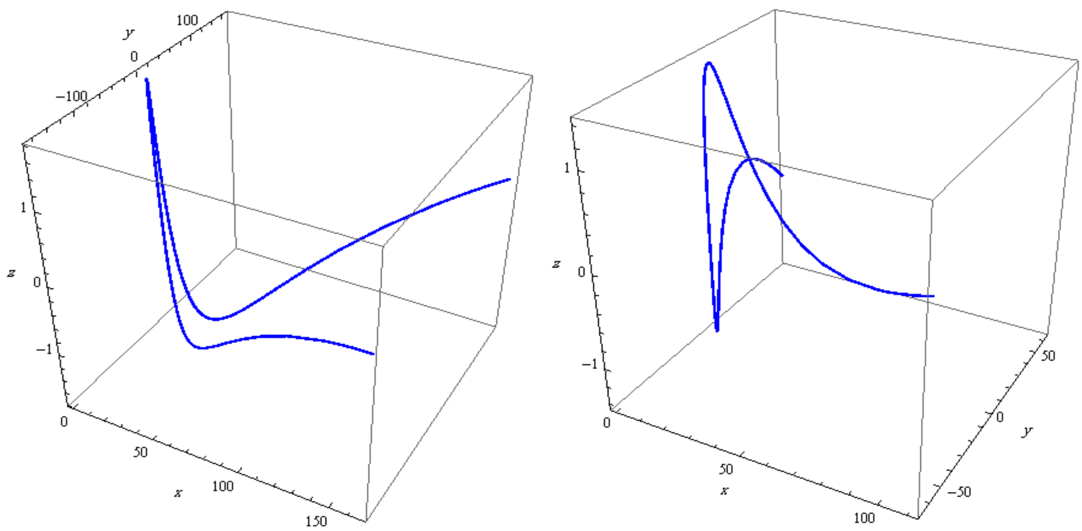
From Eq. (4.1)*, the equiform invariant trihedron of the evolute curve* β *is calculated as follows:*

$$\mathbf{V}_1 = \left(\frac{1}{2}\sinh(s), \frac{1}{2}\cosh(s), \frac{\sqrt{2}}{3}\cos(s), -\frac{\sqrt{2}}{3}\sin(s)\right),$$
$$\mathbf{V}_2 = \left(\frac{1}{3}\cosh(s), \frac{1}{3}\sinh(s), 0, 0\right),$$
$$\mathbf{V}_3 = \left(\frac{2}{3}\sinh(s), \frac{2}{3}\cosh(s), -\frac{\sqrt{2}}{2}\cos(s), \frac{\sqrt{2}}{2}\sin(s)\right).$$

Also, by using Eqs. (3.10)*, we obtain*

$$\mathcal{K}_1 = 0, \qquad \mathcal{K}_2 = 6.$$

In the following figure, one can see the projections of the evolute curves $\gamma(s)$ and $\beta(s)$ into $x1x2x3$, $x1x2x4$-spaces, respectively.



(A) (B)

Figure 1: (A) The evolute curve $\gamma(s)$, (B) The evolute curve $\beta(s)$.

CONCLUSION

In the 3-dimensional hyperbolic $\mathbb{H}^3_+(-1)$ and de Sitter $\mathbb{S}^3_1$ spaces, the equiform differential geometry of involute-evolute curve couple have been investigated. Also, Frenet apparatus for these curves have been obtained. Moreover, some characterizations of these curves using their equiform curvatures $\mathcal{K}_i$ $(i = 1, 2)$ have been introduced. Finally, some computational examples to confirm our main results are given and plotted. In future works, we plan to study the involute-evolute curve couple in different spaces like Galilean and pseudo-Galilean spaces for different queries and further improve the results in this paper, combined with the techniques and results in [20, 21, 22, 23, 24, 25, 26].

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