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SOME OSTROWSKI TYPE INEQUALITIES FOR TWO COS-INTEGRAL TRANSFORMS OF ABSOLUTELY CONTINUOUS FUNCTIONS

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ABSTRACT. For a Lebesgue integrable function $f : [a, b] \subset [0, \pi] \to \mathbb{C}$ we consider *the cos-integral transforms*

$$C_f(x) := \int_a^b f(t)\cos(x - t)\,dt,\ x \in [a, b]$$

and

$$\widetilde{C}_f(x) := \int_a^x f(t)\cos(t - a)\,dt + \int_x^b f(t)\cos(b - t)\,dt,\ x \in [a, b].$$

We provide in this paper some upper bounds for the quantities

$$|C_f(x) - f(a)\sin(x - a) - f(b)\sin(b - x)|$$

and

$$\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2} - x\right)f(x) - \widetilde{C}_f(x)\right|$$

for $x \in [a, b]$, in terms of the p-norms of the derivative f' for absolutely continuous functions $f : [a, b] \subset [0, \pi] \to \mathbb{C}$. Applications for approximating *Steklov cos-average functions* and *Steklov split cos-average functions* are also provided.

Key words and phrases: Lebesgue integral, Ostrowski inequality, Integral transforms, Absolutely continuous functions.

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1. INTRODUCTION

In 1938, A. Ostrowski [6], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

The following result, which is an improvement on Ostrowski's inequality, holds.

Theorem 1.2 (Dragomir, 2002 [3])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_\infty[a,b]$. Then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right] \\
&\le \|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a);
\end{aligned} \tag{1.2}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual norm on $L_\infty[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty} = ess\sup_{t\in[m,n]}|g(t)|<\infty.$$

The case of 1-norm is as follows:

Theorem 1.3 (Dragomir, 2002 [2])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{x-a}{b-a}\|f'\|_{[a,x],1}+\frac{b-x}{b-a}\|f'\|_{[x,b],1} \\
&\le \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}
\end{aligned} \tag{1.3}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],1}$ denotes the usual norm on $L_1[m,n]$ with $m<n$, i.e., we recall that

$$\|g\|_{[m,n],1} := \int_m^n |g(t)|\,dt<\infty.$$

The following inequality for the p-norms also holds.

Theorem 1.4 (Dragomir, 2013 [4]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.4}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[a,x],p}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[x,b],p}\right](b-a)^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}}$$
$$\times\left(\|f'\|^{\alpha}_{[a,x],p}+\|f'\|^{\alpha}_{[x,b],p}\right)^{\frac{1}{\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}\beta}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}}(b-a)^{1/q}$$

for all $x \in [a,b]$, where $\alpha > 1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[m,n],p}$ denotes the usual p-norm on $L_p[m,n]$ with $m<n$, i.e., we recall that

$$\|g\|_{[m,n],p} := \left(\int_m^n |g(t)|\,dt\right)^{1/p} < \infty.$$

More related results are presented in recent survey paper [5].

For a Lebesgue integrable function $f : [a,b] \subset [0,\pi] \to \mathbb{C}$ we consider *the cos-integral transforms*

$$C_f(x) := \int_a^b f(t)\cos(x-t)\,dt,\ x\in[a,b]$$

and

$$\widetilde{C}_f(x) := \int_a^x f(t)\cos(t-a)\,dt + \int_x^b f(t)\cos(b-t)\,dt,\ x\in[a,b].$$

Motivated by the above results, we provide in this paper some upper bounds for the quantities

$$|C_f(x)-f(a)\sin(x-a)-f(b)\sin(b-x)|$$

and

$$\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)f(x)-\widetilde{C}_f(x)\right|$$

for $x\in[a,b]$, in terms of the p-norms of the derivative f' for absolutely continuous functions $f:[a,b]\subset[0,\pi]\to\mathbb{C}$.

Applications for approximating *Steklov cos-average functions* and *Steklov split cos-average functions* are also provided.

2. ERROR BOUNDS FOR THE TRANSFORM C_f

The first main result is as follows:

Theorem 2.1. *If f is absolutely continuous on $[a,b]\subset[0,\pi]$ with $f'\in L_\infty[a,b]$, then*

$$|C_f(x)-f(a)\sin(x-a)-f(b)\sin(b-x)| \tag{2.1}$$
$$\leq 2\left[\|f'\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{2}\right)+\|f'\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{2}\right)\right]$$
$$\leq 2\|f'\|_{[a,b],\infty}\left[\sin^2\left(\frac{x-a}{2}\right)+\sin^2\left(\frac{b-x}{2}\right)\right]$$

for all $x\in[a,b]$.

If $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $f'\in L_p[a,b]$, *then*

$$\begin{aligned}(2.2)\quad &\left|C_f(x)-f(a)\sin(x-a)-f(b)\sin(b-x)\right|\\ &\leq \|f'\|_{[a,x],p}\left(\int_a^x \sin^q(x-t)\,dt\right)^{1/q}+\|f'\|_{[x,b],p}\left(\int_x^b \sin^q(t-x)\,dt\right)^{1/q}\\ &\leq \|f'\|_{[a,b],p}\left[\int_a^b \sin^q|x-t|\,dt\right]^{1/q}\end{aligned}$$

for all $x\in[a,b]$.

Also,

$$\begin{aligned}(2.3)\quad &\left|C_f(x)-f(a)\sin(x-a)-f(b)\sin(b-x)\right|\\ &\leq \max_{t\in[a,x]}[\sin(x-t)]\,\|f'\|_{[a,x],1}+\max_{t\in[x,b]}[\sin(t-x)]\,\|f'\|_{[x,b],1}\\ &\leq \|f'\|_{[a,b],1}\max\left\{\max_{t\in[a,x]}[\sin(x-t)],\max_{t\in[x,b]}[\sin(t-x)]\right\}\end{aligned}$$

for all $x\in[a,b]$.

Proof. Using the integration by parts formula, we get

$$\begin{aligned}(2.4)\quad &\int_a^b f'(t)\sin(x-t)\,dt\\ &= f(t)\sin(x-t)\Big|_a^b+\int_a^b f(t)\cos(x-t)\,dt\\ &= f(b)\sin(x-b)-f(a)\sin(x-a)+\int_a^b f(t)\cos(x-t)\,dt\\ &= C_f(x)-f(a)\sin(x-a)-f(b)\sin(b-x)\end{aligned}$$

for all $x\in[a,b]$.

By taking the modulus, we get, since $|x-t|\leq\pi$, that

$$\begin{aligned}(2.5)\quad &\left|C_f(x)-f(a)\sin(x-a)-f(b)\sin(b-x)\right|\\ &=\left|\int_a^b f'(t)\sin(x-t)\,dt\right|\leq\int_a^b |f'(t)|\,|\sin(x-t)|\,dt\\ &=\int_a^x |f'(t)|\sin(x-t)\,dt+\int_x^b |f'(t)|\sin(t-x)\,dt\\ &\leq \|f'\|_{[a,x],\infty}\int_a^x \sin(x-t)\,dt+\|f'\|_{[x,b],\infty}\int_x^b \sin(t-x)\,dt\\ &=\|f'\|_{[a,x],\infty}(1-\cos(x-a))+\|f'\|_{[x,b],\infty}(1-\cos(b-x))\\ &=2\left[\|f'\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{2}\right)+\|f'\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{2}\right)\right]\end{aligned}$$

for all $x\in[a,b]$.

Finally, observe that $\max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}=\|f'\|_{[a,b],\infty}$, which proves the last part of inequality (2.1).

Using Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get, since $|x - t| \leq \pi$, that

$$\begin{aligned}
&\int_a^x |f'(t)| \sin(x-t)\, dt + \int_x^b |f'(t)| \sin(t-x)\, dt \\
&\leq \left(\int_a^x |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^x \sin^q(x-t)\, dt\right)^{1/q} \\
&+ \left(\int_x^b |f'(t)|^p\, dt\right)^{1/p} \left(\int_x^b \sin^q(t-x)\, dt\right)^{1/q} \\
&\leq \left[\left(\left(\int_a^x |f'(t)|^p\, dt\right)^{1/p}\right)^p + \left(\left(\int_x^b |f'(t)|^p\, dt\right)^{1/p}\right)^p\right]^{1/p} \\
&\times \left[\left(\left(\int_a^x \sin^q(x-t)\, dt\right)^{1/q}\right)^q + \left(\left(\int_x^b \sin^q(t-x)\, dt\right)^{1/q}\right)^q\right]^{1/q} \\
&= \left[\int_a^x |f'(t)|^p\, dt + \int_x^b |f'(t)|^p\, dt\right]^{1/p} \left[\int_a^x \sin^q(x-t)\, dt + \int_x^b \sin^q(t-x)\, dt\right]^{1/q} \\
&= \left[\int_a^b |f'(t)|^p\, dt\right]^{1/p} \left[\int_a^b \sin^q(|x-t|)\, dt\right]^{1/q}
\end{aligned}$$

and by (2.5) we obtain (2.2).
Also,

$$\begin{aligned}
&\int_a^x |f'(t)| \sin(x-t)\, dt + \int_x^b |f'(t)| \sin(t-x)\, dt \\
&\leq \max_{t\in[a,x]} [\sin(x-t)] \int_a^x |f'(t)|\, dt + \max_{t\in[x,b]} [\sin(t-x)] \int_x^b |f'(t)|\, dt \\
&\leq \max\left\{\max_{t\in[a,x]} [\sin(x-t)], \max_{t\in[x,b]} [\sin(t-x)]\right\} \int_a^b |f'(t)|\, dt
\end{aligned}$$

and by (2.5) we obtain (2.3). ∎

Remark 2.1. In particular, if we take $x = \frac{a+b}{2}$, then we get from (2.1) that

$$\begin{aligned}
&\left|C_f\left(\frac{a+b}{2}\right) - [f(a) + f(b)] \sin\left(\frac{b-a}{2}\right)\right| \\
&\leq 2\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \sin^2\left(\frac{b-a}{4}\right) \\
&\leq 4\|f'\|_{[a,b],\infty} \sin^2\left(\frac{b-a}{4}\right).
\end{aligned} \tag{2.6}$$

Also from (2.2) we get

$$\begin{aligned}(2.7)\qquad &\left|C_f\left(\frac{a+b}{2}\right)-[f(a)+f(b)]\sin\left(\frac{b-a}{2}\right)\right| \\ &\leq \|f'\|_{\left[a,\frac{a+b}{2}\right],p}\left(\int_a^{\frac{a+b}{2}}\sin^q\left(\frac{a+b}{2}-t\right)dt\right)^{1/q} \\ &+\|f'\|_{\left[\frac{a+b}{2},b\right],p}\left(\int_{\frac{a+b}{2}}^b\sin^q\left(t-\frac{a+b}{2}\right)dt\right)^{1/q} \\ &\leq \|f'\|_{[a,b],p}\left[\int_a^b\sin^q\left|\frac{a+b}{2}-t\right|dt\right]^{1/q}.\end{aligned}$$

Moreover, from (2.3), we derive

$$\begin{aligned}(2.8)\qquad &\left|C_f\left(\frac{a+b}{2}\right)-[f(a)+f(b)]\sin\left(\frac{b-a}{2}\right)\right| \\ &\leq \max_{t\in\left[a,\frac{a+b}{2}\right]}\left[\sin\left(\frac{a+b}{2}-t\right)\right]\|f'\|_{\left[a,\frac{a+b}{2}\right],1} \\ &+\max_{t\in\left[\frac{a+b}{2},b\right]}\left[\sin\left(t-\frac{a+b}{2}\right)\right]\|f'\|_{\left[\frac{a+b}{2},b\right],1} \\ &\leq \|f'\|_{[a,b],1} \\ &\times\max\left\{\max_{t\in\left[a,\frac{a+b}{2}\right]}\left[\sin\left(\frac{a+b}{2}-t\right)\right],\max_{t\in\left[\frac{a+b}{2},b\right]}\left[\sin\left(t-\frac{a+b}{2}\right)\right]\right\}.\end{aligned}$$

Corollary 2.2. *If $f'\in L_2[a,b]$, then*

$$\begin{aligned}(2.9)\qquad &|C_f(x)-f(a)\sin(x-a)-f(b)\sin(b-x)| \\ &\leq \|f'\|_{[a,x],2}\left(\frac{x-a}{2}-\frac{1}{4}\sin(2(x-a))\right)^{1/2} \\ &+\|f'\|_{[x,b],2}\left(\frac{b-x}{2}-\frac{1}{4}\sin(2(b-x))\right)^{1/2} \\ &\leq \|f'\|_{[a,b],2}\left[\frac{b-a}{2}-\frac{1}{2}\sin(b-a)\cos(a+b-2x)\right]^{1/2},\end{aligned}$$

for all $x\in[a,b]$.

In particular,

$$\begin{aligned}(2.10)\qquad &\left|C_f\left(\frac{a+b}{2}\right)-[f(a)+f(b)]\sin\left(\frac{b-a}{2}\right)\right| \\ &\leq \left(\|f'\|_{\left[a,\frac{a+b}{2}\right],2}+\|f'\|_{\left[\frac{a+b}{2},b\right],2}\right)\left(\frac{b-a}{4}-\frac{1}{4}\sin(b-a)\right)^{1/2} \\ &\leq \|f'\|_{[a,b],2}\left[\frac{b-a}{2}-\frac{1}{2}\sin(b-a)\right]^{1/2}.\end{aligned}$$

Proof. If we take $p = q = 2$ in (2.2), then we get

$$\begin{aligned}(2.11)\qquad &\left|C_f(x) - f(a)\sin(x-a) - f(b)\sin(b-x)\right| \\ &\le \|f'\|_{[a,x],2}\left(\int_a^x \sin^2(x-t)\,dt\right)^{1/2} + \|f'\|_{[x,b],2}\left(\int_x^b \sin^2(t-x)\,dt\right)^{1/2} \\ &\le \|f'\|_{[a,b],2}\left[\int_a^b \sin^2|x-t|\,dt\right]^{1/2}.\end{aligned}$$

Observe that

$$\int_a^x \sin^2(x-t)\,dt = \frac{x-a}{2} - \frac{1}{4}\sin(2(x-a)),$$

$$\int_x^b \sin^2(t-x)\,dt = \frac{b-x}{2} - \frac{1}{4}\sin(2(b-x))$$

and

$$\begin{aligned}&\int_a^x \sin^2(x-t)\,dt + \int_x^b \sin^2(t-x)\,dt \\ &= \frac{x-a}{2} - \frac{1}{4}\sin(2(x-a)) + \frac{b-x}{2} - \frac{1}{4}\sin(2(b-x)) \\ &= \frac{b-a}{2} - \frac{1}{4}\left[\sin(2(x-a)) + \sin(2(b-x))\right] \\ &= \frac{b-a}{2} - \frac{1}{2}\sin(b-a)\cos(a+b-2x)\end{aligned}$$

and by (2.11) we get (2.9). ∎

Remark 2.2. If we assume that $[a,b] \subset \left[0, \frac{\pi}{2}\right]$, then

$$\max_{t\in[a,x]}\left[\sin(x-t)\right] = \sin(x-a), \quad \max_{t\in[x,b]}\left[\sin(t-x)\right] = \sin(b-x)$$

and by (2.3) we get

$$\begin{aligned}(2.12)\qquad &\left|C_f(x) - f(a)\sin(x-a) - f(b)\sin(b-x)\right| \\ &\le \sin(x-a)\|f'\|_{[a,x],1} + \sin(b-x)\|f'\|_{[x,b],1} \\ &\le \|f'\|_{[a,b],1}\max\{\sin(x-a), \sin(b-x)\}\end{aligned}$$

for all $x \in [a,b]$.

If we take $x = \frac{a+b}{2}$, then we get

$$(2.13)\qquad \left|C_f\left(\frac{a+b}{2}\right) - [f(a)+f(b)]\sin\left(\frac{b-a}{2}\right)\right| \le \|f'\|_{[a,b],1}\sin\left(\frac{b-a}{2}\right).$$

3. ERROR BOUNDS FOR THE TRANSFORM $\widetilde{C}_f$

We also have:

Theorem 3.1. *If f is absolutely continuous on $[a,b]\subset[0,\pi]$ with $f'\in L_\infty[a,b]$, then*

$$\begin{aligned}(3.1)\qquad &\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)f(x)-\widetilde{C}_f(x)\right|\\ &\le 2\left[\|f'\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{2}\right)+\|f'\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{2}\right)\right]\\ &\le 2\|f'\|_{[a,b],\infty}\left[\sin^2\left(\frac{x-a}{2}\right)+\sin^2\left(\frac{b-x}{2}\right)\right]\end{aligned}$$

for all $x\in[a,b]$.

If $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $f'\in L_p[a,b]$, then

$$\begin{aligned}(3.2)\qquad &\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)f(x)-\widetilde{C}_f(x)\right|\\ &\le \|f'\|_{[a,x],p}\left(\int_a^x\sin^q(t-a)\,dt\right)^{1/q}+\|f'\|_{[x,b],p}\left(\int_x^b\sin^q(b-t)\,dt\right)^{1/q}\\ &\le \|f'\|_{[a,b],p}\left(\int_a^x\sin^q(t-a)\,dt+\int_x^b\sin^q(b-t)\,dt\right)^{1/q}\end{aligned}$$

for all $x\in[a,b]$.

Also, we have

$$\begin{aligned}(3.3)\qquad &\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)f(x)-\widetilde{C}_f(x)\right|\\ &\le \sin(x-a)\|f'\|_{[a,x],1}+\sin(b-x)\|f'\|_{[x,b],1}\\ &\le \max\{\sin(x-a),\sin(b-x)\}\|f'\|_{[a,b],1}\end{aligned}$$

for all $x\in[a,b]$.

Proof. Using integration by parts, we have

$$\begin{aligned}\int_a^x f'(t)\sin(t-a)\,dt &= f(t)\sin(t-a)\big|_a^x-\int_a^x f(t)\cos(t-a)\,dt\\ &= f(x)\sin(x-a)-\int_a^x f(t)\cos(t-a)\,dt\end{aligned}$$

and

$$\begin{aligned}\int_x^b f'(t)\sin(t-b) &= f(t)\sin(t-b)\big|_x^b-\int_x^b f(t)\cos(t-b)\,dt\\ &= -f(x)\sin(x-b)-\int_x^b f(t)\cos(t-b)\,dt\\ &= f(x)\sin(b-x)-\int_x^b f(t)\cos(b-t)\,dt\end{aligned}$$

for $x\in[a,b]$.

If we add these two identities, then we get

$$\begin{aligned}&f(x)[\sin(x-a)+\sin(b-x)]-\widetilde{C}_f(x)\\ &=\int_a^x f'(t)\sin(t-a)\,dt-\int_x^b f'(t)\sin(b-t)\,dt,\end{aligned}$$

namely

$$\begin{aligned}&2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)f(x)-\widetilde{C}_f(x)\\&=\int_a^x f'(t)\sin(t-a)\,dt-\int_x^b f'(t)\sin(b-t)\,dt,\end{aligned}$$

for $x\in[a,b]$.

By taking the modulus, we get, since $|x-t|\le\pi$, that

$$\begin{aligned}&\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)f(x)-\widetilde{C}_f(x)\right|\\&\le\left|\int_a^x f'(t)\sin(t-a)\,dt\right|+\left|\int_x^b f'(t)\sin(b-t)\,dt\right|\\&\le\int_a^x|f'(t)|\sin(t-a)\,dt+\int_x^b|f'(t)|\sin(b-t)\,dt\\&\le\|f'\|_{[a,x],\infty}\int_a^x\sin(t-a)\,dt+\|f'\|_{[x,b],\infty}\int_x^b\sin(b-t)\,dt\\&=\|f'\|_{[a,x],\infty}(1-\cos(x-a))+\|f'\|_{[x,b],\infty}(1-\cos(b-x))\\&=2\left[\|f'\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{2}\right)+\|f'\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{2}\right)\right]\\&\le2\|f'\|_{[a,b],\infty}\left[\sin^2\left(\frac{x-a}{2}\right)+\sin^2\left(\frac{b-x}{2}\right)\right]\end{aligned}\tag{3.4}$$

for $x\in[a,b]$.

Using Hölder's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\int_a^x|f'(t)|\sin(t-a)\,dt\le\left(\int_a^x|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^x\sin^q(t-a)\,dt\right)^{1/q}$$

and

$$\int_x^b|f'(t)|\sin(b-t)\,dt\le\left(\int_x^b|f'(t)|^p\,dt\right)^{1/p}\left(\int_x^b\sin^q(b-t)\,dt\right)^{1/q}$$

for $x\in[a,b]$.

If we add these inequalities, then we get

$$\begin{aligned}
&\int_a^x |f'(t)| \sin(t-a)\, dt + \int_x^b |f'(t)| \sin(b-t)\, dt \\
&\le \left(\int_a^x |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^x \sin^q(t-a)\, dt\right)^{1/q} \\
&+ \left(\int_x^b |f'(t)|^p\, dt\right)^{1/p} \left(\int_x^b \sin^q(b-t)\, dt\right)^{1/q} \\
&\le \left(\int_a^x |f'(t)|^p\, dt + \int_x^b |f'(t)|^p\, dt\right)^{1/p} \\
&\times \left(\int_a^x \sin^q(t-a)\, dt + \int_x^b \sin^q(b-t)\, dt\right)^{1/q} \\
&= \left(\int_a^b |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^x \sin^q(t-a)\, dt + \int_x^b \sin^q(b-t)\, dt\right)^{1/q}
\end{aligned}$$

for $x \in [a,b]$. By using (3.4) we get (3.2).

Also, observe that

$$\begin{aligned}
&\int_a^x |f'(t)| \sin(t-a)\, dt + \int_x^b |f'(t)| \sin(b-t)\, dt \\
&\le \sin(x-a) \int_a^x |f'(t)|\, dt + \sin(b-x) \int_x^b |f'(t)|\, dt \\
&\le \max\{\sin(x-a), \sin(b-x)\} \left(\int_a^x |f'(t)|\, dt + \int_x^b |f'(t)|\, dt\right) \\
&= \max\{\sin(x-a), \sin(b-x)\} \left(\int_a^b |f'(t)|\, dt\right)
\end{aligned}$$

for $x \in [a,b]$. By using (3.4) we get (3.3). ∎

Remark 3.1. In particular, if we take $x = \frac{a+b}{2}$, then we get from (3.1) that

$$\begin{aligned}
&\left| 2\sin\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) - \widetilde{C}_f\left(\frac{a+b}{2}\right)\right| \\
&\le 2\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \sin^2\left(\frac{b-a}{4}\right) \\
&\le 4\|f'\|_{[a,b],\infty} \sin^2\left(\frac{b-a}{4}\right),
\end{aligned} \tag{3.5}$$

while from (3.2) that

$$\begin{aligned}
(3.6)\qquad & \left|2\sin\left(\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)-\widetilde{C}_f\left(\frac{a+b}{2}\right)\right| \\
& \le \|f'\|_{\left[a,\frac{a+b}{2}\right],p} \\
& \times\left(\int_a^{\frac{a+b}{2}}\sin^q(t-a)\,dt\right)^{1/q}+\|f'\|_{\left[\frac{a+b}{2},b\right],p}\left(\int_{\frac{a+b}{2}}^b\sin^q(b-t)\,dt\right)^{1/q} \\
& \le \|f'\|_{[a,b],p}\left(\int_a^{\frac{a+b}{2}}\sin^q(t-a)\,dt+\int_{\frac{a+b}{2}}^b\sin^q(b-t)\,dt\right)^{1/q}
\end{aligned}$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $f'\in L_p[a,b]$.

From (3.3) we get

$$(3.7)\qquad \left|2\sin\left(\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)-\widetilde{C}_f\left(\frac{a+b}{2}\right)\right|\le\sin\left(\frac{b-a}{2}\right)\|f'\|_{[a,b],1}.$$

Corollary 3.2. *If $f'\in L_2[a,b]$, then*

$$\begin{aligned}
(3.8)\qquad & \left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)f(x)-\widetilde{C}_f(x)\right| \\
& \le \|f'\|_{[a,x],2}\left(\frac{x-a}{2}-\frac{1}{4}\sin(2(x-a))\right)^{1/2} \\
& +\|f'\|_{[x,b],2}\left(\frac{b-x}{2}-\frac{1}{4}\sin(2(b-x))\right)^{1/2} \\
& \le \|f'\|_{[a,b],2}\left(\frac{b-a}{2}-\frac{1}{2}\sin(b-a)\cos(a+b-2x)\right)^{1/2}
\end{aligned}$$

for $x\in[a,b]$.

In particular,

$$\begin{aligned}
(3.9)\qquad & \left|2\sin\left(\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)-\widetilde{C}_f\left(\frac{a+b}{2}\right)\right| \\
& \le\left(\|f'\|_{\left[a,\frac{a+b}{2}\right],2}+\|f'\|_{\left[\frac{a+b}{2},b\right],2}\right)\left(\frac{b-a}{4}-\frac{1}{4}\sin(b-a)\right)^{1/2} \\
& \le \|f'\|_{[a,b],2}\left[\frac{b-a}{2}-\frac{1}{2}\sin(b-a)\right]^{1/2}.
\end{aligned}$$

Proof. Observe that for $x\in[a,b]$,

$$\int_a^x\sin^2(t-a)\,dt=\frac{x-a}{2}-\frac{1}{4}\sin(2(x-a))$$

and

$$\int_x^b\sin^2(b-t)\,dt=\frac{b-x}{2}-\frac{1}{4}\sin(2(b-x)).$$

Also

$$\begin{aligned}
&\int_a^x \sin^2(t-a)\,dt + \int_x^b \sin^2(b-t)\,dt \\
&= \frac{x-a}{2} - \frac{1}{4}\sin(2(x-a)) + \frac{b-x}{2} - \frac{1}{4}\sin(2(b-x)) \\
&= \frac{b-a}{2} - \frac{1}{4}\left[\sin(2(x-a)) + \sin(2(b-x))\right] \\
&= \frac{b-a}{2} - \frac{1}{2}\sin(b-a)\cos(a+b-2x),
\end{aligned}$$

$x \in [a,b]$, which proves (3.8). ∎

4. Applications for Steklov Average

The *Steklov average* (or Steklov mean function) was introduced by V. A. Steklov in1907 (see [[7]]) for the study of the problem of expanding a given function into a series of eigenvalues defined by a 2nd-order ordinary differential operator.

For $f \in C(I)$, $h > 0$, and $x \in I_1(h) = \{t : t-h, t+h \in I\}$, the operator S_h defined by

$$S_h(f,x) := \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt \tag{4.1}$$

is often called a *Steklov mean function*, although it is an operator mapping $C(I)$ into $C(I_1)$. If $I = [a,b]$, the assumption is $a \le x-h < x+h \le b$. For some recent generalizations and their properties, see [1].

For a continuous function f on $[a,b]$ and an element $x \in (a,b)$, we introduce the following *Steklov cos-average functions*

$$SC_{f,h}(x) := \frac{1}{2h}\int_{x-h}^{x+h} f(t)\cos(x-t)\,dt,$$

with $h > 0$ and such that $a \le x-h < x+h \le b$.

Also we can introduce for f continuous on $[a,b]$ and an element $x \in (a,b)$, the following *Steklov split cos-average functions*

$$\widetilde{SC}_{f,h}(x) := \frac{1}{2h}\int_{x-h}^{x} f(t)\cos(t-x+h)\,dt + \frac{1}{2h}\int_{x}^{x+h} f(t)\cos(x+h-t)\,dt,$$

with $h > 0$ and such that $a \le x-h < x+h \le b$.

From (2.1) we then get by replacing a with $x-h$ and b with $x+h$

$$\begin{aligned}
&\left|\int_{x-h}^{x+h} f(t)\cos(x-t)\,dt - [f(x-h)+f(x+h)]\sin(h)\right| \\
&\le 4\|f'\|_{[x-h,x+h],\infty}\sin^2\left(\frac{h}{2}\right)
\end{aligned} \tag{4.2}$$

with $h > 0$ and such that $0 \le a \le x-h < x+h \le b \le \pi$

If we divide by $2h$ in (4.2), then we derive the following error bound for absolutely continuous functions in $[a,b]$

$$\left|SC_{f,h}(x) - \left[\frac{f(x-h)+f(x+h)}{2}\right]\frac{\sin(h)}{h}\right| \le \frac{1}{2}h\|f'\|_{[x-h,x+h],\infty}\frac{\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)^2} \tag{4.3}$$

with $h > 0$ and such that $0 \le a \le x-h < x+h \le b \le \pi$.

From (3.1) we get by replacing a with $x-h$ and b with $x+h$

$$\begin{aligned} &\left|2\sin(h) f(x) - \int_{x-h}^{x} f(t)\cos(t-x+h)\,dt\right. \\ &\left.-\frac{1}{h}\int_{x}^{x+h} f(t)\cos(x+h-t)\,dt\right| \\ &\leq 4\|f'\|_{[x-h,x+h],\infty}\sin^2\left(\frac{h}{2}\right) \end{aligned} \tag{4.4}$$

with $h>0$ and such that $0\leq a\leq x-h<x+h\leq b\leq\pi$.

If we divide by $2h$ in (4.4), then we obtain the following error bound for absolutely continuous functions on $[a,b]$

$$\left|\frac{\sin(h)}{h} f(x) - S\widetilde{C}_{f,h}(x)\right| \leq \frac{1}{2}h\|f'\|_{[x-h,x+h],\infty}\frac{\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)^2} \tag{4.5}$$

with $h>0$ and such that $0\leq a\leq x-h<x+h\leq b\leq\pi$.

Since

$$\lim_{h\to 0+}\left(h\|f'\|_{[x-h,x+h],\infty}\frac{\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)^2}\right)=0,$$

then

$$\lim_{h\to 0+} SC_{f,h}(x) = \lim_{h\to 0+}\left\{\left[\frac{f(x-h)+f(x+h)}{2}\right]\frac{\sin(h)}{h}\right\} = f(x)$$

and

$$\lim_{h\to 0+} S\widetilde{C}_{f,h}(x) = \lim_{h\to 0+}\left(\frac{\sin(h)}{h}f(x)\right) = f(x)$$

for all $x\in(a,b)$.

5. SOME EXAMPLES

Consider the function $\ell_p(t)=t^p$, $p\geq 1$, $t\in[a,b]\subset[0,\pi]$. Then

$$C_{\ell_p}(x)=\int_a^b t^p\cos(x-t)\,dt$$

for $x\in[a,b]$.

Therefore by (2.1) we obtain

$$\begin{aligned} &\left|C_{\ell_p}(x) - a^p\sin(x-a) - b^p\sin(b-x)\right| \\ &\leq 2pb^{p-1}\left[\sin^2\left(\frac{x-a}{2}\right)+\sin^2\left(\frac{b-x}{2}\right)\right] \end{aligned} \tag{5.1}$$

for $x\in[a,b]$.

In particular, for $x=\frac{a+b}{2}$ we get

$$\left|C_{\ell_p}\left(\frac{a+b}{2}\right) - (a^p+b^p)\sin\left(\frac{b-a}{2}\right)\right| \leq 4pb^{p-1}\sin^2\left(\frac{b-a}{4}\right). \tag{5.2}$$

We have

$$\widetilde{C}_{\ell_p}(x)=\int_a^x t^p\cos(t-a)\,dt+\int_x^b t^p\cos(b-t)\,dt$$

for $x\in[a,b]$ and by (3.1) we derive

$$\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)x^p-\widetilde{C}_{\ell_p}(x)\right| \le 2pb^{p-1}\left[\sin^2\left(\frac{x-a}{2}\right)+\sin^2\left(\frac{b-x}{2}\right)\right] \tag{5.3}$$

for $x\in[a,b]$.

In particular, for $x=\frac{a+b}{2}$ we obtain

$$\left|2\sin\left(\frac{b-a}{2}\right)\left(\frac{a+b}{2}\right)^p-\widetilde{C}_{\ell_p}\left(\frac{a+b}{2}\right)\right|\le 4pb^{p-1}\sin^2\left(\frac{b-a}{4}\right). \tag{5.4}$$

Also, we consider the function $f(t)=\exp t^p=\exp\ell_p(t)$, $t\in[a,b]\subset[0,\pi]$, $p\ge 1$. We have $f'(t)=pt^{p-1}\exp\ell_p(t)$ and $\|f'\|_{[a,b],\infty}=pb^{p-1}\exp(b^p)$. Then

$$C_{\exp\ell_p}(x):=\int_a^b\exp\ell_p(t)\cos(x-t)\,dt,\ x\in[a,b]$$

and

$$\widetilde{C}_{\exp\ell_p}(x):=\int_a^x\exp\ell_p(t)\cos(t-a)\,dt+\int_x^b\exp\ell_p(t)\cos(b-t)\,dt,\ x\in[a,b].$$

From (2.1) we have

$$\left|C_{\exp\ell_p}(x)-\exp(a^p)\sin(x-a)-\exp(b^p)\sin(b-x)\right| \le 2pb^{p-1}\exp(b^p)\left[\sin^2\left(\frac{x-a}{2}\right)+\sin^2\left(\frac{b-x}{2}\right)\right] \tag{5.5}$$

for all $x\in[a,b]$, which gives for $x=\frac{a+b}{2}$ that

$$\left|C_{\exp\ell_p}\left(\frac{a+b}{2}\right)-[\exp(a^p)+\exp(b^p)]\sin\left(\frac{b-a}{2}\right)\right| \le 4pb^{p-1}\exp(b^p)\sin^2\left(\frac{b-a}{4}\right). \tag{5.6}$$

From (3.1) we get

$$\left|2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-x\right)\exp\ell_p(x)-\widetilde{C}_{\exp\ell_p}(x)\right| \le 2pb^{p-1}\exp(b^p)\left[\sin^2\left(\frac{x-a}{2}\right)+\sin^2\left(\frac{b-x}{2}\right)\right] \tag{5.7}$$

for all $x\in[a,b]$, which gives for $x=\frac{a+b}{2}$ that

$$\left|2\sin\left(\frac{b-a}{2}\right)\exp\ell_p\left(\frac{a+b}{2}\right)-\widetilde{C}_{\exp\ell_p}\left(\frac{a+b}{2}\right)\right| \le 4pb^{p-1}\exp(b^p)\sin^2\left(\frac{b-a}{4}\right). \tag{5.8}$$

Now, for the function $f=\ell_p$, $p\ge 1$ on $[a,b]\subset[0,\pi]$, we consider

$$SC_{\ell_p,h}(x):=\frac{1}{2h}\int_{x-h}^{x+h}t^p\cos(x-t)\,dt$$

and

$$S\widetilde{C}_{\ell_p,h}(x) := \frac{1}{2h}\int_{x-h}^{x} t^p \cos(t-x+h)\,dt + \frac{1}{2h}\int_{x}^{x+h} t^p \cos(x+h-t)\,dt,$$

with $h>0$ and such that $0 \le a \le x-h < x+h \le b \le \pi$.

From (4.2) we get

$$\left|SC_{\ell_p,h}(x) - \left[\frac{(x-h)^p + (x+h)^p}{2}\right]\frac{\sin(h)}{h}\right| \le \frac{1}{2}ph(x+h)^{p-1}\frac{\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)^2} \tag{5.9}$$

with $h>0$ and such that $0 \le a \le x-h < x+h \le b \le \pi$.

From (4.5) we get

$$\left|\frac{\sin(h)}{h}x^p - S\widetilde{C}_{f,h}(x)\right| \le \frac{1}{2}ph(x+h)^{p-1}\frac{\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)^2} \tag{5.10}$$

with $h>0$ and such that $0 \le a \le x-h < x+h \le b \le \pi$.

6. SOME NUMERICAL EXAMPLES

In these section we provide some numerical experiments to illustrate the example for power function considered above.

6.1. Steklov Average with Integer p.

For $p=2$, we have

$$\begin{aligned} SC_{\ell_2,h}(x) &:= \frac{1}{2h}\int_{x-h}^{x+h} t^2\cos(x-t)\,dt \\ &= \frac{1}{2h}\left[(2x^2+2h^2-4)\sin(h) + 4h\cos(h)\right], \end{aligned}$$

with $h>0$ and such that $0 \le a \le x-h < x+h \le b \le \pi$. From (5.9) we get the error bound

$$\left|SC_{\ell_2,h}(x) - \frac{(x^2+h^2)\sin(h)}{h}\right| \le \frac{h(x+h)\sin^2\left(\frac{h}{2}\right)}{(\frac{h}{2})^2} \tag{6.1}$$

with $h>0$ and such that $0 \le a \le x-h < x+h \le b \le \pi$.

Table 6.1 details the values of $SC_{\ell_2,h}$, the absolute error and its upper bound from (6.1).

		$\mathbf{x_1} = 0.1$	$\mathbf{x_2} = 0.6$	$\mathbf{x_3} = 1.1$	$\mathbf{x_4} = 1.6$	$\mathbf{x_5} = 2.1$	$\mathbf{x_6} = 2.6$
SC	$h = 0.1$	0.0133067	0.3627236	1.2113077	2.5590588	4.405977	6.7520623
AbsErr		0.00666	0.00666	0.00666	0.00666	0.00666	0.00666
UpperB		0.0199833	0.0699417	0.1199	0.1698584	0.2198167	0.2697751
SC	$h = 0.01$	0.0100332	0.3600273	1.2100132	2.5599907	4.4099598	6.7599207
AbsErr		$6.67E-05$	$6.67E-05$	$6.67E-05$	$6.67E-05$	$6.67E-05$	$6.67E-05$
UpperB		0.0011	0.0060999	0.0110999	0.0160999	0.0210998	0.0260998
SC	$h = 0.001$	0.0100003	0.3600003	1.2100001	2.5599999	4.4099996	6.7599992
AbsErr		$6.67E-07$	$6.67E-07$	$6.67E-07$	$6.67E-07$	$6.67E-07$	$6.67E-07$
UpperB		0.000101	0.000601	0.001101	0.001601	0.002101	0.002601
SC	$h = 0.0001$	0.01	0.36	1.21	2.56	4.41	6.76
AbsErr		$6.67E-09$	$6.67E-09$	$6.67E-09$	$6.67E-09$	$6.67E-09$	$6.67E-09$
UpperB		$1.00E-05$	$6.00E-05$	$1.10E-04$	$1.60E-04$	$2.10E-04$	$2.60E-04$
SC	$h = 0.00001$	0.01	0.36	1.21	2.56	4.41	6.76
AbsErr		$6.67E-11$	$6.67E-11$	$6.67E-11$	$6.67E-11$	$6.67E-11$	$6.67E-11$
UpperB		$1.00E-06$	$6.00E-06$	$1.10E-05$	$1.60E-05$	$2.10E-05$	$2.60E-05$
SC	$h = 0.000001$	0.01	0.36	1.21	2.56	4.41	6.76
AbsErr		$6.66E-13$	$6.67E-13$	$6.67E-13$	$6.67E-13$	$6.67E-13$	$6.66E-13$
UpperB		$1.00E-07$	$6.00E-07$	$1.10E-06$	$1.60E-06$	$2.10E-06$	$2.60E-06$

Table 6.1: Numerical results for $p = 2$ and various x and h.

6.2. **Steklov Average with non-Integer p.** For non-integer $p = \frac{3}{2}$, we have

$$SC_{\ell_{\frac{3}{2}},h}(x) := \frac{1}{2h}\int_{x-h}^{x+h} t^{\frac{3}{2}} \cos(x-t)\, dt, \tag{6.2}$$

with $h > 0$ and such that $0 \leq a \leq x - h < x + h \leq b \leq \pi$. The integral (6.2) has no analytical expression and therefore requires approximation or numerical integration techniques. From (5.1) we get the Ostrowski approximation for the integral in (6.2), namely

$$\begin{aligned} SC_{\ell_{\frac{3}{2}},h}(x) &:= \frac{1}{2h}\int_{x-h}^{x+h} t^{\frac{3}{2}} \cos(x-t)\, dt \\ &\simeq \frac{\sin(h)}{2h}\left(\left[(x-h)^{\frac{3}{2}} + (x+h)^{\frac{3}{2}}\right]\right), \end{aligned} \tag{6.3}$$

with $h > 0$ and such that $0 \leq a \leq x - h < x + h \leq b \leq \pi$.

Table 6.2 details the values of $SC_{\ell_{\frac{3}{2}},10^{-3}}$, for $N = 32$ and various x, using (6.3) and various quadrature (Newton-Coates) rules (including absolute differences).

Int Type		$\mathbf{x} = 0.1$	$\mathbf{x} = 0.6$	$\mathbf{x} = 1.1$	$\mathbf{x} = 1.6$	$\mathbf{x} = 2.1$	$\mathbf{x} = 2.6$
Ostrowski		0.031624	0.464758	1.153690	2.023858	3.043189	4.192374
Trapezoidal	N=32	0.031382	0.464942	1.154563	2.025506	3.045657	4.195691
		(0.000242)	(0.000215)	(0.000182)	(0.001648)	(0.002468)	(0.003318)
Simpson's	N=32	0.031379	0.464940	1.154572	2.025523	3.045684	4.195727
		(0.000245)	(0.000217)	(0.000184)	(0.001666)	(0.002495)	(0.003353)
Mid Point	N=32	0.031382	0.464758	1.154563	2.025506	3.045657	4.195691
		(0.000242)	(0.000215)	(0.000182)	(0.001648)	(0.002468)	(0.003318)

Table 6.2: Numerical results comparing various quadrature (Newton-Coates) rules to the Ostrowski approximation for $h = 10^{-3}$, $p = \frac{3}{2}$ and various x.

Table 6.3 details the values of $SC_{\ell_{\frac{3}{2}},h}$, for $x = 0.1$ with various h and N, using (6.3) and various quadrature (Newton-Coates) rules (including absolute differences).

Int Type		$\mathbf{h} = 10^{-1}$	$\mathbf{h} = 10^{-2}$	$\mathbf{h} = 10^{-3}$	$\mathbf{h} = 10^{-4}$	$\mathbf{h} = 10^{-5}$	$\mathbf{h} = 10^{-6}$
Ostrowski		0.044647	0.031741	0.031624	0.031623	0.031623	0.031623
Trapezoidal	N=8	0.035871	0.031663	0.031405	0.031623	0.031623	0.031623
(Absolute Error)		(0.008776)	$(7.790E-05)$	(0.000219)	$(7.784E-09)$	$(7.785E-11)$	$(7.467E-13)$
	N=16	0.035752	0.031662	0.031389	0.031623	0.031623	0.031623
		(0.008894)	$(7.881E-05)$	(0.000234)	$(7.875E-09)$	$(7.876E-11)$	$(7.559E-13)$
	N=32	0.035722	0.031662	0.031382	0.031623	0.031623	0.031623
		(0.008925)	$(7.904E-05)$	(0.000242)	$(7.898E-09)$	$(7.899E-11)$	$(7.582E-13)$
Simpson's	N=8	0.035719	0.031662	0.031395	0.031623	0.031623	0.031623
		(0.008928)	$(7.912E-05)$	(0.000229)	$(7.906E-09)$	$(7.907E-11)$	$(7.589E-13)$
	N=16	0.035713	0.031662	0.031384	0.031623	0.031623	0.031623
		(0.008934)	$(7.912E-05)$	(0.000240)	$(7.906E-09)$	$(7.907E-11)$	$(7.589E-13)$
	N=32	0.035712	0.031662	0.031379	0.031623	0.031623	0.031623
		(0.008935)	$(7.912E-05)$	(0.000245)	$(7.906E-09)$	$(7.907E-11)$	$(7.589E-13)$
Mid Point	N=8	0.035634	0.031661	0.031405	0.031623	0.031623	0.031623
		(0.009013)	$(7.973E-05)$	(0.000219)	$(7.967E-09)$	$(7.968E-11)$	$(7.650E-13)$
	N=16	0.035692	0.031662	0.031389	0.031623	0.031623	0.031623
		(0.008955)	$(7.927E-05)$	(0.000235)	$(7.921E-09)$	$(7.922E-11)$	$(7.605E-13)$
	N=32	0.035707	0.031662	0.031382	0.031623	0.031623	0.031623
		(0.008940)	$(7.915E-05)$	(0.000242)	$(7.910E-09)$	$(7.911E-11)$	$(7.593E-13)$

Table 6.3: Numerical results comparing various quadrature (Newton-Coates) rules to the Ostrowski approximation for $p = \frac{3}{2}$, $x = 0.1$ and various h and N.

Table (6.4) details the values of $SC_{\ell_p,10^{-3}}$, for $x = 0.1$, $N = 32$ and various p, using (6.3) and various quadrature (Newton-Coates) rules (including absolute differences).

Int Type		$\mathbf{p} = \frac{3}{2}$	$\mathbf{p} = \frac{5}{3}$	$\mathbf{p} = \frac{7}{4}$	$\mathbf{p} = \frac{9}{4}$	$\mathbf{p} = \frac{7}{3}$	$\mathbf{p} = \frac{5}{2}$
Ostrowski		0.031624	0.021546	0.017784	0.005624	0.004642	0.003163
Trapezoidal	N=32	0.031382 (0.000242)	0.021322 (0.000224)	0.017576 (0.000208)	0.005517 (0.000107)	0.004549 $(9.370E-05)$	0.003092 $(7.062E-05)$
Simpson's	N=32	0.031379 (0.000245)	0.021320 (0.000226)	0.017574 (0.000210)	0.005516 (0.000108)	0.004548 $(9.471E-05)$	0.003091 $(7.138E-05)$
Mid Point	N=32	0.031382 (0.000242)	0.021322 (0.000224)	0.017576 (0.000208)	0.005517 (0.000107)	0.004549 $(9.372E-05)$	0.003092 $(7.064E-05)$

Table 6.4: Numerical results comparing various quadrature (Newton-Coates) rules to the Ostrowski approximation for $h = 10^{-3}$, $x = 0.1$ and various p.

The plots in Figure 1 depict the Absolute Error for p = 2, 3 and 4 and various x and h.

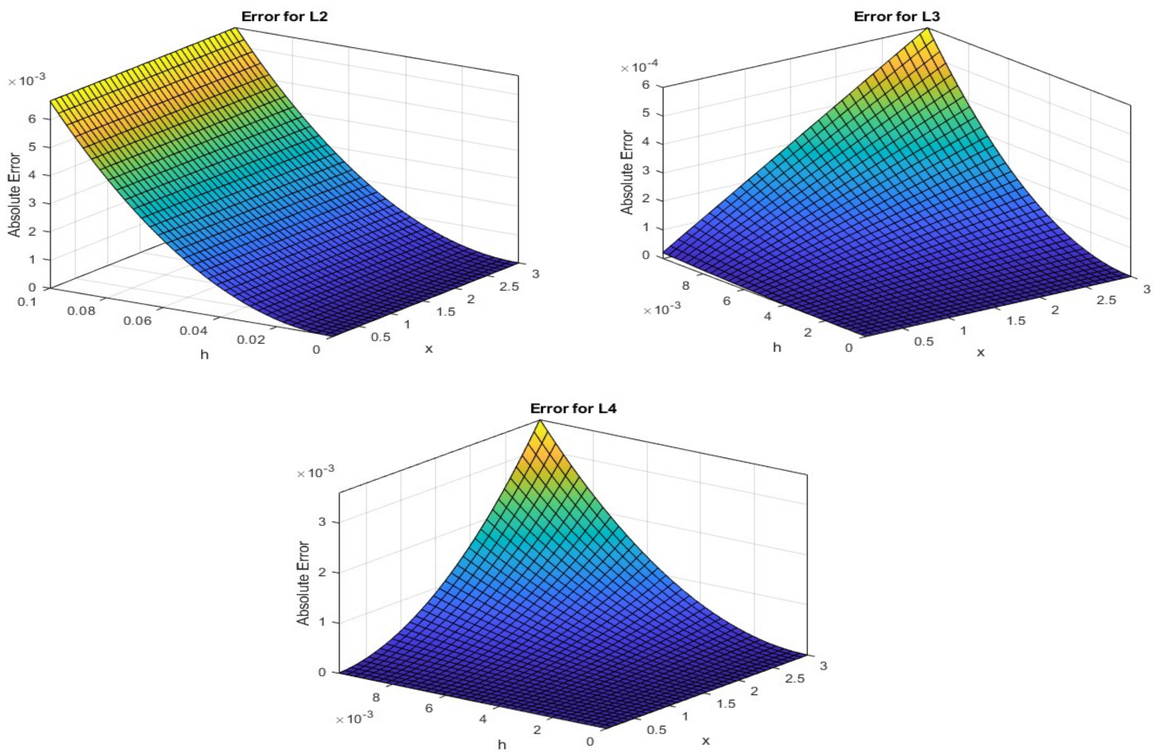

Figure 1: Graphical representation of the Absolute Error for $p = 2, 3$ and 4 and various x and h.

7. CONCLUSION

In this paper we established some sharp Ostrowski type inequalities for two cos-integral transforms. Error bounds for the Steklov average were also provided. Some numerical experiments were conducted as well.

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