Aust. J. Math. Anal. Appl.
Vol. **20** (2023), No. 2, Art. 2, 17 pp.
AJMAA

CONSTRAINT QUALIFICATIONS FOR MULTIOBJECTIVE PROGRAMMING PROBLEMS ON HADAMARD MANIFOLDS

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Received 6 September, 2022; accepted 11 July, 2023; published 1 August, 2023.

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ABSTRACT. The study of optimization methods on manifolds has emerged as an immensely significant topic in mathematics due its ubiquitous applicability as well as various computational advantages associated with it. Motivated by this fact, the present article is devoted to the study of a class of constrained multiobjective programming problems (MOPP) in the framework of Hadamard manifolds. We present the generalized Guignard constraint qualification (GGCQ) in the framework of Hadamard manifolds for (MOPP). Employing (GGCQ), we derive Karush-Kuhn-Tucker type necessary optimality criteria for (MOPP). Moreover, we present several other constraint qualifications (CQs) on Hadamard manifolds, namely, Abadie's CQ, generalized Abadie's CQ, Cottle-type CQ, Slater-type CQ, linear CQ, linear objective CQ and Mangasarian-Fromovitz CQ. Further, we establish various relations between these constraint qualifications. In particular, we show that these constraint qualifications, in turn, become sufficient conditions ensuring that (GGCQ) is satisfied.

Key words and phrases: Multiobjective programming; Constraint qualification; Hadamard manifolds; KKT conditions.

2020 *Mathematics Subject Classification.* 90C30, 90C48, 90C29, 58A05.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In recent times, it has frequently been observed that a lot of nonlinear mathematical programming problems arise in several real world applications which require to be formulated on smooth manifolds (see, [11, 12]). As a result, despite the fact that generalizing the methods and algorithms of optimization from Euclidean spaces to the framework of manifolds is a non-trivial task, it is fairly obvious to realize the necessity of such endeavours from a practical point of view. In this regard, Rapcsák [9] and Udrişte [14] introduced a generalization of the convexity notion, namely, geodesic convexity in the framework of Riemannian manifolds. In such setting, the usual linear space is replaced by a Riemannian manifold, and a line segment is redefined in terms of 'geodesic'. Generalizing and extending the theories of optimization from Euclidean space to the framework of Riemannian manifold is significantly advantageous. For instance, by employing suitable Riemannian geometry perspective, a constrained optimization problem can be often treated as an unconstrained problem. Additionally, by incorporating suitable Riemannian metric, one can conveniently transform many nonconvex programming problems into convex programming problems, (see, for instance, [10, 11]). For all these reasons, in recent times, various scholars have generalized and extended numerous concepts and ideas in mathematical programming from Euclidean plane to Riemannian manifolds; see for instance, [1, 2, 19, 20, 16] and the references cited therein.

In theory of optimization, Karush-Kuhn-Tucker (in short, KKT) type optimality criteria is of utmost importance, both from theoretical as well as numerical point of view. Constraint qualifications are certain conditions imposed on a nonlinear programming problem that ensure the satisfaction of KKT conditions at an optimal point. Thus, constraint qualifications play a very crucial role in deducing KKT type optimality conditions. KKT type optimality criteria and their applications have been studied extensively by many notable authors. We refer to the the paper [5] for a brief history of such conditions in the Euclidean space setting.

Maeda [7] studied differentiable multiobjective optimization problems and constraint qualifications in $\mathbb{R}^n$. Constraint qualifications for nonsmooth multiobjective optimization problems in $\mathbb{R}^n$ were explored by Li [6]. Yang et al. [21] investigated optimality criteria for nonlinear optimization problems on Riemannian manifolds. Intrinsic formulation of KKT conditions and constraint qualifications on smooth manifolds were discussed by Bergman and Herzog [3].

Motivated by the results derived in [3, 6, 7] as well as by the importance of optimization in the context of smooth manifolds, in this article, we focus on studying a class of smooth multiobjective programming problems on Hadamard manifolds (MOPP). The novelty of our work is discussed as follows. We present the generalized Guignard constraint qualification (GGCQ) in Hadamard manifold setting for (MOPP). Employing (GGCQ), we derive Karush-Kuhn-Tucker type necessary optimality criteria for (MOPP). Further, we present some other constraint qualifications, namely, Abadie's constraint qualification, generalized Abadie's constraint qualification, Cottle-type constraint qualification, Slater-type constraint qualification, linear constraint qualification, linear objective constraint qualification and Mangasarian-Fromovitz constraint qualification. Moreover, we derive several interesting relations between these constraint qualifications. In particular, we show that these constraint qualifications are in fact, sufficient conditions which ensure that (GGCQ) is satisfied. The results presented in this paper generalize and extend the corresponding results derived by Maeda [7] in the framework of an even more general space, namely, Hadamard manifolds.

The organization of the paper is as follows. In Section 2, we recall some mathematical notation and preliminary concepts that will be used throughout this article. In Section 3, we present the generalized Guignard constraint qualification in Hadamard manifold framework and employ it to derive KKT type necessary conditions for (MOPP). In Section 4, we present several other constraint qualifications for (MOPP), and establish sufficient conditions ensuring (GGCQ). In Section 5, conclusions are drawn as well as some future research directions are mentioned.

2. NOTATION AND MATHEMATICAL PRELIMINARIES

Throughout this article, the standard notation $\mathbb{N}$ and $\mathbb{R}^n$ signify the set of all naturals and the n-dimensional Euclidean space, respectively. The Euclidean inner product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. For any pair of elements $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, the following notational conventions are used

$$u \leqq v \iff u_j \leqq v_j, \quad \forall j = 1, 2, \ldots, n,$$
$$u \prec v \iff u_j < v_j, \quad \forall j = 1, 2, ..., n.$$
$$u \preceq v \iff \begin{cases} u_j \leqq v_j, & \text{for all } j = 1, 2, ..., n; j \neq k, \\ u_k < v_k, & \text{for at least one } k \in \{1, 2, .., n\}. \end{cases}$$

The notation $u \not\prec v$ (respectively, $u \not\preceq v$) indicates the negation of $u \prec v$ (respectively, $u \preceq v$).

Let the notation $\mathscr{H}$ denote an n-dimensional smooth manifold. A Riemannian metric on a smooth manifold $\mathscr{H}$ is a positive definite and symmetric 2-tensor field $\mathscr{G}$. A Riemannian metric induces an inner product on every tangent space $T_p\mathscr{H}$ and this inner product is denoted by the symbol $\mathscr{G}(u, v) = \langle u, v \rangle_p$ for every $p \in \mathscr{H}$ and $u, v \in T_p\mathscr{H}$. Any smooth manifold endowed with some Riemannian metric is referred to as a Riemannian manifold. For any element $p \in \mathscr{H}$, the exponential map $\exp_p : T_p\mathscr{H} \to \mathscr{H}$ is defined as $\exp_p(q) = \gamma_{p,q}(1)$ for any $q \in T_p\mathscr{H}$, where $\gamma_{p,q}$ is the geodesic that starts at the point $p \in \mathscr{H}$ with a velocity q.

A Riemannian manifold $\mathscr{H}$ is termed as geodesic complete, if for any arbitrary element $u \in \mathscr{H}$, the exponential map $\exp_u(v)$ is defined for every $v \in T_p\mathscr{H}$. Any Riemannian manifold that is complete, simply connected and has a nonpositive sectional curvature everywhere is termed as a Cartan-Hadamard manifold, or simply, Hadamard manifold. Let $p \in \mathscr{H}$ be any arbitrary element of a Hadamard manifold $\mathscr{H}$. Then the exponential map $\exp_p : T_p\mathscr{H} \to \mathscr{H}$ is a diffeomorphism. Further, the inverse exponential map $\exp_p^{-1} : \mathscr{H} \to T_p\mathscr{H}$ satisfies $\exp_p^{-1}(p) = 0_p$. Moreover, for every element $q \in \mathscr{H}$, there always exists some unique minimal geodesic $\gamma_{p,q} : [0, 1] \to \mathscr{H}$, which satisfies $\gamma_{p,q}(t) = \exp_p(t\exp_p^{-1}(q))$. The gradient of a smooth map $\Phi : \mathscr{H} \to \mathbb{R}$, denoted by grad Ψ, is a vector field on $\mathscr{H}$ defined through $d\Phi(X) = \langle \operatorname{grad} \Phi, X \rangle = X(\Phi)$, where X is also a vector field on the manifold $\mathscr{H}$.
The following definition is from Udrişte [14].

Definition 2.1. Any subset S of a Hadamard manifold $\mathscr{H}$ is said to be a geodesic convex set in $\mathscr{H}$, if for every pair of distinct elements $x, y \in S$ and for any geodesic $\gamma_{x,y} : [0, 1] \to \mathscr{H}$ joining the points x and y, we have

$$\gamma_{x,y}(t) \in S, \quad \forall t \in [0, 1],$$

where, $\gamma_{x,y}(t) = \exp_x (t \exp_x^{-1}(y))$.

The following definition is from Rapcsák [9].

Definition 2.2. Let S be a geodesic convex subset of a Hadamard manifold $\mathscr{H}$ and let $g : S \to \mathbb{R}$ be a real valued function on S. Then g is referred to as a geodesic convex function at $y \in S$ if and only if the following holds

$$g(z) - g(y) \geq \left\langle \operatorname{grad} g(y),\ \exp_y^{-1}(z) \right\rangle, \quad \forall z \in S.$$

Also, g is referred to as a geodesic strictly convex function at $y \in S$ if the preceding inequality is strict, for $y \neq z$.

For more comprehensive discussions on geodesic convex functions on Hadamard manifolds, we refer to [1, 2, 9, 13, 14, 15, 17, 18] and the references cited therein. Henceforth, throughout this article, we shall use the symbol $\mathscr{H}$ to denote a Hadamard manifold of dimension n, unless otherwise specified.

3. CONSTRAINT QUALIFICATIONS FOR (MOPP)

In this article, the following constrained multiobjective programming problem on Hadamard manifold (MOPP) is considered:

$$\text{(MOPP)} \quad \text{Minimize } f(z) := (f_1(z), ..., f_r(z)),$$
$$\text{subject to} \quad g_j(z) \leq 0, \quad j = 1, 2, \ldots, s.$$

Here, the functions $f_i : \mathscr{H} \to \mathbb{R}$, $(i \in \{1, 2, ..., r\})$ and $g_j : \mathscr{H} \to \mathbb{R}$, $(j \in \{1, 2, ..., s\})$, are all real valued smooth functions that are defined on some n-dimensional Hadamard manifold $\mathscr{H}$. The feasible set for (MOPP), denoted by $\mathcal{D}$, is given by

$$\mathcal{D} := \{z \in \mathscr{H} : g_j(z) \leq 0, \quad \forall j = 1, 2, \ldots, s\}.$$

We denote the set of all active inequality constraints at a feasible element $z \in \mathcal{D}$ by $\mathscr{A}(z)$, that is,

$$\mathscr{A}(z) := \{j \in \{1, 2, ..., s\} : g_j(z) = 0\}.$$

The following definitions from Maeda [7] will be used in the sequel.

Definition 3.1. Any feasible element $\widetilde{z} \in \mathcal{D}$ is called a Pareto efficient solution of (MOPP), if there exists no other feasible element $z \in \mathcal{D}$ which satisfies the following:

$$f(z) \preceq f(\widetilde{z}).$$

Definition 3.2. Any feasible element $\widetilde{z} \in \mathcal{D}$ is called a weak Pareto efficient solution of (MOPP), if there exists no other feasible element $z \in \mathcal{D}$ which satisfies the following:

$$f(z) \prec f(\widetilde{z}).$$

Remark 3.1. It follows readily from the above definitions that every Pareto efficient solution of (MOPP) is a weak Pareto efficient solution of (MOPP). The converse, however, is not true in general.

For any arbitrary feasible element $\widetilde{z} \in \mathcal{D}$, we define the following sets S^k and S for $k = 1, \ldots, r$, that will be used throughout this article.

$$S^k := \{z \in \mathscr{H} \mid f_i(z) \leqq f_i(\widetilde{z}),\ \forall i = 1, \ldots, r,\ i \neq k,\ g_j(z) \leqq 0, \forall j = 1, 2, \ldots, s\},$$
$$S := \{z \in \mathscr{H} \mid f_i(z) \leqq f_i(\widetilde{z}),\ \forall i = 1, \ldots, r,\ g_j(z) \leqq 0,\ \forall j = 1, 2, \ldots, s\}.$$

In the following definition, we define the Bouligand tangent cone (in other words, contingent cone) for any subset of a Hadamard manifold $\mathscr{H}$.

Definition 3.3. Let $A \subseteq \mathscr{H}$ and z be any element in the closure of the set A. Then the Bouligand tangent cone (in other words, contingent cone) of the set A at z, denoted by $\mathscr{T}(A; z)$, is defined by

$$\mathscr{T}(A; z) := \{w \in T_z\mathscr{H} : \exists t_m \downarrow 0, \exists w_m \in T_z\mathscr{H}, \\ w_m \to w, \exp_z(t_m w_m) \in A, \forall m \in \mathbb{N}\}.$$

In the following definition, we extend the notion of linearizing cone from Maeda [7] on Hadamard manifolds for (MOPP).

Definition 3.4. For any $\widetilde{z} \in \mathcal{D}$, the linearizing cone to the set S at the element $\widetilde{z}$ is the set defined as follows

$$T^{\mathrm{Lin}}(S; \widetilde{z}) := \{w \in T_{\widetilde{z}}\mathscr{H} : \langle \operatorname{grad} f_i(\widetilde{z}), w\rangle \leqq 0 \quad \forall i = 1, 2, \ldots, r, \\ \langle \operatorname{grad} g_j(\widetilde{z}), w\rangle \leqq 0, \quad \forall j \in \mathscr{A}(\widetilde{z})\}.$$

In the following theorem we establish an interesting relationship between the contingent cone $\mathscr{T}(S; \widetilde{z})$ and linearizing cone $T^{\mathrm{Lin}}(S; \widetilde{z})$.

Theorem 3.1. *For any arbitrary feasible element $\widetilde{z} \in \mathcal{D}$ the following holds.*

$$\bigcap_{k=1}^{r} \operatorname{cl}\operatorname{co}\mathscr{T}\left(S^k; \widetilde{z}\right) \subseteq T^{\mathrm{Lin}}(S; \widetilde{z}).$$

Proof. To begin with, we claim that

$$\mathscr{T}\left(S^k; \widetilde{z}\right) \subseteq T^{\mathrm{Lin}}\left(S^k; \widetilde{z}\right), \quad \forall k = 1, 2, \ldots, r.$$

Let us assume that $v \in T_{\widetilde{z}}\mathscr{H}$ is an arbitrary element in $\mathscr{T}\left(S^k; \widetilde{z}\right)$ for any fixed $k = 1, 2, \ldots, r$. Then by the definition of contingent cone, $\exists t_m \downarrow 0$, $\exists v_m \in T_z\mathscr{H}$, $v_m \to v$, $\forall m \in \mathbb{N}$ such that

$$\exp_{\widetilde{z}}(t_m v_m) \in S^k. \tag{3.1}$$

Let us construct a sequence $\{z_m\}_{m=1}^{\infty}$ in the following manner

$$z_m = \exp_{\widetilde{z}}(t_m v_m), \quad \forall m \in \mathbb{N}.$$

Then for every $m \in \mathbb{N}$, we have the following

$$f_i(z_m) = f_i(\exp_{\widetilde{z}}(t_m v_m)) \leqq f_i(\widetilde{z}), \qquad \forall i = 1, 2, \ldots, r \text{ and } i \neq k, \\ g_j(z_m) = g_j(\exp_{\widetilde{z}}(t_m v_m)) \leqq 0 = g_j(\widetilde{z}), \quad \forall j \in \mathscr{A}(\widetilde{z}).$$

By the Taylor expansion of f_i at $\widetilde{z}$, for every $i = 1, 2, \ldots r$, $i \neq k$, we have the following

$$f_i(\exp_{\widetilde{z}}(t_m v_m)) = f_i(\widetilde{z}) + t_m \langle \operatorname{grad} f_i(\widetilde{z})), v_m\rangle + o(t_m). \tag{3.2}$$

From (3.2) it follows that for every $i = 1, 2, \ldots r$, $i \neq k$, we have

$$\frac{f_i(\exp_{\widetilde{z}}(t_m v_m)) - f_i(\widetilde{z})}{t_m} = \langle \operatorname{grad} f_i(\widetilde{z}), v_m\rangle + \frac{o(t_m)}{t_m}. \tag{3.3}$$

Since $f_i(z_m) = f_i(\exp_{\widetilde{z}}(t_m v_m)) \leqq f_i(\widetilde{z})$, for every $i = 1, 2, \ldots, r$, $i \neq k$, then as $t_m \to 0$, it follows from equation (3.3) that

$$\langle \operatorname{grad} f_i(\widetilde{z}), v_m\rangle \leqq 0, \quad i = 1, 2, \ldots, r,\ i \neq k. \tag{3.4}$$

Similarly, we can establish that

$$\langle \operatorname{grad} g_j(\widetilde{z}), v_m\rangle \leqq 0, \quad j \in \mathscr{A}(\widetilde{z}). \tag{3.5}$$

By employing the continuity and linearity properties of inner product, we obtain the following

$$\begin{aligned} &\langle \operatorname{grad} f_i(\widetilde{z}), v\rangle \leqq 0, \quad \forall i = 1,2,\ldots,r,\ i \neq k, \\ &\langle \operatorname{grad} g_j(\widetilde{z}), v\rangle \leqq 0, \quad \forall j \in \mathscr{A}(\widetilde{z}). \end{aligned} \tag{3.6}$$

From (3.6) it follows that $v \in T^{\mathrm{Lin}}\left(S^k;\widetilde{z}\right)$. This implies that

$$\mathscr{T}\left(S^k;\widetilde{z}\right) \subseteq T^{\mathrm{Lin}}\left(S^k;\widetilde{z}\right), \quad k = 1,2,\ldots,r.$$

The linearizing cone $T^{\mathrm{Lin}}\left(S^k;\widetilde{z}\right)$ is a closed convex cone. Hence it follows that

$$\bigcap_{k=1}^{r} \operatorname{cl}\operatorname{co}\mathscr{T}\left(S^k;\widetilde{z}\right) \subseteq \bigcap_{k=1}^{r} T^{\mathrm{Lin}}\left(S^k;\widetilde{z}\right) = T^{\mathrm{Lin}}(S;\widetilde{z}). \tag{3.7}$$

This completes the proof. ∎

Remark 3.2. Theorem 3.1 is a generalization of Lemma 3.1 from [7] from Euclidean space to the novel context of Hadamard manifolds.

Now, we present the generalized Guignard constraint qualification in the framework of Hadamard manifolds for (MOPP).

Definition 3.5. Let $\widetilde{z} \in \mathcal{D}$. Then the generalized Guignard constraint qualification (in short, (GGCQ)) is said to be satisfied at $\widetilde{z}$ if and only if the following holds

$$T^{\mathrm{Lin}}(S;\widetilde{z}) \subseteq \bigcap_{i=1}^{r} \operatorname{cl}\operatorname{co}\mathscr{T}\left(S^i;\widetilde{z}\right).$$

The following theorem provides a necessary condition for efficiency of a feasible point of (MOPP).

Theorem 3.2. *Let $\widetilde{z} \in \mathcal{D}$ be an arbitrary feasible point of (MOPP) at which (GGCQ) holds. If $\widetilde{z} \in \mathcal{D}$ is an efficient solution of (MOPP), then the following system of inequalities*

$$\begin{aligned} &\langle \operatorname{grad} f_i(\widetilde{z}), w\rangle \leqq 0, \quad \forall i \in \{1,\ldots,r\}, \\ &\langle \operatorname{grad} f_i(\widetilde{z}), w\rangle < 0, \quad \textit{for at least one } i, \\ &\langle \operatorname{grad} g_j(\widetilde{z}), w\rangle \leqq 0, \quad \forall j \in \mathscr{A}(\widetilde{z}), \end{aligned}$$

has no solution $w \in T_{\widetilde{z}}\mathscr{H}$.

Proof. On the contrary, let us assume that there exists a vector $w \in T_{\widetilde{z}}\mathscr{H}$ such that the following system of inequalities is satisfied

$$\begin{aligned} &\langle \operatorname{grad} f_i(\widetilde{z}), w\rangle \leqq 0, \quad \forall i \in \{1,\ldots,r\}, \\ &\langle \operatorname{grad} f_i(\widetilde{z}), w\rangle < 0, \quad \text{for at least one } i, \\ &\langle \operatorname{grad} g_j(\widetilde{z}), w\rangle \leqq 0, \quad \forall j \in \mathscr{A}(\widetilde{z}). \end{aligned}$$

Thus we have

$$w \in T^{Lin}(S;\widetilde{z}).$$

Without loss of generality, let us consider that

$$\begin{aligned} &\langle \operatorname{grad} f_1(\widetilde{z}), w\rangle < 0, \\ &\langle \operatorname{grad} f_i(\widetilde{z}), w\rangle \leqq 0, \quad i = 2,3,\ldots,r. \end{aligned}$$

Since (GGCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, it follows that

$$w \in \operatorname{cl}\operatorname{co}\mathscr{T}\left(S^1;\widetilde{z}\right).$$

Then it follows that there exists some sequence $\{w_m\}_{m=1}^{\infty} \subseteq \operatorname{co} \mathscr{T}(S^1; \widetilde{z})$, such that

$$\lim_{m\to\infty} w_m = w.$$

Then for any element w_m $(m = 1, 2, \ldots)$ of the sequence $\{w_m\}_{m=1}^{\infty}$, there exist $L_m \in \mathbb{R}$, $\lambda_{m_k} \geqq 0$ and $w_{m_k} \in \mathscr{T}(S^1; \widetilde{z})$, $k = 1, 2, \ldots, L_m$, such that

$$\sum_{k=1}^{L_m} \lambda_{m_k} = 1, \quad \sum_{k=1}^{L_m} \lambda_{m_k} w_{m_k} = w_m.$$

For every $m \in \mathbb{N}$ and $k = 1, 2, \ldots, L_m$, since $w_{m_k} \in \mathscr{T}(S^1, \widetilde{z})$, by Definition 3.3 there exist sequences $\left\{w_{m_k}^n\right\}_{n=1}^{\infty}$, $w_{m_k}^n \in T_{\widetilde{z}}\mathscr{H}$, $\forall n \in \mathbb{N}$ and $\left\{t_{m_k}^n\right\}_{n=1}^{\infty}$, $t_{m_k}^n (> 0) \in \mathbb{R}$ $\forall n \in \mathbb{N}$, with $t_{m_k}^n \downarrow 0$, such that

$$\lim_{n\to\infty} w_{m_k}^n = w_{m_k}, \quad \exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n) \in S^1.$$

Let us set $x_{m_k}^n$ as follows

$$x_{m_k}^n := \exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n), \quad \forall n \in \mathbb{N}.$$

Then for every $n \in \mathbb{N}$, we have the following

$$\begin{aligned} f_i\left(x_{m_k}^n\right) &= f_i\left(\exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n)\right) \leqq f_j\left(\widetilde{z}\right), \quad \forall i \in \{2, 3, \ldots, r\}, \\ g_j\left(x_{m_k}^n\right) &= g_j\left(\exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n)\right) \leqq 0, \qquad \forall j \in \mathscr{A}(\widetilde{z}). \end{aligned} \tag{3.8}$$

Again, since $\widetilde{z} \in \mathcal{D}$ is an efficient solution of (MOPP), it follows that

$$f_1\left(x_{m_k}^n\right) = f_1\left(\exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n)\right) \geqq f_1\left(\widetilde{z}\right), \quad \forall n \in \mathbb{N}.$$

From the Taylor expansion of f_i at $\widetilde{z}$, for each $i \in \{2, 3, \ldots, r\}$, we have the following

$$f_i\left(\exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n)\right) = f_i(\widetilde{z}) + t_{m_k}^n \left\langle \operatorname{grad} f_i(\widetilde{z}), w_{m_k}^n\right\rangle + o\left(t_{m_k}^n\right).$$

Then it follows that for each $i \in \{2, 3, \ldots, r\}$, we have

$$\frac{f_i\left(\exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n)\right) - f_i(\widetilde{z})}{t_{m_k}^n} = \left\langle \operatorname{grad} f_i(\widetilde{z}), w_{m_k}^n\right\rangle + \frac{o\left(t_{m_k}^n\right)}{t_{m_k}^n}. \tag{3.9}$$

Since $f_i\left(x_{m_k}^n\right) = f_i\left(\exp_{\widetilde{z}}(t_{m_k}^n w_{m_k}^n)\right) \leqq f_i\left(\widetilde{z}\right)$, for every $i \in \{2, 3, \ldots, r\}$, then as $t_{m_k}^n \to 0$, it follows from equation (3.9) that

$$\left\langle \operatorname{grad} f_i\left(\widetilde{z}\right), w_{m_k}\right\rangle \leqq 0, \quad \forall i \in \{2, 3, \ldots, r\}.$$

Similarly, we can show that

$$\begin{aligned} &\left\langle \operatorname{grad} f_1\left(\widetilde{z}\right), w_{m_k}\right\rangle \geqq 0, \\ &\left\langle \operatorname{grad} f_i\left(\widetilde{z}\right), w_{m_k}\right\rangle \leqq 0, \quad \forall i \in \{2, 3, \ldots, r\}, \\ &\left\langle \operatorname{grad} g_j\left(\widetilde{z}\right), w_{m_k}\right\rangle \leqq 0, \quad \forall j \in \mathscr{A}(\widetilde{z}). \end{aligned}$$

From the continuity and linearity property of the inner product it follows that

$$\begin{aligned} &\left\langle \operatorname{grad} f_1\left(\widetilde{z}\right), w\right\rangle \geqq 0, \\ &\left\langle \operatorname{grad} f_i\left(\widetilde{z}\right), w\right\rangle \leqq 0, \quad \forall i \in \{2, 3, \ldots, r\}, \\ &\left\langle \operatorname{grad} g_j\left(\widetilde{z}\right), w\right\rangle \leqq 0, \quad \forall j \in \mathscr{A}(\widetilde{z}), \end{aligned}$$

which is a contradiction. This completes the proof. ∎

Remark 3.3. If $\mathscr{H} = \mathbb{R}^n$, Theorem 3.2 reduces to Theorem 3.1 from [7]. Thus, Theorem 3.2 generalizes Theorem 3.1 from [7] from Euclidean space to an even more general space, that is, Hadamard manifolds.

In the following theorem we deduce Karush-Kuhn-Tucker type necessary optimality criteria for (MOPP) using (GGCQ).

Theorem 3.3. *Let $\widetilde{z} \in \mathcal{D}$ be an arbitrary feasible point of (MOPP) at which (GGCQ) holds. If $\widetilde{z} \in \mathcal{D}$ is an efficient solution of (MOPP), then there exist Lagrange multipliers α_i $(i = 1, \ldots, r)$, and λ_j $(j = 1, \ldots, s)$ such that*

$$\sum_{i=1}^{r} \alpha_i \operatorname{grad} f_i(\widetilde{z}) + \sum_{j=1}^{s} \lambda_j \operatorname{grad} g_j(\widetilde{z}) = 0,$$
$$\text{and, } \alpha_i > 0, \quad \forall i \in \{1, \ldots r\},$$
$$g_j(\widetilde{z}) \leqq 0, \quad \lambda_j \geqq 0, \quad \lambda_j g_j(\widetilde{z}) = 0, \quad \forall j \in \{1, \ldots, s\}.$$

Proof. Since $\widetilde{z} \in \mathcal{D}$ is an efficient solution of (MOPP) and (GGCQ) holds at $\widetilde{z}$, it follows from Theorem 3.2 that the following system of inequalities

$$\langle \operatorname{grad} f_i(\widetilde{z}), w \rangle \leqq 0, \quad \forall i \in \{1, \ldots, r\},$$
$$\langle \operatorname{grad} f_i(\widetilde{z}), w \rangle < 0, \quad \text{for at least one } i,$$
$$\langle \operatorname{grad} g_j(\widetilde{z}), w \rangle \leqq 0, \quad \forall j \in \mathscr{A}(\widetilde{z}),$$

has no solution $w \in T_{\widetilde{z}}\mathscr{H}$. By using Tucker's theorem of alternative ([8] pp. 29-30), it follows that there exist $\alpha_i > 0$ $(i \in \{1, \ldots, r\})$, $\lambda_j \geqq 0$ $(j \in \mathscr{A}(\widetilde{z}))$, such that

$$\sum_{i=1}^{r} \alpha_i \operatorname{grad} f_i(\widetilde{z}) + \sum_{j \in \mathscr{A}(\widetilde{z})} \lambda_j \operatorname{grad} g_j(\widetilde{z}) = 0. \tag{3.10}$$

By setting $\lambda_j = 0$ $(j \notin \mathscr{A}(\widetilde{z}))$, we have

$$\sum_{i=1}^{r} \alpha_i \operatorname{grad} f_i(\widetilde{z}) + \sum_{j=1}^{s} \lambda_j \operatorname{grad} g_j(\widetilde{z}) = 0. \tag{3.11}$$

On the other hand, since $g_j(\widetilde{z}) = 0$ for every $j \in \mathscr{A}(\widetilde{z})$, we have

$$\lambda_j g_j(\widetilde{z}) = 0, \quad \forall j \in \{1, 2, \ldots, s\}.$$

This completes the proof. ∎

Remark 3.4. If the manifold $\mathscr{H}$ is $\mathbb{R}^n$, Theorem 3.3 reduces to Theorem 3.2 from [7]. Thus, Theorem 3.3 generalizes Theorem 3.2 from [7] from Euclidean space to the framework of Hadamard manifolds.

The following examples illustrate the significance of (GGCQ) and Theorem 3.3.

Example 3.1. *Let us take into consideration the set $\mathscr{H} \subset \mathbb{R}$ defined as $\mathscr{H} := \{x \in \mathbb{R},\ x > 0\}$. Then $\mathscr{H}$ is a Riemannian manifold (see for instance, [9], and Example 4.4 of [12]). The tangent plane at any point $x \in \mathscr{H}$, denoted by $T_x\mathscr{H}$, is the set of all reals $\mathbb{R}$. $\mathscr{H}$ is equipped with the Riemannian metric as defined below*

$$\langle p, q \rangle_x = \langle \mathscr{G}(x)p, q \rangle, \quad \forall p, q \in T_x\mathscr{H} = \mathbb{R}^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$ and $\mathscr{G}(x) = 1/x^2$. $\mathscr{H}$ is also a Hadamard manifold. The exponential map $exp_x : T_x\mathscr{H} \to \mathscr{H}$ for any $u \in T_x\mathscr{H}$ is given by $exp_x(u) = xe^{\frac{u}{x}}, \quad \forall u \in \mathscr{H}$.
We take into consideration the following smooth multiobjective programming problem (P) on $\mathscr{H}$.

$$(P) \quad \textit{Minimize } f(x) = (f_1(x), f_2(x)) := (-(x-e)^3, (x-e)^3),$$
$$\textit{subject to } \; g(x) := -x \leq e,$$

where, the functions $f_i : \mathscr{H} \to \mathbb{R}$, $(i = 1, 2)$ and $g : \mathscr{H} \to \mathbb{R}$, are smooth real valued functions defined on $\mathscr{H}$. The feasible set F for the problem is

$$F = \{x \in \mathbb{R},\ x \geq e\}.$$

Let us pick the feasible point $\widetilde{z} = e \in F$. Clearly $\widetilde{z}$ is a Pareto efficient solution of (P). Then it follows that

$$\begin{aligned}
\operatorname{grad} f_1(x) &= \mathscr{G}(x)^{-1}\left(-3(x-e)^2\right) = -3x^2(x-e)^2,\\
\operatorname{grad} f_2(x) &= \mathscr{G}(x)^{-1}\left(3(x-e)^2\right) = 3x^2(x-e)^2,\\
\operatorname{grad} g(x) &= \mathscr{G}(x)^{-1}(-1) = -x^2.
\end{aligned}$$

Using above equations, it can be verified that

$$T^{\mathrm{Lin}}(S;\widetilde{z}) = [0,\infty). \tag{3.12}$$

Again, for the problem (P), we can show by simple calculations that $S^1 = \{e\}$ and $S^2 = [e,\infty)$. Then it follows that $\mathscr{T}(S^1;e) = \{0\}$ and $\mathscr{T}(S^2;e) = [0,\infty)$. Then we obtain the following

$$\bigcap_{i=1}^{2} \operatorname{cl}\operatorname{co}\mathscr{T}\left(S^i;\widetilde{z}\right) = \{0\}. \tag{3.13}$$

From (3.12) *and* (3.13)*, it follows that (GGCQ) is not satisfied at the point $\widetilde{z} = e \in F$. However, it can be verified that for any choice of Lagrange multipliers $\alpha_i > 0$ $(i = 1, 2)$ and for $\lambda = 0$, we have*

$$\begin{aligned}
\sum_{i=1}^{2}\alpha_i \operatorname{grad} f_j(\widetilde{z}) + \lambda \operatorname{grad} g(\widetilde{z}) &= 0,\\
g(\widetilde{z}) \leqq 0,\ \lambda g(\widetilde{z}) &= 0.
\end{aligned} \tag{3.14}$$

Thus, it is verified that satisfaction of (GGCQ) is not a sufficient condition for Theorem 3.3.

Example 3.2. *Let us take into consideration the set $\mathscr{H} \subset \mathbb{R}$ defined as $\mathscr{H} := \{x \in \mathbb{R},\ x > 0\}$. Then $\mathscr{H}$ is a Hadamard manifold, as explained in Example 3.1.*
We take into consideration the following smooth multiobjective programming problem (MP) on $\mathscr{H}$.

$$\begin{aligned}
\text{(MP)}\quad \text{Minimize } & f(x) = (f_1(x), f_2(x)) := (2\sqrt{x}, \log x),\\
\text{subject to } & g(x) := 1 - \log x \leq 0,
\end{aligned}$$

where, the functions $f_i : \mathscr{H} \to \mathbb{R}$, $(i = 1, 2)$ and $g : \mathscr{H} \to \mathbb{R}$, are smooth real valued functions defined on $\mathscr{H}$. The feasible set F for the problem is

$$F = \{x \in \mathbb{R},\ x \geq e\}.$$

Let us pick the feasible point $\widetilde{z} = e \in F$. Clearly $\widetilde{z}$ is a Pareto efficient solution of (MP). Then it follows that

$$\begin{aligned}
\operatorname{grad} f_1(x) &= \mathscr{G}(x)^{-1}\left(\frac{1}{\sqrt{x}})\right) = x\sqrt{x},\\
\operatorname{grad} f_2(x) &= \mathscr{G}(x)^{-1}\left(\frac{1}{x}\right) = x,\\
\operatorname{grad} g(x) &= \mathscr{G}(x)^{-1}\left(-\frac{1}{x}\right) = -x.
\end{aligned}$$

Using above equations, it can be verified that

$$T^{\mathrm{Lin}}(S;\widetilde{z}) = \{0\}. \tag{3.15}$$

Again, for the problem (P), we can show by simple calculations that $S^1 = \{e\}$ *and* $S^2 = \{e\}$*. Then it follows that* $\mathscr{T}(S^1; e) = \{0\}$ *and* $\mathscr{T}(S^2; e) = \{0\}$*. Then we obtain the following*

$$\bigcap_{i=1}^{2} \operatorname{cl} \operatorname{co} \mathscr{T}\left(S^i, \widetilde{z}\right) = \{0\}. \tag{3.16}$$

From (3.15) *and* (3.16)*, it follows that (GGCQ) is satisfied at the point* $\widetilde{z} = e \in F$*. Then it can be verified that exist Lagrange multipliers* $\alpha_1 = 1$*,* $\alpha_2 = \sqrt{e}$*,* $\lambda = 2\sqrt{e}$ *such that*

$$\sum_{i=1}^{2} \alpha_i \operatorname{grad} f_j\left(\widetilde{z}\right) + \lambda \operatorname{grad} g\left(\widetilde{z}\right) = 0, \tag{3.17}$$

Thus, the necessary optimality conditions in Theorem 3.3 is verified.

4. SUFFICIENT CONDITIONS FOR (GGCQ)

In this section, we present several constraint qualifications in the framework of Hadamard manifolds for (MOPP). Subsequently, we establish various interesting relationships between these constraint qualifications. In particular, we show that these constraint qualifications become sufficient conditions ensuring that (GGCQ) is satisfied.
The following definitions are extensions of different constraint qualifications from Maeda [7] from Euclidean spaces to Hadamard manifolds for (MOPP).

Definition 4.1. Let $\widetilde{z} \in \mathcal{D}$. Then the Abadie's constraint qualification (in short, (ACQ)) is said to hold at the feasible point $\widetilde{z}$ if we have the following

$$T^{\mathrm{Lin}}\left(S; \widetilde{z}\right) \subseteq \mathscr{T}\left(S; \widetilde{z}\right). \tag{4.1}$$

Definition 4.2. Let $\widetilde{z} \in \mathcal{D}$. Then the generalized Abadie's constraint qualification (in short, (GACQ)) is said to hold at the feasible point $\widetilde{z}$ if we have the following

$$T^{\mathrm{Lin}}\left(S; \widetilde{z}\right) \subseteq \bigcap_{i=1}^{r} \mathscr{T}\left(S^i; \widetilde{z}\right). \tag{4.2}$$

Definition 4.3. Let $\widetilde{z} \in \mathcal{D}$. Then the Cottle-type constraint qualification (in short, (CTCQ)) is said to hold at the feasible point $\widetilde{z}$ if for every $k = 1, 2, \ldots, r$, the following system of inequalities

$$\begin{aligned} &\langle \operatorname{grad} f_i\left(\widetilde{z}\right), v\rangle < 0, \quad i = 1, 2, \ldots, r \text{ and } i \neq k, \\ &\langle \operatorname{grad} g_j\left(\widetilde{z}\right), v\rangle < 0, \quad j \in \mathscr{A}\left(\widetilde{z}\right), \end{aligned} \tag{4.3}$$

admits of a solution $v \in T_{\widetilde{z}}\mathscr{H}$.

Definition 4.4. Let $\widetilde{z} \in \mathcal{D}$. Then the Slater-type constraint qualification (in short, (STCQ) is said to hold at the feasible point $\widetilde{z}$ if each of the functions f_i ($i = 1, 2, \ldots, r$), and g_j ($j = 1, 2, \ldots, s$), are geodesic convex and for every $k = 1, 2, \ldots, r$, the following system of inequalities

$$\begin{aligned} &f_i(z) < f_i\left(\widetilde{z}\right), \qquad i = 1, 2, \ldots, r \text{ and } i \neq k, \\ &g_j(z) < 0, \qquad\qquad j = 1, 2, \ldots, s, \end{aligned} \tag{4.4}$$

admits of a solution $z \in \mathscr{H}$.

Definition 4.5. Let $\widetilde{z} \in \mathcal{D}$. Then the linear constraint qualification (in short, (LCQ)) is said to hold at the feasible point $\widetilde{z}$ if each of the functions f_i ($i = 1, 2, \ldots, r$), and g_j ($j \in \mathscr{A}\left(\widetilde{z}\right)$) are linear.

Definition 4.6. Let $\widetilde{z} \in \mathcal{D}$. Then the linear objective constraint qualification (in short, (LOCQ)) is said to hold at the feasible point $\widetilde{z}$ if each of the functions f_i $(i = 1, 2, \ldots, r)$ are linear and the following system of inequalities

$$\langle \operatorname{grad} f_i(\widetilde{z}), v\rangle \leqq 0, \quad i = 1, 2, \ldots, r,$$
$$\langle \operatorname{grad} g_j(\widetilde{z}), v\rangle < 0, \quad j \in \mathscr{A}(\widetilde{z}),$$

admits of a solution $v \in T_{\widetilde{z}}\mathscr{H}$.

Definition 4.7. Let $\widetilde{z} \in \mathcal{D}$. Then the Mangasarian-Fromovitz constraint qualification (in short, (MFCQ)) is said to hold at the feasible point $\widetilde{z}$ if $\{\operatorname{grad} f_i,\ i = 1, 2, \ldots, r\}$ is a linearly independent set and following system of inequalities

$$\langle \operatorname{grad} f_i(\widetilde{z}), v\rangle = 0, \quad i = 1, 2, \ldots, r,$$
$$\langle \operatorname{grad} g_j(\widetilde{z}), v\rangle < 0, \quad j \in \mathscr{A}(\widetilde{z}),$$

admits of a solution $v \in T_{\widetilde{z}}\mathscr{H}$.

Remark 4.1. It readily follows from the above definitions that, if (ACQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, then (GACQ) is satisfied at $\widetilde{z} \in \mathcal{D}$. Moreover, if (GACQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, then (GGCQ) is automatically satisfied at $\widetilde{z} \in \mathcal{D}$.

In the following theorem we establish a relationship between the linear constraint qualification (LCQ) and the Abadie's constraint qualification (ACQ).

Theorem 4.1. *Let $\widetilde{z} \in \mathcal{D}$ be any feasible element of (MOPP). If (LCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, then (ACQ) is satisfied at $\widetilde{z} \in \mathcal{D}$.*

Proof. Let $v \in T_{\widetilde{z}}\mathscr{H}$ be an arbitrary element of $T^{\mathrm{Lin}}(S; \widetilde{z})$. Then from the definition of linearizing cone, it follows that

$$\langle \operatorname{grad} f_i(\widetilde{z}), v\rangle \leqq 0, \quad i = 1, 2, \ldots, r, \tag{4.5}$$

$$\langle \operatorname{grad} g_j(\widetilde{z}), v\rangle \leqq 0, \quad j \in \mathscr{A}(\widetilde{z}). \tag{4.6}$$

Let us assume that (LCQ) is satisfied at $\widetilde{z}$. Since $T^{\mathrm{Lin}}(S; \widetilde{z})$ is closed, there exists a sequence $\{v_n\}_{n=1}^{\infty}$ such that $v_n \to v$ as $n \to \infty$. Let us consider a sequence $t_n \downarrow 0$. Now let us define a sequence $\{z_n\}$ in the following manner

$$z_n := \exp_{\widetilde{z}}(t_n v_n), \quad \forall n \in \mathbb{N}.$$

Clearly, $z_n \to \widetilde{z}$ as $n \to \infty$. Then it follows that each of the functions f_i $(i = 1, 2, \ldots, r)$, and g_j $(j \in \mathscr{A}(\widetilde{z}))$, are linear. Combining this with inequalities (4.5) and (4.6), we have

$$f_i(z_n) = f_i(\exp_{\widetilde{z}}(t_n v_n)) = f_i(\widetilde{z}) + t_n \langle \operatorname{grad} f_i(\widetilde{z}), v_n\rangle \leqq f_i(\widetilde{z}),\ \forall i = 1, 2, \ldots, r, \tag{4.7}$$

$$g_j(z_n) = g_j(\exp_{\widetilde{z}}(t_n v_n)) = g_j(\widetilde{z}) + t_n \langle \operatorname{grad} g_j(\widetilde{z}), v_n\rangle \leqq g_j(\widetilde{z}) = 0,\ \forall j \in \mathscr{A}(\widetilde{z}). \tag{4.8}$$

For every $j \notin \mathscr{A}(\widetilde{z})$, it follows from the continuity of g_j that

$$g_j(z_n) = g_j(\exp_{\widetilde{z}}(t_n v_n)) < 0, \quad \text{for sufficiently large } n. \tag{4.9}$$

From inequalities (4.8) and (4.9), it follows that

$$g_j(z_n) = g_j(\exp_{\widetilde{z}}(t_n v_n)) \leq 0, \quad \text{for sufficiently large } n, \quad \forall j = 1, 2, \ldots, s. \tag{4.10}$$

From (4.7), (4.8) and (4.10) we have

$$z_n = \exp_{\widetilde{z}}(t_n v_n) \in S, \quad \text{all } n \text{ sufficiently large.} \tag{4.11}$$

This implies that

$$v \in \mathscr{T}(S; \widetilde{z}).$$

This completes the proof. ∎

In the following theorem we establish a relationship between the linear objective constraint qualification (LOCQ) and the Abadie's constraint qualification (ACQ).

Theorem 4.2. *Let $\widetilde{z} \in \mathcal{D}$ be any feasible element of (MOPP). If (LOCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, then (ACQ) is satisfied at $\widetilde{z} \in \mathcal{D}$.*

Proof. From the given hypothesis, (LOCQ) is satisfied at $\widetilde{z}$. Hence each of the functions f_i $(i = 1, 2, \ldots, r)$ are linear. Furthermore, there exists some $w \in T_{\widetilde{z}}\mathscr{H}$ such that

$$\begin{aligned} &\langle \operatorname{grad} f_i(\widetilde{z}), w \rangle \leqq 0, \quad i = 1, 2, \ldots, r, \\ &\langle \operatorname{grad} g_j(\widetilde{z}), w \rangle < 0, \quad j \in \mathscr{A}(\widetilde{z}). \end{aligned} \tag{4.12}$$

Let us assume that $v \in T_{\widetilde{z}}\mathscr{H}$ be any arbitrary element of $T^{\mathrm{Lin}}(S; \widetilde{z})$. We define a sequence $\{v_n\}_{n=1}^{\infty}$ as follows

$$v_n = v + \tau_n w, \quad \forall n \in \mathbb{N},$$

where $\{\tau_n\} \downarrow 0$. Clearly, $v_n \to v$ as $n \to \infty$. Then it follows that for every n, we have

$$\langle \operatorname{grad} f_i(\widetilde{z}), v_n \rangle = \langle \operatorname{grad} f_i(\widetilde{z}), v \rangle + \tau_n \langle \operatorname{grad} f_i(\widetilde{z}), w \rangle \leqq 0, \quad i = 1, 2, \ldots, r, \tag{4.13}$$

$$\langle \operatorname{grad} g_j(\widetilde{z}), v_n \rangle = \langle \operatorname{grad} g_j(\widetilde{z}), v \rangle + \tau_n \langle \operatorname{grad} g_j(\widetilde{z}), w \rangle < 0, \quad j \in \mathscr{A}(\widetilde{z}). \tag{4.14}$$

For every v_n $(n \in \mathbb{N})$, we consider a sequence $\{\lambda_{n_k}\}_{k=1}^{\infty} \downarrow 0$. Now, we define a sequence $\{z_{n_k}\}_{k=1}^{\infty}$ as follows

$$z_{n_k} := \exp_{\widetilde{z}}(\lambda_{n_k} v_n), \quad \forall k \in \mathbb{N}.$$

Clearly, $\{z_{n_k}\} \to \widetilde{z}$ as $k \to \infty$. Since each of the functions f_i $(i = 1, 2, \ldots, r)$ are linear, it follows from (4.13) that

$$f_i(z_{n_k}) = f_i(\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) = f_i(\widetilde{z}) + \lambda_{n_k} \langle \operatorname{grad} f_i(\widetilde{z}), v_n \rangle \leqq f_i(\widetilde{z}). \tag{4.15}$$

Again, from (4.14), it follows that for every $j \in \mathscr{A}(\widetilde{z})$,

$$\begin{aligned} g_j(z_{n_k}) = g_j(\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) &= g_j(\widetilde{z}) + \lambda_{n_k} \langle \operatorname{grad} g_j(\widetilde{z}), v_n \rangle + o(|\lambda_{n_k}|) \\ &< g_j(\widetilde{z}), \end{aligned} \tag{4.16}$$

for sufficient large values of k, and $\lim\limits_{k \to \infty} \frac{o(|\lambda_{n_k}|)}{|\lambda_{n_k}|} = 0$. Now, for every $j \notin \mathscr{A}(\widetilde{z})$, it follows from the continuity of g_j that

$$g_j(z_{n_k}) = g_j(\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) < 0, \quad \text{for sufficiently large } k. \tag{4.17}$$

Then it follows from (4.15), (4.16) and (4.17) that

$$z_{n_k} = (\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) \in S, \quad \text{for sufficiently large } k. \tag{4.18}$$

This implies that

$$v \in \mathscr{T}(S; \widetilde{z}).$$

This completes the proof. ∎

In the following theorem we establish relationship between the Mangasarian-Fromovitz constraint qualification (MFCQ) and the Cottle-type constraint qualification (CTCQ).

Theorem 4.3. *Let $\widetilde{z} \in \mathcal{D}$ be any arbitrary feasible element of (MOPP). If (MFCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, then (CTCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$.*

Proof. According to the given hypothesis, (MFCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$. This implies that $\{\text{grad}\, f_i,\ i = 1, 2, \ldots, r\}$ is a linearly independent set. Moreover, there exists some $w \in T_{\widetilde{z}}\mathscr{H}$ such that

$$\langle \text{grad}\, f_i\,(\widetilde{z})\,, w\rangle = 0, \quad i = 1, 2, \ldots, r, \tag{4.19}$$

$$\langle \text{grad}\, g_j\,(\widetilde{z})\,, w\rangle < 0, \quad j \in \mathscr{A}\,(\widetilde{z})\,. \tag{4.20}$$

On the contrary, let us assume that (CTCQ) is not satisfied at $\widetilde{z} \in \mathcal{D}$. Then there exists some $k \in \{1, 2, \ldots, r\}$ such that the following system of inequalities

$$\langle \text{grad}\, f_i\,(\widetilde{z})\,, v\rangle < 0, \quad i = 1, 2, \ldots, r \text{ and } i \neq k, \tag{4.21}$$

$$\langle \text{grad}\, g_j\,(\widetilde{z})\,, v\rangle < 0, \quad j \in \mathscr{A}\,(\widetilde{z})\,, \tag{4.22}$$

does not admit of a solution $v \in T_{\widetilde{z}}\mathscr{H}$. Then from Gordon's theorem of alternative (see, [8]), it follows from (4.21) and (4.22) that there exist real numbers $\alpha_i \geqq 0$, $i = 1, 2, \ldots, r$, $i \neq k$, and $\lambda_j \geqq 0$, $j \in \mathscr{A}\,(\widetilde{z})$, not all zero such that the following equation is satisfied

$$\sum_{\substack{i=1\\ i\neq k}}^{r} \alpha_i \,\text{grad}\, f_i\,(\widetilde{z}) + \sum_{j\in\mathscr{A}(\widetilde{z})} \lambda_j \,\text{grad}\, g_j\,(\widetilde{z}) = 0. \tag{4.23}$$

From (4.19) and (4.23) it follows that

$$\sum_{j\in\mathscr{A}(\widetilde{z})} \lambda_j \,\langle \text{grad}\, g_j\,(\widetilde{z})\,, w\rangle = 0. \tag{4.24}$$

Combining (4.20) and (4.24) we have $\lambda_j = 0$, for every $j \in \mathscr{A}\,(\widetilde{z})$. Then from (4.23), we have

$$\sum_{\substack{i=1\\ i\neq k}}^{r} \alpha_i \,\text{grad}\, f_k\,(\widetilde{z}) = 0. \tag{4.25}$$

From the linear independence of the set $\{\text{grad}\, f_i,\ i = 1, 2, \ldots, r\}$, it follows that $\alpha_i = 0$, for every $i = 1, 2, \ldots, r$ and $i \neq k$, which is a contradiction. This completes the proof. ∎

In the following theorem we establish a relationship between the Slater-type constraint qualification (STCQ) and the Cottle-type constraint qualification (CTCQ).

Theorem 4.4. *Let $\widetilde{z} \in \mathcal{D}$ be any feasible element of (MOPP). If (STCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, then* (CTCQ) *is satisfied at $\widetilde{z} \in \mathcal{D}$.*

Proof. From the given hypothesis, (STCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$. Then it follows that each of the functions f_i $(i = 1, 2, \ldots, r)$, and g_j $(j = 1, 2, \ldots, s)$, are geodesic convex and for every $k = 1, 2, \ldots, r$, the following system of inequalities

$$\begin{aligned} &f_i(z) < f_i\,(\widetilde{z})\,, && i = 1, 2, \ldots, r \text{ and } i \neq k,\\ &g_j(z) < 0, && j = 1, 2, \ldots, s, \end{aligned} \tag{4.26}$$

admits of a solution $z_k \in \mathscr{H}$. From the geodesic convexity of the functions f_i, $(i = 1, 2, \ldots, r$, and g_j, $(j = 1, 2, \ldots, s)$, it follows from (4.26) that

$$\left\langle \text{grad}\, f_i\,(\widetilde{z})\,, \exp_{\widetilde{z}}^{-1}(z_k)\right\rangle \leqq f_i\,(z_k) - f_i\,(\widetilde{z}) < 0, \quad \forall i = 1, 2, \ldots, r, \tag{4.27}$$

$$\left\langle \text{grad}\, g_j\,(\widetilde{z})\,, \exp_{\widetilde{z}}^{-1}(z_k)\right\rangle \leqq g_j\,(z_k) - g_j\,(\widetilde{z}) < 0, \quad \forall j \in \mathscr{A}(\widetilde{z}). \tag{4.28}$$

Let us define $v_k := \exp_{\widetilde{z}}^{-1}(z_k)$. Then it follows that for every $k = 1, 2, \ldots, r$, we have the following

$$\begin{aligned} &\langle \operatorname{grad} f_i(\widetilde{z}), v_k\rangle < 0, \quad i = 1, 2, \ldots, r \text{ and } i \neq k, \\ &\langle \operatorname{grad} g_j(\widetilde{z}), v_k\rangle < 0, \quad j \in \mathscr{A}(\widetilde{z}). \end{aligned} \tag{4.29}$$

From (4.29) it follows that (CTCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$. This completes the proof. ∎

In the following theorem we establish a relationship between the Cottle-type constraint qualification (CTCQ) and the generalized Guignard constraint qualification (GGCQ).

Theorem 4.5. *Let $\widetilde{z} \in \mathcal{D}$ be any feasible element of (MOPP). If (CTCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$, then (GGCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$.*

Proof. From the given hypothesis, (CTCQ) is satisfied at $\widetilde{z} \in \mathcal{D}$. Hence, it follows that there exists some $w \in T_{\widetilde{z}}\mathscr{H}$ such that the following system of inequalities is satisfied for every $k = 1, 2, \ldots, r$.

$$\langle \operatorname{grad} f_i(\widetilde{z}), w\rangle < 0, \quad i = 1, 2, \ldots, r \text{ and } i \neq k, \tag{4.30}$$

$$\langle \operatorname{grad} g_j(\widetilde{z}), w\rangle < 0, \quad j \in \mathscr{A}(\widetilde{z}). \tag{4.31}$$

Let $v \in T_{\widetilde{z}}\mathscr{H}$ be an arbitrary element of $T^{\text{Lin}}(S; \widetilde{z})$. Then from the definition of linearizing cone, it follows that

$$\begin{aligned} &\langle \operatorname{grad} f_i(\widetilde{z}), v\rangle \leqq 0, \quad i = 1, 2, \ldots, r, \\ &\langle \operatorname{grad} g_j(\widetilde{z}), v\rangle \leqq 0, \quad j \in \mathscr{A}(\widetilde{z}). \end{aligned} \tag{4.32}$$

To begin with, we claim that $v \in \mathscr{T}(S^1; \widetilde{z})$. Let us consider a sequence $\{\tau_n\}_{n=1}^{\infty} \downarrow 0$. Then we define a sequence $\{v_n\}_{n=1}^{\infty}$ as follows

$$v_n := v + \tau_n w. \tag{4.33}$$

Clearly, $v_n \to v$ as $n \to \infty$. From (4.30), (4.31) and (4.33), it follows that

$$\langle \operatorname{grad} f_i(\widetilde{z}), v_n\rangle = \langle \operatorname{grad} f_i(\widetilde{z}), v\rangle + \tau_n \langle \operatorname{grad} f_i(\widetilde{z}), w\rangle < 0, \tag{4.34}$$

$$\langle \operatorname{grad} g_j(\widetilde{z}), v_n\rangle = \langle \operatorname{grad} \dot{g}_j(\widetilde{z}), v\rangle + \tau_n \langle \operatorname{grad} g_j(\widetilde{z}), w\rangle < 0. \tag{4.35}$$

For every element of the sequence $\{v_n\}$, $(n = 1, 2, \ldots)$, we consider a sequence $\{\lambda_{n_k}\}_{k=1}^{\infty} \downarrow 0$. Now, we construct a sequence $\{z_{n_k}\}_{k=1}^{\infty}$ converging to $\widetilde{z}$ by

$$z_{n_k} := \exp_{\widetilde{z}}(\lambda_{n_k} v_n), \quad \forall k \in \mathbb{N}.$$

Clearly, $\{z_{n_k}\} \to \widetilde{z}$ as $k \to \infty$. Then for sufficiently large values of k, we have the following for every $i = 2, 3, \ldots, r$.

$$\begin{aligned} f_i(z_{n_k}) = f_i(\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) &= f_i(\widetilde{z}) + \lambda_{n_k} \langle \operatorname{grad} f_i(\widetilde{z}), v_n\rangle + o(|\lambda_{n_k}|) \\ &< f_i(\widetilde{z}). \end{aligned} \tag{4.36}$$

Similarly, for sufficiently large values of k, we have the following for every $j \in \mathscr{A}(\widetilde{z})$.

$$\begin{aligned} g_j(z_{n_k}) = g_j(\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) &= g_j(\widetilde{z}) + \lambda_{n_k} \langle \operatorname{grad} g_j(\widetilde{z}), v_n\rangle + o(|\lambda_n|) \\ &< g_j(\widetilde{z}) = 0. \end{aligned} \tag{4.37}$$

Now, for every $j \notin \mathscr{A}(\widetilde{z})$, it follows from the continuity of g_j that

$$g_j(z_{n_k}) = g_j(\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) < 0, \quad \text{for sufficiently large } k. \tag{4.38}$$

Then it follows from (4.36), (4.37) and (4.38) that

$$z_{n_k} = (\exp_{\widetilde{z}}(\lambda_{n_k} v_n)) \in S^1, \quad \text{for sufficiently large } k. \tag{4.39}$$

Without any loss of generality, we may assume that $z_{n_k} \in S^1$ for all k. This implies that

$$v_n \in \mathscr{T}\left(S^1; \widetilde{z}\right).$$

Since $v_n \to v$ as $n \to \infty$ and the set $\mathscr{T}\left(S^1; \widetilde{z}\right)$ is closed, it follows that

$$v \in \mathscr{T}\left(S^1; \widetilde{z}\right).$$

By following exactly same procedure, we can show that for every $k = 2, 3, \ldots, r$, we have $v \in \mathscr{T}\left(S^k; \widetilde{z}\right)$. Then it follows that

$$v \in \bigcap_{k=1}^{r} \mathscr{T}\left(S^k; \widetilde{z}\right) \subseteq \bigcap_{k=1}^{r} \operatorname{cl} \operatorname{co} \mathscr{T}\left(S^k; \widetilde{z}\right).$$

This completes the proof. ∎

In the following theorem we summarize all the results that have been derived in this section.

Theorem 4.6. *Let $\widetilde{z} \in \mathcal{D}$ be a Pareto efficient solution of (MOPP). Let us assume that any of the constraint qualifications ((ACQ), (GACQ), (CTCQ), (STCQ), (LCQ), (LOCQ), (MFCQ)) is satisfied at $\widetilde{z} \in \mathcal{D}$. Then there exist multipliers $\alpha \in \mathbb{R}^r$ and $\lambda \in \mathbb{R}^s$ such that*

$$\sum_{i=1}^{r} \alpha_i \operatorname{grad} f_i(\widetilde{z}) + \sum_{j=1}^{s} \lambda_j \operatorname{grad} g_j(\widetilde{z}) = 0,$$
$$\langle \lambda, g(\widetilde{z}) \rangle = 0, \ \alpha > 0, \ \lambda \geqq 0.$$

Remark 4.2. If $\mathscr{H} = \mathbb{R}^n$, Theorem 4.6 reduces to Theorem 4.1 from [7]. Thus, Theorem 4.6 generalizes Theorem 4.1 from [7] from Euclidean space to an even more general space of Hadamard manifolds.

The relationships between various constraint qualifications are summarized in the form of a schematic diagram in Fig 1.

$$\begin{array}{ccccc} LCQ & \longrightarrow & ACQ & \longleftarrow & LOCQ \\ & & \downarrow & & \\ & & GACQ & & \\ & & \downarrow & & \\ & & GGCQ & & \\ & & \uparrow & & \\ MFCQ & \longrightarrow & CTCQ & \longleftarrow & STCQ \end{array}$$

Figure 1: Diagram showing the relationships between various constraint qualifications.

5. CONCLUSIONS AND FUTURE DIRECTIONS

This article dealt with the study of a class of constrained multiobjective programming problems on Hadamard manifolds (MOPP). The importance and novelty of our work lies in the fact that we have provided several constraint qualifications as well as several interesting interrelations among them in the framework of Hadamard manifolds, which was not previously explored. In particular, we have presented the generalized Guignard constraint qualification (GGCQ) in the framework of Hadamard manifolds for (MOPP). We have employed (GGCQ) to derive Karush-Kuhn-Tucker type necessary optimality criteria for (MOPP). Further, we have introduced several other constraint qualifications (CQs), such as, Abadie's CQ, generalized Abadie's CQ, Cottle CQ, Slater CQ, linear CQ, linear objective CQ and Mangasarian-Fromovitz CQ in Hadamard manifold setting. Moreover, we have derived various interesting relations between these constraint qualification. Subsequently, we have established these constraint qualification, in turn, become sufficient conditions ensuring that (GGCQ) is satisfied.

The various results that are presented in this article leave various avenues for future research. For example, it would be interesting to extend this work for nonsmooth multiobjective programming problems on Hadamard manifolds. In particular, we aim to explore constraint qualifications for both smooth as well as nonsmooth semi-infinite programming problems on Hadamard manifolds in our upcoming works.

ACKNOWLEDGMENTS

The first author is supported by the Council of Scientific and Industrial Research, New Delhi, India, through grant number 09/1023(0044)/2021-EMR-I.

DECLARATIONS

The authors declare that there is no actual or potential conflict of interest in relation to this article.

REFERENCES

[1] P.-A. ABSIL, C. G. BAKER and K. A. GALLIVAN, Trust-region methods on Riemannian manifolds, *Found. Comput. Math.* **7** (2007), pp. 303-330.

[2] P.-A. ABSIL, R. MAHONY and R. SEPULCHRE, *Optimization Algorithms on Matrix Manifolds*, Princeton University Press, Princeton, NJ, 2009.

[3] R. BERGMANN and R. HERZOG, Intrinsic formulation of KKT conditions and constraint qualifications on smooth manifolds, *SIAM J. Optim.* **29** (2019), pp. 2423-2444.

[4] I. EKELAND, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), pp. 324-353.

[5] T. KJELDSEN, A contextualized historical analysis of the Kuhn–Tucker theorem in nonlinear programming: the impact of World War II. *Hist. Math.* **27** (2000), pp. 331-361.

[6] X.F. LI, Constraint qualifications in nonsmooth multiobjective optimization, *J. Optim. Theory Appl.* **106** (2000), pp. 373-398.

[7] T. MAEDA, Constraint qualifications in multiobjective optimization problems: Differentiable case, *J. Optim. Theory Appl.* **80** (1994), pp. 483-500.

[8] O.L. MANGASARIAN, *Nonlinear Programming*, SIAM Classics in Applied Mathematics, vol. 10. McGraw-Hill, New York, 1969. Reprint Philadelphia, 1994.

[9] T. RAPCSÁK, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$, Springer, Berlin, 2013.

[10] E.A. PAPA QUIROZ and P.R. OLIVEIRA, Proximal point methods for quasiconvex and convex functions with Bregman distances on Hadamard manifolds, *J. Convex Anal.* **16** (2009), pp. 49-69.

[11] E.A. PAPA QUIROZ, E.M. QUISPE and P.R. OLIVEIRA, Steepest descent method with a generalized Armijo search for quasiconvex functions on Riemannian manifolds, *J. Math. Anal. Appl.* **341** (2008), pp. 467-477.

[12] E.A. PAPA QUIROZ and P.R. OLIVEIRA, Full convergence of the proximal point method for quasiconvex functions on Hadamard manifolds, *ESAIM Control Optim. Calc. Var.* **18** (2012), pp. 483-500.

[13] S. TREANŢĂ, B.B. UPADHYAY, A. GHOSH and K. NONLAOPON, Optimality conditions for multiobjective mathematical programming problems with equilibrium constraints on Hadamard manifolds, *Mathematics* **10** (2022), pp. 3516.

[14] C. UDRIŞTE, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Springer, Berlin, 2013.

[15] B.B. UPADHYAY and A. GHOSH, On constraint qualifications for mathematical programming problems with vanishing constraints on Hadamard manifolds, *J. Optim. Theory Appl.*, (2023). https://doi.org/10.1007/s10957-023-02207-2

[16] B.B. UPADHYAY, A. GHOSH, P. MISHRA and S. TREANŢĂ, Optimality conditions and duality for multiobjective semi-infinite programming problems on Hadamard manifolds using generalized geodesic convexity, *RAIRO Oper. Res.* **56** (2022), pp. 2037-2065.

[17] B.B. UPADHYAY, A. GHOSH and I.M. STANCU-MINASIAN, Second-order optimality conditions and duality for multiobjective semi-infinite programming problems on Hadamard manifolds, *Asia-Pac. J. Oper. Res.*, (2023). https://doi.org/10.1142/S0217595923500197

[18] B.B. UPADHYAY, A. GHOSH and S. TREANŢĂ, Optimality conditions and duality for nonsmooth multiobjective semi-infinite programming problems on Hadamard manifolds, *Bull. Iran. Math. Soc.* **49** (2023), pp. 1-36. https://doi.org/10.1007/s41980-023-00791-7

[19] B.B. UPADHYAY, S.K. MISHRA and S.K. PORWAL, Explicitly geodesic B-preinvex functions on Riemannian Manifolds, *Trans. Math. Program. Appl.* **2** (2015), pp. 1-14.

[20] B.B. UPADHYAY, I.M. STANCU-MINASIAN, P. MISHRA and R.N. MOHAPATRA, On generalized vector variational inequalities and nonsmooth vector optimization problems on Hadamard manifolds involving geodesic approximate convexity, *Adv. Nonlinear Var. Inequal.* **25** (2022), pp. 1-25.

[21] W.H. YANG, L.-H. ZHANG and R. SONG, Optimality conditions for the nonlinear programming problems on Riemannian manifolds, *Pac. J. Optim.* **10** (2014), pp. 415-434.