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ON INFINITE UNIONS AND INTERSECTIONS OF SETS IN A METRIC SPACE

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ABSTRACT. The aim of this paper is to examine infinite unions and intersections of sets in a general metric space, with a view to explaining when an infinite intersection of open sets is an open set and when an infinite union of closed sets is a closed set.

Key words and phrases: Metric space; Interior point; Limit point; Open set; Closed set; Infinite union; Infinite intersection.

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1. INTRODUCTION

In literature (see, for instance, [1, 2, 3, 5, 7, 8, 9, 10]), when the basic topological properties of $\mathbf{R}$ or a general metric space are discussed, it is noted that an infinite intersection of open sets in the space need not be an open set, and similarly, an infinite union of closed sets in the space need not be a closed set, either, and both are shown by examples. In order to gain some more insight into the behavior of infinite unions and intersections of sets in a general metric space, and with an eye to explaining when an infinite intersection of open sets is an open set, and similarly, when an infinite union of closed sets is a closed set, we examine when a common interior point of an infinite collection of sets in a general metric space is an interior point of the intersection of the sets, and we derive a relevant "if and only if" criterion. Using the previous criterion, we explain when an infinite intersection of open sets is an open set and when an infinite union of closed sets is a closed set, and we arrive at an "if and only if" criterion for each case. Next, we examine when a limit point of an infinite union of sets in a general metric space is a limit point of (at least) one of the sets in the union, deriving again a relevant "if and only if" criterion, by means of which we provide a sufficient condition for an infinite union of closed sets to be a closed set. We also give illustrative examples that, for the sake of understanding, are taken from the familiar metric space of real numbers with the absolute value metric.

In what follows, we assume that (M, d) is a metric space, I is a nonempty set of indices, and $\{G_i : i \in I\}$ is a collection of subsets of M, which we denote by G.

2. INTERSECTIONS AND INTERIOR POINTS

Let $x \in M$ be a common interior point of all sets in the collection G; i.e., x is an interior point of G_i for every $i \in I$. Clearly, x is independent of i. Since x is an interior point of all sets in G, there exists $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subseteq G_i$ for every $i \in I$, where $B_{\varepsilon_i}(x)$ is the open ball about x with radius ε_i, in M. For any given $i \in I$, we consider the set $E_i(x)$ such that $E_i(x) = \{\varepsilon_i > 0 : B_{\varepsilon_i}(x) \subseteq G_i\}$; i.e., the set $E_i(x)$ contains every radius $\varepsilon_i > 0$ for which the open ball $B_{\varepsilon_i}(x)$ is contained in G_i. Since x is a common interior point, the set $E_i(x)$ is nonempty and its points are positive real numbers for every $i \in I$. Hence, if $E_i(x)$ is bounded above, then the completeness of $\mathbf{R}$ implies that the supremum $\sup E_i(x)$ of $E_i(x)$ is a real number; in particular, a positive real number, while if $E_i(x)$ is unbounded above, then its supremum is positive infinity, and we write $\sup E_i(x) = \infty$. In general, $\sup E_i(x)$ depends on i, since it depends on the particular set G_i. Next, we assume that there exists $i \in I$ such that $\sup E_i(x)$ is a real number. An obvious sufficient condition for this to be true is that there exists $i \in I$ such that G_i is bounded in M. However, this condition is not necessary, because if we consider the metric space $(\mathbf{R}, |\cdot|)$ of real numbers with the absolute value metric and the collection $\{(-\infty, \frac{1}{n}) : n \in \mathbf{N}\}$, then $0 \in (-\infty, \frac{1}{n})$ for every $n \in \mathbf{N}$, and since $(-\infty, \frac{1}{n})$ is an open set; in particular, an open interval, for every $n \in \mathbf{N}$, it follows that 0 is an interior point of all sets in the collection, and clearly, $\sup E_n(0) = \frac{1}{n}$, which is a real number for every $n \in \mathbf{N}$, even though $(-\infty, \frac{1}{n})$ is unbounded below; thus, unbounded, in $\mathbf{R}$.

Next, we consider the set $E(x)$ such that $E(x) = \{\sup E_i(x) < \infty : i \in I\}$; i.e., the set $E(x)$ contains those suprema that are real numbers. Since there exists a $\sup E_i(x)$ that is a real number, it follows that $E(x)$ is nonempty. Further, since for every $i \in I$, we have $\sup E_i(x) > 0$, it follows that $E(x)$ is bounded below by 0. Hence, the completeness of $\mathbf{R}$ implies that the infimum $\inf E(x)$ of $E(x)$ is a real number and is non-negative, as 0 is a lower bound of $E(x)$. In general, $\inf E(x)$ depends on the point x, but it is independent of i, since it is taken over every $i \in I$ for which $\sup E_i(x) < \infty$. We are now ready to prove the following lemma.

Lemma 2.1. *Let $x \in M$ be an interior point of $G_i \subseteq M$ for every $i \in I$. The point x is an interior point of $\bigcap_{i\in I} G_i$ if and only if $\inf E(x) > 0$.*

Proof. (i) If x is an interior point of $\bigcap_{i\in I} G_i$, then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \bigcap_{i\in I} G_i$, whence for every $i \in I$, we have $B_\varepsilon(x) \subseteq G_i$. Consequently, for every $i \in I$, we have $\varepsilon \in E_i(x)$, whence $\sup E_i(x) \geq \varepsilon$. As a result, ε is a lower bound of $E(x)$, and thus $\inf E(x) \geq \varepsilon > 0$, whence $\inf E(x) > 0$.

(ii) If $\inf E(x) > 0$, then, since $\mathbf{R}$ is dense in $\mathbf{R}$, there exists a real number ε such that $\inf E(x) > \varepsilon > 0$. Besides, $\inf E(x)$ is a lower bound of $E(x)$, and thus $\sup E_i(x) \geq \inf E(x)$ for every $i \in I$ for which $\sup E_i(x)$ is a real number. Clearly, the previous inequality holds a fortiori for those $i \in I$ for which $\sup E_i(x) = \infty$. Consequently, $\sup E_i(x) \geq \inf E(x)$ for every $i \in I$. Hence, for every $i \in I$, we have $\sup E_i(x) \geq \inf E(x) > \varepsilon > 0$, whence $\sup E_i(x) > \varepsilon$. As a result, for every $i \in I$, we have that ε is not an upper bound of $E_i(x)$, and thus there exists $\varepsilon_i \in E_i(x)$ such that $\varepsilon_i > \varepsilon$, whence $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x)$, and since for every $i \in I$, we have $B_{\varepsilon_i}(x) \subseteq G_i$ (as a result of $\varepsilon_i \in E_i(x)$), it follows that $B_\varepsilon(x) \subseteq G_i$ for every $i \in I$, whence $B_\varepsilon(x) \subseteq \bigcap_{i\in I} G_i$. Consequently, x is an interior point of $\bigcap_{i\in I} G_i$. ∎

Corollary 2.2. *Let $x \in M$ be an interior point of $G_i \subseteq M$ for every $i \in I$. The point x is not an interior point of $\bigcap_{i\in I} G_i$ if and only if $\inf E(x) = 0$.*

Proof. As explained, $\inf E(x) \geq 0$ for every common interior point x of all sets in G. Hence, the negation of lemma 2.1 yields the statement of the corollary. ∎

Example 2.1. *Let $(\mathbf{R}, |\cdot|)$ be the metric space of real numbers with the absolute value metric and let the infinite collection of intervals $\{(-\frac{1}{n}, 1 + \frac{1}{n}] : n \in \mathbf{N}\}$ of $\mathbf{R}$.*We observe that, for every $n \in \mathbf{N}$, we have $(-\frac{1}{n}, \frac{1}{n}) \subset (-\frac{1}{n}, 1 + \frac{1}{n}]$, whence 0 is an interior point of $(-\frac{1}{n}, 1 + \frac{1}{n}]$. Following our notation, for every $n \in \mathbf{N}$, we have $\sup E_n(0) = \frac{1}{n} < \infty$, and thus $\inf E(0) = \inf\{\frac{1}{n} : n \in \mathbf{N}\} = 0$. Hence, by corollary 2.2, the point 0 is not an interior point of the infinite intersection $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, 1 + \frac{1}{n}]$. Next, we observe that, for every $n \in \mathbf{N}$, we have

$$(\frac{1}{4} - (\frac{1}{4} + \frac{1}{n}), \frac{1}{4} + (\frac{1}{4} + \frac{1}{n})) = (-\frac{1}{n}, \frac{1}{2} + \frac{1}{n}) \subset (-\frac{1}{n}, 1 + \frac{1}{n}],$$

whence $\frac{1}{4}$ is an interior point of $(-\frac{1}{n}, 1 + \frac{1}{n}]$. Further, for every $n \in \mathbf{N}$, we have

$$\sup E_n(\frac{1}{4}) = \min\{\frac{1}{4} - (-\frac{1}{n}), 1 + \frac{1}{n} - \frac{1}{4})\} = \min\{\frac{1}{4} + \frac{1}{n}, \frac{3}{4} + \frac{1}{n}\} = \frac{1}{4} + \frac{1}{n},$$

and thus $\inf E(\frac{1}{4}) = \inf\{\frac{1}{4} + \frac{1}{n} : n \in \mathbf{N}\} = \frac{1}{4} > 0$. Hence, by lemma 2.1, the point $\frac{1}{4}$ is an interior point of the infinite intersection $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, 1 + \frac{1}{n}]$.

3. INFINITE INTERSECTIONS OF OPEN SETS AND INFINITE UNIONS OF CLOSED SETS

If $E(x)$ is finite, then it contains its infimum, and since the points of $E(x)$ are positive real numbers, so is its infimum; i.e., $\inf E(x) > 0$. Thus, by lemma 2.1, every common interior point of all sets in the collection G is also an interior point of the intersection $\bigcap_{i\in I} G_i$. Clearly, if I is finite, then so is $E(x)$.[1] As a result, if I is finite, then every common interior point of all sets in the collection G is an interior point of the intersection $\bigcap_{i\in I} G_i$, too. If, additionally, all sets in G are open in M, then every common point is an interior point, and thus it is an interior point of the intersection $\bigcap_{i\in I} G_i$, too, which is therefore an open set in M. We have thus arrived at the known statement that a finite intersection of open sets in a metric space is an open set.

Note, however, that the converse is not necessarily true; it may happen that $E(x)$ is finite and yet I is infinite, since for every $i \in I$, we have $\sup E_i(x) \in E(x)$ if and only if $\sup E_i(x) < \infty$.

On the other hand, if I is infinite and all sets in the collection G are open in M, then, again, every common point of all sets in the collection is an interior point of each of the sets. Besides, the intersection $\bigcap_{i\in I} G_i$ is open in M if and only if every common point of all sets in the collection G is an interior point of the intersection $\bigcap_{i\in I} G_i$, and since every common point of all sets is an interior point of each of the sets, it follows from lemma 2.1 that the intersection $\bigcap_{i\in I} G_i$ is open in M if and only if $\inf E(x) > 0$ for every common point x of all sets in the collection G.

Next, let us suppose that I is infinite and all sets in the collection G are closed in M. We will derive a necessary and sufficient condition for the union $\bigcup_{i\in I} G_i$ to be closed in M. The union $\bigcup_{i\in I} G_i$ is closed in M if and only if its complement $M\backslash \bigcup_{i\in I} G_i$ in M is open. By De Morgan's laws, $M\backslash \bigcup_{i\in I} G_i = \bigcap_{i\in I} M\backslash G_i$. Hence, $\bigcup_{i\in I} G_i$ is closed if and only if $\bigcap_{i\in I} M\backslash G_i$ is open. Since for every $i \in I$, the set G_i is closed, it follows that $M\backslash G_i$ is open. Hence, by the previous discussion, $\bigcap_{i\in I} M\backslash G_i$ is open if and only if $\inf E(x) > 0$ for every common point x of all sets $M\backslash G_i$. Therefore, $\bigcup_{i\in I} G_i$ is closed if and only if $\inf E(x) > 0$ for every common point x of all sets $M\backslash G_i$.

Besides, since all sets $M\backslash G_i$ are open, they contain only interior points, and thus any common point x of all these sets is an interior point of each of the sets. Consequently, $\inf E(x) \geq 0$ for every common point x of all sets $M\backslash G_i$. Next, negating the previous "if and only if" statement yields that $\bigcup_{i\in I} G_i$ is not closed if and only if there exists a common point x of all sets $M\backslash G_i$ such that $\inf E(x) = 0$.

Example 3.1. *Let* $(\mathbf{R}, |\cdot|)$ *be the metric space of real numbers with the absolute value metric and let the infinite collection* $\{\bigcup_{k=1}^{n}\{\frac{1}{k}\} : n \in \mathbf{N}\}$. *We will examine whether or not the union* $\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\}$ *is closed in* $\mathbf{R}$.By De Morgan's laws, for every $n \in \mathbf{N}$, we have

$$\mathbf{R}\backslash \bigcup_{k=1}^{n}\{\frac{1}{k}\} = \bigcap_{k=1}^{n}\mathbf{R}\backslash\{\frac{1}{k}\} = \bigcap_{k=1}^{n}(-\infty, \frac{1}{k}) \cup (\frac{1}{k}, \infty).$$

For every $k \in \mathbf{N}$, the set $(-\infty, \frac{1}{k}) \cup (\frac{1}{k}, \infty)$ is open in $\mathbf{R}$, as the union of open sets (intervals). Hence, for every $n \in \mathbf{N}$, the set $\bigcap_{k=1}^{n}(-\infty, \frac{1}{k}) \cup (\frac{1}{k}, \infty)$ is a finite intersection of open sets, and thus it is open in $\mathbf{R}$. As a result, for every $n \in \mathbf{N}$, the set $\mathbf{R}\backslash \bigcup_{k=1}^{n}\{\frac{1}{k}\}$ is open in $\mathbf{R}$. Consequently, for every $n \in \mathbf{N}$, the set $\bigcup_{k=1}^{n}\{\frac{1}{k}\}$ is closed in $\mathbf{R}$. Besides, $\mathbf{R}\backslash \bigcup_{k=1}^{n}\{\frac{1}{k}\} = \mathbf{R}\backslash\{1, \frac{1}{2}, \ldots, \frac{1}{n}\}$ and, for every $n \in \mathbf{N}$, we have that $0 \in \mathbf{R}\backslash\{1, \frac{1}{2}, \ldots, \frac{1}{n}\}$. Hence, in line with our notation, for every $n \in \mathbf{N}$, we have $\sup E_n(0) = \frac{1}{n} < \infty$, and thus

$$\inf E(0) = \inf\{\frac{1}{n} : n \in \mathbf{N}\} = 0.$$

We have thus found a point $0 \in \mathbf{R}\backslash \bigcup_{k=1}^{n}\{\frac{1}{k}\}$ for every $n \in \mathbf{N}$, such that $\inf E(0) = 0$. Consequently, $\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\} = \bigcup_{k=1}^{\infty}\{\frac{1}{k}\} = \{1, \frac{1}{2}, \ldots\}$ is not a closed set in $\mathbf{R}$.

Example 3.2. *Let* $(\mathbf{R}, |\cdot|)$ *be the metric space of real numbers with the absolute value metric and let the infinite collection* $\{\{0\} \cup (\bigcup_{k=1}^{n}\{\frac{1}{k}\}) : n \in \mathbf{N}\}$. *We will examine whether or not the union* $\bigcup_{n=1}^{\infty}\{0\} \cup (\bigcup_{k=1}^{n}\{\frac{1}{k}\})$ *is closed in* $\mathbf{R}$.We observe that, for every $n \in \mathbf{N}$, we have

$$\mathbf{R}\backslash(\{0\} \cup (\bigcup_{k=1}^{n}\{\frac{1}{k}\})) = \mathbf{R}\backslash \bigcup_{k=1}^{n}\{0\} \cup \{\frac{1}{k}\},$$

and by De Morgan's laws,

$$\mathbf{R}\backslash \bigcup_{k=1}^{n}\{0\} \cup \{\frac{1}{k}\} = \bigcap_{k=1}^{n}\mathbf{R}\backslash(\{0\} \cup \{\frac{1}{k}\}) = \bigcap_{k=1}^{n}(-\infty, 0) \cup (0, \frac{1}{k}) \cup (\frac{1}{k}, \infty).$$

Hence,

$$\mathbf{R}\backslash(\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\}))=\bigcap_{k=1}^{n}(-\infty,0)\cup(0,\frac{1}{k})\cup(\frac{1}{k},\infty).$$

For every $k\in\mathbf{N}$, the set $(-\infty,0)\cup(0,\frac{1}{k})\cup(\frac{1}{k},\infty)$ is open in $\mathbf{R}$, as the union of open sets (intervals). As a result, for every $n\in\mathbf{N}$, the set $\bigcap_{k=1}^{n}(-\infty,0)\cup(0,\frac{1}{k})\cup(\frac{1}{k},\infty)$ is a finite intersection of open sets, and thus it is open in $\mathbf{R}$. Consequently, for every $n\in\mathbf{N}$, the set $\mathbf{R}\backslash(\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\}))$ is open, and thus $\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\})$ is closed in $\mathbf{R}$. Besides, for every $n\in\mathbf{N}$, we have $\mathbf{R}\backslash(\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\}))=\mathbf{R}\backslash\{1,\frac{1}{2},\dots,\frac{1}{n},0\}$. Hence, given any $x\in\mathbf{R}\backslash(\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\}))$ for every $n\in\mathbf{N}$, we have that $x\in\mathbf{R}\backslash\{1,\frac{1}{2},\dots,\frac{1}{n},0\}$, whence $x<0$ or $x>1$ or $x\in[0,1]\backslash\{1,\frac{1}{2},\dots,\frac{1}{n},0\}$. If $x<0$, then, in line with our notation, $\sup E_n(x)=-x$ (independent of n), and thus

$$\inf E(x)=\inf\{-x:n\in\mathbf{N}\}=-x>0.$$

Similarly, if $x>1$, then $\sup E_n(x)=x-1$ (independent of n), and thus

$$\inf E(x)=\inf\{x-1:n\in\mathbf{N}\}=x-1>0.$$

Finally, if $x\in[0,1]\backslash\{1,\frac{1}{2},\dots,\frac{1}{n},0\}$, then x belongs to one of the open intervals with endpoints 0 and $\frac{1}{n}$, $\frac{1}{n}$ and $\frac{1}{n-1}$,...,$\frac{1}{2}$ and 1. We observe that, in the limit $n\to\infty$, the open interval $(0,\frac{1}{n})$ and any interval of the form $(\frac{1}{n-k},\frac{1}{n-k-1})$, where k is a fixed, i.e., independent of n, non-negative integer, become empty. Besides, since $x\in\mathbf{R}\backslash\{1,\frac{1}{2},\dots,\frac{1}{n},0\}$ for every $n\in\mathbf{N}$, we have that $x\in\mathbf{R}\backslash\{1,\frac{1}{2},\dots,\frac{1}{n},0\}$ for arbitrarily large n. It is thus clear that x cannot belong to $(0,\frac{1}{n})$ or to any interval of the form $(\frac{1}{n-k},\frac{1}{n-k-1})$, either. As a result, x belongs to an open interval of the form $(\frac{1}{m+1},\frac{1}{m})$, where m is a fixed, i.e., independent of n, positive integer. Consequently, $\sup E_n(x)=\min\{x-\frac{1}{m+1},\frac{1}{m}-x\}$, which is independent of n, and thus

$$\inf E(x)=\inf\{\min\{x-\frac{1}{m+1},\frac{1}{m}-x\}:n\in\mathbf{N}\}=\min\{x-\frac{1}{m+1},\frac{1}{m}-x\}>0,$$

as a result of $x\in(\frac{1}{m+1},\frac{1}{m})$. Hence, in all three cases for x, we have that $\inf E(x)>0$. Therefore, the union

$$\bigcup_{n=1}^{\infty}\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\})=\{0\}\cup(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\})=\{0\}\cup(\bigcup_{k=1}^{\infty}\{\frac{1}{k}\})=\{0,1,\frac{1}{2},\frac{1}{3},\dots\}$$

is a closed set in $\mathbf{R}$.

4. UNIONS AND LIMIT POINTS

Definition 4.1. A point $x\in M$ is said to be a limit point of $X\subseteq M$ if for every $\varepsilon>0$, the open ball $B_\varepsilon(x)$ contains a point of X other than x; i.e., $B_\varepsilon(x)\cap(X\backslash\{x\})\neq\emptyset$.2

Theorem 4.1. *A point $x\in M$ is a limit point of $X\subseteq M$ if and only if there exists a sequence (x_n) in $X\backslash\{x\}$ that converges to x.*

The proof of the previous sequential criterion is given in standard real analysis textbooks (see, for instance [2, p. 104]), for the metric space of real numbers with the absolute value metric, and it is straightforwardly generalized to any metric space. It is worth noting that in proving

The definition 4.1 is logically equivalent to the definition given by Cantor in 1872 [6], but for subsets of real numbers, since the concept of metric space was not known at that time [11]. However, several authors prefer to use the term cluster point instead of limit point for a point that satisfies the definition 4.1, and they define limit points imposing the weaker condition that the open ball $B_\varepsilon(x)$ should contain (at least) one point of X.

the direct part of the theorem, the Axiom of Choice must be invoked in order to construct the sequence (x_n) (see, for instance, [4]).

Next, in order to derive a necessary and sufficient condition for a limit point of an infinite union to be a limit point of (at least) one of the sets in the union, we will use the following, simple auxiliary lemma.

Lemma 4.2. *Let X be a set, let (x_n) be a sequence of points of X, and let $P(n)$ be a property (a predicate) in $n \in \mathbf{N}$. Infinitely many terms of (x_n) have the property $P(n)$3 if and only if there exists a subsequence of (x_n) whose terms have the property $P(n)$.*

Proof. (i) Let infinitely many terms of (x_n) have the property $P(n)$. As a result, there exists a term x_{n_1} of (x_n) that has the property $P(n)$. Further, there exists $n_2 \in \mathbf{N}$ such that $n_2 > n_1$ and such that the term x_{n_2} has the property $P(n)$; otherwise, at most n_1 terms of (x_n), thus finitely many terms of (x_n), have the property $P(n)$, which is a contradiction. Next, we assume that, given any $k \in \mathbf{N}$, the term x_{n_k} of (x_n) has the property $P(n)$. As a result, there exists $n_{k+1} \in \mathbf{N}$ such that $n_{k+1} > n_k$ and such that the term $x_{n_{k+1}}$ of (x_n) has the property $P(n)$; otherwise, at most n_k terms of (x_n), thus finitely many terms of (x_n), have the property $P(n)$, which is a contradiction. Consequently, by induction on $k \in \mathbf{N}$, there exists a strictly increasing sequence (n_k) of natural numbers such that the term x_{n_k} of (x_n) has the property $P(n)$. Therefore, there exists a subsequence (x_{n_k}) of (x_n) whose terms have the property $P(n)$.

(ii) Let there exist a subsequence (x_{n_k}) of (x_n) whose terms have the property $P(n)$. Then, as a result of k being any natural number, infinitely many terms of (x_n) have the property $P(n)$. ∎

We are now ready to prove the following lemma.

Lemma 4.3. *Let $x \in M$ be a limit point of $\bigcup_{i \in I} G_i$. There exists $j \in I$ such that x is a limit point of G_j if and only if there exists a sequence (x_n) in $(\bigcup_{i \in I} G_i)\backslash\{x\}$ that converges to x and G_j contains infinitely many terms of (x_n).*

Proof. Let $x \in M$ be a limit point of $\bigcup_{i \in I} G_i$.

(i) Let there exist $j \in I$ such that x is a limit point of G_j. Since x is a limit point of $\bigcup_{i \in I} G_i$, it follows from the sequential criterion 4.1 that there exists a sequence (y_n) in $(\bigcup_{i \in I} G_i)\backslash\{x\}$ that converges to x. Similarly, since x is a limit point of G_j, there exists a sequence (z_n) in $G_j\backslash\{x\}$ that converges to x, too. Next, let (x_n) be the sequence defined by $x_{2n-1} = y_n$ and $x_{2n} = z_n$, for every $n \in \mathbf{N}$. Thus, the subsequence (x_{2n-1}) of (x_n) is in $(\bigcup_{i \in I} G_i)\backslash\{x\}$ and the subsequence (x_{2n}) of (x_n) is in $G_j\backslash\{x\} \subseteq (\bigcup_{i \in I} G_i)\backslash\{x\}$; hence, (x_n) is in $(\bigcup_{i \in I} G_i)\backslash\{x\}$. Further, (x_{2n-1}) converges to x, as a result of (y_n) converging to x, and (x_{2n}) also converges to x, as a result of (z_n) converging to x, too. As a result, (x_n) converges to x. Also, for every $n \in \mathbf{N}$, we have $x_{2n} = z_n \in G_j\backslash\{x\} \subseteq G_j$; hence, G_j contains infinitely many terms of (x_n). Therefore, there exists a sequence (x_n) in $(\bigcup_{i \in I} G_i)\backslash\{x\}$ that converges to x and G_j contains infinitely many terms of (x_n).

(ii) Let there exist a sequence (x_n) in $(\bigcup_{i \in I} G_i)\backslash\{x\}$ that converges to x and let G_j contain infinitely many terms of (x_n). Since G_j contains infinitely many terms of (x_n), infinitely many terms of (x_n) have the property $P(n) =$ "the nth term of (x_n) belongs to G_j", and thus by lemma 4.2, there exists a subsequence of (x_n) that is in G_j. Further, since the terms of (x_n) are different from x, so are the terms of the said subsequence. Consequently, there exists a subsequence of (x_n) that is in $G_j\backslash\{x\}$. Moreover, the said subsequence converges to x, as a result of (x_n) itself converging to x. Finally, since any subsequence of a sequence is a sequence in its own right, it follows that there exists a sequence in $G_j\backslash\{x\}$ that converges to x, and thus by the sequential criterion 4.1, the point x is a limit point of G_j. ∎

"Infinitely many terms of (x_n) have the property $P(n)$" means that the predicate $P(n)$ is satisfied, i.e., it is true, for infinitely many points of the domain $\mathbf{N}$ of (x_n).

The negation of lemma 4.3 yields the following corollary.

Corollary 4.4. *Let $x \in M$ be a limit point of $\bigcup_{i\in I} G_i$. The point x is not a limit point of any of the sets in the union $\bigcup_{i\in I} G_i$ if and only if for every sequence in $(\bigcup_{i\in I} G_i)\backslash\{x\}$ that converges to x, every set in the union $\bigcup_{i\in I} G_i$ contains only finitely many terms of the sequence.*

If I is finite, then so is the union $\bigcup_{i\in I} G_i$, which is also nonempty, as a result of I being nonempty. Hence, if $x \in M$ is a limit point of the union $\bigcup_{i\in I} G_i$, then, by the sequential criterion 4.1, there exists a sequence in $(\bigcup_{i\in I} G_i)\backslash\{x\}$ that converges to x, and since there exist finitely many sets in the union, at least one of the sets contains infinitely many terms of the sequence; otherwise, each of the finitely many sets in the union contains finitely many terms of the sequence, and thus the sequence has finitely many terms, which is a contradiction. Consequently, by lemma 4.3, the point x is also a limit point of some set in the union. However, if I is infinite, then it may happen that for every sequence in $(\bigcup_{i\in I} G_i)\backslash\{x\}$ that converges to x, each of the infinitely many sets in the union contains only finitely, and some of them possibly zero, terms of the sequence, and thus by the corollary 4.4, the point x is not a limit point of any of the sets in the union $\bigcup_{i\in I} G_i$. Indeed, referring to example 3.1, we observe that the infinite union $\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\}$ of the sets $\bigcup_{k=1}^{n}\{\frac{1}{k}\}$, where $n \in \mathbf{N}$, is written as

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\} = \bigcup_{k=1}^{\infty}\{\frac{1}{k}\} = (\bigcup_{k=1}^{\infty}\{\frac{1}{k}\})\backslash\{0\} = (\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\})\backslash\{0\},$$

and the sequence $(x_m) = (\frac{1}{m})$ is in $\bigcup_{k=1}^{\infty}\{\frac{1}{k}\} = (\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\})\backslash\{0\}$ and converges to 0. As a result, the point 0 is a limit point of the infinite union of the sets $\bigcup_{k=1}^{n}\{\frac{1}{k}\}$, where $n \in \mathbf{N}$. Further, given any sequence (y_m) in $(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\{\frac{1}{k}\})\backslash\{0\}$ that converges to 0, we observe that any of the sets $\bigcup_{k=1}^{n}\{\frac{1}{k}\}$ in the union contains at most n terms of (y_m); thus, it contains finitely many terms of (y_m). Hence, by corollary 4.4, the point 0 is not a limit point of any of the sets in the union.

In conclusion, a limit point of a nonempty and finite union is also a limit point of at least one of the sets in the union, but a limit point of an infinite union may not be a limit point of any of the sets in the union.

On the other hand, if $x \in M$ is a limit point of G_j, for some $j \in I$, then, by the sequential criterion 4.1, there exists a sequence in $G_j\backslash\{x\}$ that converges to x, and since $G_j\backslash\{x\} \subseteq (\bigcup_{i\in I} G_i)\backslash\{x\}$, it follows that there exists a sequence in $(\bigcup_{i\in I} G_i)\backslash\{x\}$ that converges to x, whence x is a limit point of $\bigcup_{i\in I} G_i$. Hence, a limit point of a set in the union $\bigcup_{i\in I} G_i$ is also a limit point of the union itself, regardless of whether the union is finite (and nonempty) or infinite.

From the previous discussion, it is clear that the derived set of a union contains the union of the derived sets of the sets in the union. Furthermore, if the union is finite, then the previous inclusion is equality.

We will conclude this section by proving the following sufficient condition, which follows from lemma 4.3 and the property of a set to be closed if and only if it contains its limit points.

Lemma 4.5. *Let G_i be closed in M, for every $i \in I$. If for every limit point $x \in M$ of the union $\bigcup_{i\in I} G_i$, there exists a sequence (x_n) in $(\bigcup_{i\in I} G_i)\backslash\{x\}$ that converges to x and $j \in I$ such that G_j contains infinitely many terms of (x_n), then $\bigcup_{i\in I} G_i$ is closed in M.*

Proof. Given any limit point x of the union $\bigcup_{i\in I} G_i$, if there exists a sequence (x_n) in $(\bigcup_{i\in I} G_i)\backslash\{x\}$ converging to x and some set G_j containing infinitely many terms of (x_n), then, by lemma 4.3, the point x is a limit point of G_j. Also, by assumption, G_j is closed; thus, it contains its limit points. As a result, $x \in G_j \subseteq \bigcup_{i\in I} G_i$; i.e., $x \in \bigcup_{i\in I} G_i$. Hence, the union $\bigcup_{i\in I} G_i$ contains its limit points, and thus it is closed in M. ∎

Lemma 4.5 is not a necessary condition, though. As shown in example 3.2, in the metric space of real numbers with the absolute value metric, the set $\{0\} \cup (\bigcup_{k=1}^{n}\{\frac{1}{k}\})$ is closed, for every $n \in \mathbf{N}$, and so is the union $\bigcup_{n=1}^{\infty}\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\})$. Further, the sequence $(x_m) = (\frac{1}{m})$ converges to 0 and is in $(\bigcup_{n=1}^{\infty}\{0\} \cup (\bigcup_{k=1}^{n}\{\frac{1}{k}\}))\setminus\{0\}$. As a result, 0 is a limit point of the previous union. Further, given any sequence (y_m) that converges to 0 and is in $(\bigcup_{n=1}^{\infty}\{0\} \cup (\bigcup_{k=1}^{n}\{\frac{1}{k}\}))\setminus\{0\}$, any of the sets in the union $\bigcup_{n=1}^{\infty}\{0\} \cup (\bigcup_{k=1}^{n}\{\frac{1}{k}\})$ contains at most n terms of (y_m) (as 0 is not a term of (y_m)); thus, it contains finitely many terms of (y_m). Hence, there does not exist a sequence in $(\bigcup_{n=1}^{\infty}\{0\}\cup(\bigcup_{k=1}^{n}\{\frac{1}{k}\}))\setminus\{0\}$ that converges to 0 and a set in the union that contains infinitely many terms of the sequence.

REFERENCES

[1] R. B. ASH, *Real Variables with Basic Metric Space Topology*, Dover Publications, New York, (2009).

[2] R. G. BARTLE and D. R. SHERBERT, *Introduction to Real Analysis*, John Wiley & Sons, Inc., New Jersey, (2011).

[3] D. GOPAL and A. DESHMUKH and A. S. RANADIVE and S. YADAV, *An Introduction to Metric Spaces*, Taylor & Francis Group, LLC, Oxfordshire, (2021).

[4] K. HRBACEK and T. JECH, *Introduction to Set Theory*, Marcel Dekker, Inc., New York, (1999).

[5] R. MAGNUS, *Metric Spaces: A Companion to Analysis*, Springer, Berlin/Heidelberg, (2022).

[6] G. H. MOORE, The emergence of open sets, closed sets, and limit points in analysis and topology, *Hist. Math.*, **35** (2008), No. 3, pp. 220âĂŞ241, [Online: https://www.sciencedirect.com/science/article/pii/S0315086008000050].

[7] M. A. PONS, *Real Analysis for the Undergraduate, With an Invitation to Functional Analysis*, Springer, Berlin/Heidelberg, (2014).

[8] C. C. PUGH, *Real Mathematical Analysis*, Springer, Berlin/Heidelberg, (2015).

[9] W. RUDIN, *Principles of Mathematical Analysis*, McGraw-Hill, Inc., New York, (1976).

[10] M. Ó SEARCÓID, *Metric Spaces*, Springer, Berlin/Heidelberg, (2007).

[11] E. S. ŞUHUBI, *Functional Analysis*, Springer, Berlin/Heidelberg, (2003).