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THE AUTOMATIC CONTINUITY OF N-HOMOMORPHISMS IN CERTAIN *-BANACH ALGEBRAS

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ABSTRACT. In this study, we prove the automatic continuity of surjective n-homomorphism between complete p-normed algebras. We show also that if $\mathfrak{A}$ and $\mathfrak{B}$ are complete *-p-normed algebras, $\mathfrak{B}$ is *simple and $\psi : \mathfrak{A} \to \mathfrak{B}$ is a surjective n-homomorphism under certain conditions, then ψ is continuous.

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1. INTRODUCTION

In this paper, the algebras considered are assumed complex, commutative, and not necessarily unitary.

Definition 1.1. Let $\mathfrak{A}$ be a vector space and p a real number $(0 < p \leq 1)$. A real function $\| \cdot \|_p$: $\mathfrak{A} \rightarrow \mathbb{R}^+$ is called a p-norm if :

- $\|x\|_p \geq 0.$
- $\|x\|_p = 0 \Longleftrightarrow x = 0.$
- $\|\lambda x\|_p = |\lambda|^p \|x\|_p \; \forall x \in \mathfrak{A}$ and $\forall \lambda \in \mathbb{C}.$
- $\|x + y\|_p \leq \|x\|_p + \|y\|_p \; \forall x, y \in \mathfrak{A}$

Definition 1.2. A(complex) p-normed algebra is a pair ($\mathfrak{A}$,$\| \cdot \|_p$) where $\mathfrak{A}$ is a complex algebra and $\| \cdot \|_p$ is a p-norm on $\mathfrak{A}$ which is sub-multiplicative,i.e. for all $x, y \in \mathfrak{A}$ we have $\|xy\|_p \leq \|x\|_p \|y\|_p$

A complete p-normed algebra is a p-normed algebra which is complete as a normed space.

2. PRELIMINARIES

It is convenient to begin by recalling some definitions and known results.
If $\mathfrak{A}$ does not have a unit, then we can adjoin one as follows:

Proposition 2.1. *A p-normed algebra without a unit can be embedded into a unital p-normed algebra $\mathfrak{A}^{\#}$ as an ideal of codimension one.*

Proof. Let $\mathfrak{A}^{\#} = \mathfrak{A} \oplus \mathbb{C}$ Direct sum of $\mathfrak{A}$ and the field of complex numbers.
$\mathfrak{A}^{\#}$ is a vector space under the usual operations :
$+ : \mathfrak{A}^{\#} \times \mathfrak{A}^{\#} \longrightarrow \mathfrak{A}^{\#}$

$((x, \alpha), (y, \beta)) \longrightarrow (x + y, \alpha + \beta)$

$. : \mathfrak{C} \times \mathfrak{A}^{\#} \longrightarrow \mathfrak{A}^{\#}$

$(\lambda, (x, \alpha)) \rightarrow (\lambda x, \lambda\alpha)$

In addition to, $\mathfrak{A}^{\#}$ is an algebra when defining a multiplication in $\mathfrak{A}^{\#}$ by :
$\odot : \mathfrak{A}^{\#} \times \mathfrak{A}^{\#}. \longrightarrow \mathfrak{A}^{\#}$

$(x, \alpha), (y, \beta)) \longrightarrow (x, \alpha) \odot (y, \beta)$
$(x, \alpha) \odot (y, \beta) := (x, \alpha)(y, \beta) := (xy + \beta x + \alpha y, \alpha\beta)$

The operation $\odot$ is closed on $\mathfrak{A}^{\#}$, and $\left(\mathfrak{A}^{\#}, +, ., \odot\right)$ is algebra with unit element $(0, 1)$.
Now, define the function $\| \cdot \|_p$ on $\mathfrak{A}^{\#}$ by :
$\| \cdot \|_p : \mathfrak{A}^{\#} \longrightarrow \mathbb{R}^+$

$(x, \alpha) \longrightarrow \|(x, \alpha)\|_p = \|x\|_p + |\alpha|$
then ($\mathfrak{A}^{\#}$,$\| \cdot \|_p$) is p-normed algebra.
Let $B = \{(x, 0) : x \in A\}$, and
Identify :
$\phi : A \rightarrow B$

$x \rightarrow (x, 0)$
$\|(x, 0)\|_p = \|x\|_p + |0 \,| = \|x\|_p$ hence ϕ is isometric isomorphe.

We write $(x,\lambda)=(x,0)+\lambda(0,1)$, since B is an ideal in $A\times\mathbb{C}$ of codimension 1.

∎

Now, define the spectrum and the spectral radius:

Let $\mathfrak{A}$ be an algebra :

(1) If $\mathfrak{A}$ is unital with unit $e_{\mathfrak{A}}$ then the spectrum and the spectral radius of x are defined by :

$$\operatorname{sp}_{\mathfrak{A}}(x) := \{\lambda\in\mathbb{C} : \lambda e_{\mathfrak{A}} - x \notin \operatorname{Inv}\mathfrak{A}\} \tag{2.1}$$

$$\rho_{\mathfrak{A}}(x) := \sup\{|\lambda| : \lambda\in\operatorname{sp}_{\mathfrak{A}}(x)\} \tag{2.2}$$

where Inv $\mathfrak{A}$ is the set of invertible elements of $\mathfrak{A}$.

(2) If $\mathfrak{A}$ is nonunital, we define the quasi-product $\centerdot$ on $\mathfrak{A}$ by

$$x\centerdot y = x+y-xy \quad (x,y\in\mathfrak{A})$$

An element x of $\mathfrak{A}$ is called quasi-invertible if there is $y\in\mathfrak{A}$ such that $x\centerdot y=0$ and $x\centerdot y=0$. The set of all quasi-invertible elements of $\mathfrak{A}$ is denoted by $q-\operatorname{Inv}\mathfrak{A}$.
Let $\mathfrak{A}^{\#}$ the Banach algebra obtained by adjoining a unit to $\mathfrak{A}$, called the unitization of $\mathfrak{A}$.
We define spectrum in non-unital Banach algebra :
$\operatorname{sp}_{\mathfrak{A}}(x)=\{0\}\cup\left\{\lambda\in\mathbb{C}\backslash\{0\} : \frac{1}{\lambda}x\notin q-\operatorname{Inv}\mathfrak{A}\right\}$ and it is easy to see that $sp_{\mathfrak{A}}(x)=sp_{\mathfrak{A}^{\#}}((x,0))$ and $\rho_A(x)=\rho_{A^{\#}}((x,0))$

Definition 2.1. An involution $*$ on an algebra $\mathfrak{A}$ is a mapping $*:\mathfrak{A}\to\mathfrak{A}$ satisfying :

$$(x+y)^*=x^*+y^* \qquad (\lambda x)^*=\bar{\lambda}x^*$$
$$(xy)^*=y^*x^*$$

with involution $*$, $\mathfrak{A}$ is called $*$-algebra.

Remark 2.1. If $\mathfrak{A}$ is involutive, defining an involution on $\mathfrak{A}^{\#}$ by : $(x,\lambda)^*:=\left(x^*,\bar{\lambda}\right),\forall(x,\lambda)\in\mathfrak{A}^{\#}$

Definition 2.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras. A linear map $\psi:\mathfrak{A}\to\mathfrak{B}$ is called an n-homomorphism if for each $\alpha_1,\ldots,\alpha_n\in\mathfrak{A}$ then $\psi(\alpha_1\ldots\alpha_n)=\psi(\alpha_1)\ldots\psi(\alpha_n)$.

An ideal J of $*$-algebra is called a $*$-ideal if $J^*\subseteq J$ (then $J^*=J$).
Recall that an algebra $\mathfrak{A}$ is called simple if it has no proper ideals. An $*$-algebra $\mathfrak{A}$ is called $*$ -simple if it has no proper $*$ -ideals.

Proposition 2.2. [7] *Let $\mathfrak{A}$ be an *-simple algebra, if $\mathfrak{A}$ is not simple. Then there exists a unitary simple subalgebra J of $\mathfrak{A}$ such that $A=J\oplus J^*$*

Definition 2.3. Let $\mathfrak{A}$ be an algebra, $\mathfrak{A}$ is called factorizable if for each $\gamma\in\mathfrak{A}$ there are $\alpha,\beta\in A$ such that $\gamma=\alpha\beta$.

Lemma 2.3. [9] *Let $\mathfrak{A}$ be a Banach algebra such that $xy=yx$. Then $\rho(x+y)\le\rho(x)+\rho(y)$ and $\rho(xy)\le\rho(x)\rho(y)$ for all $x,y\in\mathfrak{A}$*

Definition 2.4. The (Jacobson) radical of an algebra $\mathfrak{A}$ is denoted by $\operatorname{rad}\mathfrak{A}$ where $\operatorname{rad}\mathfrak{A}$ is the intersection of all maximal left (right) ideals in $\mathfrak{A}$.
Recall that an algebra $\mathfrak{A}$ is called semisimple if $\operatorname{rad}\mathfrak{A}=\{0\}$.

Lemma 2.4. [5]. *Let $\mathfrak{B}$ be a Banach algebra, let $p(z)$ be a polynomial with coefficients in $\mathfrak{B}$, and let $R>0$. Then*

$$\rho_{\mathfrak{B}}(p(1))^2 \leq \sup_{|z|=R} \rho_{\mathfrak{B}}(p(z)) \sup_{|z|=\frac{1}{R}} \rho_{\mathfrak{B}}(p(z)) \tag{2.3}$$

Lemma 2.5. *Let $\mathfrak{A}$ be a Banach algebra. Then*

1. *given $x \in \mathfrak{A}$ and suppose that $\rho_{\mathfrak{A}}(x_1 x_2 \cdots x_{n-1} x) = 0$ for all $x_1, x_2, \ldots, x_{n-1} \in \mathfrak{A}$, then $x \in \operatorname{rad} \mathfrak{A}$.*
2. *given $x \in \mathfrak{A}$ and suppose that $\rho_{\mathfrak{A}}(x x_1 x_2 \cdots x_{n-1}) = 0$ for all $x_1, x_2, \ldots, x_{n-1} \in \mathfrak{A}$, then $x \in \operatorname{rad} \mathfrak{A}$.*

Recall the concept of separating space of a linear operator, let $\mathfrak{A}$ and $\mathfrak{B}$ be two Banach algebras, and let $\psi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a linear mapping. The separating space of ψ is defined by :

$$\mathfrak{S}(\psi) = \{\beta \in \mathfrak{B} : \text{ there exists } (\alpha_m)_m \text{ in } \mathfrak{A} \text{ such that } \alpha_m \to 0 \text{ and } \psi(\alpha_m) \to \beta\} \tag{2.4}$$

We know that $\mathfrak{S}(\psi)$ is a closed linear subspace of $\mathfrak{B}$. By the closed graph theorem, ψ is continuous if and only if $\mathfrak{S}(\psi) = \{0\}$ [2, 5.1.2]

Proposition 2.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be topological algebras and $\psi : \mathfrak{A} \to \mathfrak{B}$ be a dense range n-homomorphism with $\psi(\mathfrak{A})$ is factorizable. Then $\mathfrak{S}(\psi)$ is a closed (two-sided) ideal in $\mathfrak{B}$.*

Proof. By [[2], Proposition 5.1.2], $\mathfrak{S}(\psi)$ is a closed linear subspace of $\mathfrak{B}$. Let $y \in \mathfrak{S}(\psi)$ and $x \in \mathfrak{A}$. There exists a net $\{x_m\}$ in $\mathfrak{A}$ such that $x_m \to 0$ and $\psi(x_m) \to y$. Since $\psi(\mathfrak{A})$ is a factorizable algebra, there are $x'_1, \ldots, x'_{n-1} \in \mathfrak{A}$ such that $\psi(x) = \psi(x'_1) \cdots \psi\left(x'_{n-1}\right)$. Since $x'_1 \cdots x'_{n-1} x_m \to 0$ and $\psi\left(x'_1 \cdots x'_{n-1} x_m\right) \to \psi(x'_1) \cdots \psi\left(x'_{n-1}\right) y = \psi(x) y$, it follows that $\psi(x) y \in \mathfrak{S}(\psi)$. Similarly, $y\psi(x) \in \mathfrak{S}(\psi)$
If $y' \in \mathfrak{B}$ then there exists a net $\{x'_k\}$ in $\mathfrak{A}$ such that $\psi(a'_k) \to y'$ and so $\psi(x'_k)\, y \to y' y$. Since $\psi(x'_k)\, y \in \mathfrak{S}(\psi)$ and $\mathfrak{S}(\psi)$ is closed, it follows that $y' y \in \mathfrak{S}(\psi)$. Similarly, $y y' \in \mathfrak{S}(\psi)$. Hence $\mathfrak{S}(\psi)$ is an ideal in $\mathfrak{B}$ ∎

3. Main result

Theorem 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be complete p-normed algebras, and let $\psi : \mathfrak{A} \to \mathfrak{B}$ be a surjective n-homomorphism, and suppose that $\mathfrak{B}$ is semisimple and factorizable. Then ψ is automatically continuous.*

Proof. Let $\mathfrak{A}$ be a complete p-normed algebra and $x_m \to 0$ in $\mathfrak{A}$ such that $\psi(x_m) \to y$ in $\mathfrak{B}$
Let $x \in \mathfrak{A}$ with $\psi(x) = y$, and for $m \geq 1$, and let $P_m(z) = z\psi(x_m) + (\psi(x) - \psi(x_m))$
Then for all $z \in \mathbb{C}$: $\rho_{\mathfrak{B}}(P_m(z)) \leq \|P_m(z)\|_p \leq |z| \, \|\psi(x_m)\|_p + \|\psi(x) - \psi(x_m)\|_p$

for all $z \in \mathbb{C}$:

$$\begin{aligned} \rho_{\mathfrak{B}}\left(P_m(z)^{n-1}\right) &\leq \rho_{\mathfrak{A}}\left((z x_m + (x - x_m))^{n-1}\right) \leq \left\|(z x_m + (x - x_m))^{n-1}\right\|_p \\ &\leq \|z x_m + (x - x_m)\|_p^{n-1} \leq \left(|z| \, \|x_m\|_p + \|x - x_m\|_p\right)^{n-1} \end{aligned}$$

If $\lambda \in sp_{\mathfrak{B}}(P_m(z))$ then $\lambda^{n-1} \in sp_{\mathfrak{B}}(P_m(z)^{n-1})$
Hence $\rho_{\mathfrak{B}}(P_m(z)) \leq |z| \, \|x_m\|_p + \|x - x_m\|_p$ for all $m \geq 1$, and all $R > 0$:

$$\rho_B(y)^2 \leq \left(R \|x_m\|_p + \|x - x_m\|_p\right) \left(R^{-1} \|\psi(x_m)\|_p + \|\psi(x) - \psi(x_m)\|_p\right)$$

Letting first $m \to \infty$, and then $R \to \infty$, it follows that $\rho_{\mathfrak{B}}(y) = 0$.

$\mathfrak{B}$ is factorizable, then for every $y' \in \mathfrak{B}$ there are $y'_1, \ldots, y'_{n-1} \in \mathfrak{B}$ such that $y' = y'_1 \ldots y'_{n-1}$
By choosing $x'_i \in \mathfrak{A}, i = 1, \ldots, n-1$, with $\psi(x'_i) = y'_i, i = 1, \ldots, n-1$,

we have $x_1' \ldots x_{n-1}' x_m \rightarrow 0$ in $\mathfrak{A}$ and $\psi\left(x_1' \ldots x_{n-1}' x_m\right) \rightarrow y_1 \ldots y_{n-1}' y = y' y$ in $\mathfrak{B}$ $\rho_{\mathfrak{B}}\left(y' y\right) = 0$.
Since y' is arbitrary, by Lemma 2.5, it follows that $y \in \operatorname{rad} \mathfrak{B}$, and hence $y = 0$
∎

Theorem 3.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be complete p-normed algebras with $\mathfrak{B}$ is an unital, strongly semi-simple algebra. If $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a dense range n-homomorphism such that $\psi(\mathfrak{A})$ is factorizable, then ψ has a closed graph.*

Proof. Let M be a maximal ideal of $\mathfrak{B}$. Since $\mathfrak{B}$ is an unital complete p-normed algebra, M is closed and so, by $[1, 6.14(3)]$, $\mathfrak{B}/M$ is a complete p-normed algebra. Since ideals in $\mathfrak{B}/M$ are in the form of J/M, where J is an ideal in $\mathfrak{B}$ containing M, the only ideals of $\mathfrak{B}/M$ are zero (that is, M) and $\mathfrak{B}/M$. Hence $\mathfrak{B}/M$ is simple.
Let $\pi : \mathfrak{A} \rightarrow \mathfrak{B}/M$, which is the composition of ψ, and the canonical map from $\mathfrak{B}$ onto $\mathfrak{B}/M$. By Proposition 2.6, $\mathfrak{S}(\pi)$ is an ideal of $\mathfrak{B}/M$. On the other hand, by Lemma 2.4 we have

$$\rho_{\mathfrak{B}/M}\left(\pi(x)^{n-1}\right) \leq \rho_{\mathfrak{A}}\left(x^{n-1}\right) \quad (x \in \mathfrak{A})$$

If $\lambda \in \operatorname{sp}_{\mathfrak{B}/M}(\pi(x))$ then $\lambda^{n-1} \in \operatorname{sp}_{\mathfrak{B}/M}\left(\pi(x)^{n-1}\right)$ and so $\rho_{\mathfrak{B}/M}(\pi(x)) \leq \rho_{\mathfrak{A}}(x)$. If $e_{\mathfrak{B}/M} \in \mathfrak{S}(\pi)$ then there exists a net $\{x_k\}$ in $\mathfrak{A}$ such that $x_k \rightarrow 0$ in $\mathfrak{A}$ and $\pi\left(x_k\right) \rightarrow e_{\mathfrak{B}/M}$ in $\mathfrak{B}$.
Moreover,

$$\begin{aligned} 1 = \rho_{\mathfrak{B}/M}\left(e_{\mathfrak{B}/M}\right) &\leq \rho_{\mathfrak{B}/M}\left(\pi\left(x_k\right)\right) + \rho_{\mathfrak{B}/M}\left(e_{\mathfrak{B}/M} - \pi\left(x_k\right)\right) \\ &\leq \rho_{\mathfrak{A}}\left(x_k\right) + \rho_{\mathfrak{B}/M}\left(e_{\mathfrak{B}/M} - \pi\left(x_k\right)\right). \end{aligned}$$

or $\rho_{\mathfrak{A}}$ and $\rho_{\mathfrak{B}/M}$ are continuous at zero and so

$$\rho_{\mathfrak{A}}\left(x_k\right) + \rho_{\mathfrak{B}/M}\left(e_{\mathfrak{B}/M} - \pi\left(x_k\right)\right) \rightarrow 0$$

which is a contradiction. Hence $e_{\mathfrak{B}/M} \notin \mathfrak{S}(\pi)$. Since $\mathfrak{B}/M$ is simple, it follows that $\mathfrak{S}(\pi) = M$, that is, π is continuous and hence $\pi\left(x_k\right) \rightarrow 0$, which implies that $y \in M$. Since M is an arbitrary maximal ideal, we conclude that $y \in \Re(\mathfrak{B})$. Since $\mathfrak{B}$ is strongly semisimple, we have $y = 0$.
∎

Theorem 3.3. *Let ψ be a surjective n-homomorphism from a complete p-normed algebra $\mathfrak{A}$ onto a complete *-p-normed algebra $\mathfrak{B}$, and suppose that $\mathfrak{B}$ is *-simple. Then ψ is continuous.*

Proof. Since $\mathfrak{B}$ is a *-simple algebra, there exists a unitary simple subalgebra J of $\mathfrak{B}$ such that: $\mathfrak{B} = J \oplus J^\star$; of the following algebraic isomorphism: $J \simeq \mathfrak{B}/J^\star$.
We deduce that J is a maximal ideal of $\mathfrak{B}$. Hence J(resp. $J^\star$) is closed in $\mathfrak{B}$. Hence, J (resp.$J^\star$) is a complete p-normed subalgebra.
Let :$\mathrm{Pr}_1 : \mathfrak{B} \longrightarrow J$ (resp. $Pr_2 : \mathfrak{B} \longrightarrow J^*$) the canonical projection of $\mathfrak{B}$ on J (resp. of $\mathfrak{B}$ on $J^\star$).
Since Pr_1 (resp. Pr_2) is a continuous epimorphism, $Pr_1 \circ \psi$ (resp. $Pr_2 \circ \psi$) is continuous.
As a result, $\psi = (Pr_1 + Pr_2) \circ \psi = Pr_1 \circ \psi + Pr_2 \circ \psi$ is continuous.
∎

Theorem 3.4. *Let ψ be a surjective n-homomorphism from a complete p-normed algebra $\mathfrak{A}$ onto a complete *-p-normed algebra $\mathfrak{B}$. If $\mathfrak{B}$ is *-semi-simple then ψ is continuous.*

Proof. Let M un ideal *-maximum of $\mathfrak{B}$ and $\pi : \mathfrak{B} \longrightarrow \mathfrak{B}/M$ the canonical surjection. As π is surjective and continuous, it, therefore, follows that $\pi \circ \psi$ is a surjective homomorphism in the quotient algebra $\mathfrak{B}/M$ which is *-simple. Since M is a closed ideal of $\mathfrak{B}$, $\mathfrak{B}/M$ is a complete p-normed algebra. So, by Theorem 3.2, $\pi \circ \psi$ is continuous. as a result , $\mathfrak{S}(\pi \circ \psi) = (\overline{0})$,or $\overline{0}$

is the class of 0.
Or $\mathfrak{S}\,(\pi \circ \psi) = \overline{\pi(\mathfrak{S}(\psi))}$ whence $\pi(\mathfrak{S}(\psi)) = \{0\}$,
which implies $\mathfrak{S}(\psi) \subseteq M$
Since M is arbitrary, then $\mathfrak{S}(\psi) \subseteq \cap M$ or $\cap M = \mathrm{Rad}_*(\mathfrak{A}) = \{0\}$ whence ψ is continuous. ∎

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